Local newforms with global applications in the Jacobi theory

by Ralf Schmidt

Abstract

The study of spherical representations of the Jacobi group begun in [Sch1] is continued. Using certain *index shifting operators*, the notion of *age* of such a representation is introduced, as well as the notion of a *local newform*. The precise structure of the space of spherical vectors, in particular its dimension, is determined in terms of the age. The age of the spherical principal series representations is computed, and the local newforms amongst these are determined. Restricting to the classical situation, it is shown that the local index shifting operators essentially coincide with well-known Hecke operators on classical modular forms. This leads to some global applications of the local results.

