MID II — SOLUTIONS

Q1]...[10 points] Compute the derivatives of the following functions. Show all your work, writing down the names of key differentiation rules you are using. You do **not** have to spend time simplifying the expressions in your answers.

$$(x^3 - 2x^2 + 17)\cos(x)\,.$$

<u>PRODUCT RULE</u>

$$\frac{d}{dx}\left((x^3-2x^2+17)\cos(x)\right) = \frac{d(x^3-2x^2+17)}{dx}\cdot\cos(x) + (x^3-2x^2+17)\cdot\frac{d\cos(x)}{dx}$$

$$= (3x^2-2x)\cos(x) + (x^3-2x^2+17)(-\sin(x))$$

$$x^2 - \frac{3}{x} + \frac{7}{\sqrt{x}} - 13$$

<u>Power (& Sum) Rule</u>

$$\frac{d}{dx}\left(x^2 - \frac{3}{x} + \frac{7}{\sqrt{x}} - 13\right)$$

$$= \frac{d}{dx}\left(x^2 - 3x^{-1} + 7x^{-\frac{1}{2}} - 13\right) = 2x - 3(-1)x^{-2} + 7\left(-\tfrac{1}{2}\right)x^{-3/2} - 0$$

$$= 2x + 3x^{-2} - \frac{7}{2}x^{-3/2}$$

$$\frac{2x^2+7}{2\tan(x)+1}$$

<u>Quotient Rule</u>

$$\frac{d}{dx}\left(\frac{2x^2+7}{2\tan(x)+1}\right) = \frac{\frac{d(2x^2+7)}{dx}\cdot(2\tan(x)+1) - (2x^2+7)\frac{d}{dx}(2\tan(x)+1)}{(2\tan(x)+1)^2}$$

$$= \frac{4x(2\tan(x)+1) - (2x^2+7)\,2\sec^2(x)}{(2\tan(x)+1)^2}$$

Q2]... [10 points] Find $\frac{dy}{dx}$ for each of the following functions. Remember to show the details of your work.

$$\sin^2(3x^2+9)$$

Let $u = \sin(3x^2+9)$ $\qquad y = u^2 \qquad u = \sin(v)$

Let $v = 3x^2+9$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}$$ --- Chain Rule (x2)

$$= \frac{du^2}{du} \cdot \frac{d\sin(v)}{dv} \cdot \frac{d(3x^2+9)}{dx}$$

$$= 2u \cdot \cos(v) \cdot 6x$$

$$= \underline{2\sin(3x^2+9) \cdot \cos(3x^2+9) \cdot 6x}$$

$$\sqrt{\sec(\tan(x))}$$

Let $u = \sec(\tan(x))$ $\qquad y = \sqrt{u} = u^{1/2}$

Let $v = \tan(x)$ $\qquad u = \sec(v)$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}$$ --- ch. Rule (x2)

$$= \frac{d\sqrt{u}}{du} \cdot \frac{d\sec(v)}{dv} \cdot \frac{d\tan(x)}{dx}$$

$$= \frac{1}{2}u^{-\frac{1}{2}} \cdot \underline{\sec(v)\tan(v)} \cdot \sec^2(x)$$

$$= \frac{\sec(\tan(x)) \cdot \tan(\tan(x)) \cdot \sec^2(x)}{2\sqrt{\sec(\tan(x))}}$$

* A lot of people were confused at this point. Look over this step carefully.

Q3]... [10 points] Find the following limits. Show all your work, carefully stating any important limits from class that you may be using. *You can* **not** *use theorems (eg. L'Hôpital's Rule) that we have not yet seen in class!*

$$\lim_{t\to 0} \frac{\tan^2(5t)}{t^2}$$

$$\lim_{t\to 0} \frac{\tan^2(5t)}{t^2} = \lim_{t\to 0} \left(\frac{\sin(5t)}{t} \frac{1}{\cos(5t)}\right)^2 \quad \text{--- } \tan = \frac{\sin}{\cos}$$

$$= \lim_{t\to 0} \left(\frac{\sin(5t)}{5t} \cdot \frac{5}{\cos(5t)}\right)^2$$

$$= \left[\lim_{t\to 0}\left(\frac{\sin(5t)}{(5t)}\right)\right]^2 \cdot \frac{5^2}{\left[\lim_{t\to 0}(\cos(5t))\right]^2} \quad \text{... Limit laws (because all Limits here exist!)}$$

$$= 1^2 \cdot \frac{5^2}{1^2} \quad \text{--- by } \boxed{\cos(0)=1} \text{ \& continuity of } \cos(\theta)$$

and by $\boxed{\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) = 1}$ **

$$= \boxed{25}$$

$$\lim_{x\to 1} \frac{\sin(x-1)}{x^2-1}$$

$$\lim_{x\to 1} \frac{\sin(x-1)}{x^2-1} = \lim_{x\to 1}\left(\frac{\sin(x-1)}{(x-1)} \cdot \frac{1}{(x+1)}\right)$$

$$= \lim_{x\to 1}\left(\frac{\sin(x-1)}{(x-1)}\right) \cdot \frac{1}{\lim_{x\to 1}(x+1)} \quad \text{... by Limit laws (since all new Limits exist!)}$$

$$= 1 \cdot \frac{1}{2} \quad \text{--- by ** again}$$

Let $\theta = x-1$

as $x \to 1$ then $\theta \to 0$

$$= \boxed{\frac{1}{2}}$$

Q4]. . . [10 points] Write down the *limit of a difference quotient* definition of the derivative of a function $f(x)$.

$$f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$$

State the *product rule* for differentiation (formula only is enough).

$$\frac{d}{dx}(fg) = \frac{df}{dx} g + f \frac{dg}{dx}$$

Give a proof of the product rule.

$$\frac{d(fg)}{dx} = \lim_{h \to 0} \left(\frac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h} \right)$$

add & subtract same term!

$$= \lim_{h \to 0} \left[\frac{f(x+h)\,g(x+h) \underbrace{- f(x)\,g(x+h) + f(x)\,g(x+h)} - f(x)\,g(x)}{h} \right]$$

$$= \lim_{h \to 0} \left[\left(\frac{f(x+h) - f(x)}{h} \right) g(x+h) + f(x) \left(\frac{g(x+h) - g(x)}{h} \right) \right]$$

$$= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right) \lim_{h \to 0} \left(g(x+h) \right) + f(x) \lim_{h \to 0} \left(\frac{g(x+h) - g(x)}{h} \right)$$

by continuity of $g(x)$ (follows since $g(x)$ is differentiable). --- by Limit Laws (since all 3 limits above exist!)

$$= \frac{df}{dx} g(x) + f(x) \frac{dg}{dx}$$

done!

Q5]... [10 points] Find the derivative y' of the implicitly defined curve.

$$y^2 = 2 + x^2$$

$$\Rightarrow 2y\,y' = 0 + 2x$$

$$\Rightarrow y' = \frac{2x}{2y} \Rightarrow \boxed{y' = \frac{x}{y}}$$

Find the equation of the straight lines which pass through the point $(0, 1)$ and are tangent to the curve $y^2 = 2 + x^2$ above. How many such tangent lines are there?

Point (x,y) on $y^2 = 2+x^2$ is so that tangent line through it also contains the point $(0,1)$ (which does not lie on $y^2 = 2+x^2$).

$$\text{Slope} = y' = \frac{x}{y} \quad \text{from above}$$

$$\text{Slope} = \frac{y-1}{x-0} \Rightarrow \frac{y-1}{x} = \frac{x}{y}$$

$$\Rightarrow y^2 - y = x^2$$

But $y^2 = 2+x^2$ ... point (x,y) lies on the curve $y^2 = 2+x^2$

Thus

$$2 + \cancel{x^2} - y = \cancel{x^2}$$

$$\Rightarrow \boxed{2 = y} \Rightarrow 4 = 2+x^2 \Rightarrow x^2 = 2 \Rightarrow (x = \pm\sqrt{2})$$

$(\sqrt{2}, 2)$ and $(-\sqrt{2}, 2)$ Two points $\Rightarrow$ 2 tangent lines,

$$\boxed{(y-2) = \frac{\sqrt{2}}{2}(x - \sqrt{2})} \text{ — one tangent line}$$

$$\boxed{(y-2) = \frac{-\sqrt{2}}{2}(x + \sqrt{2})} \text{ — 2nd tangent line}$$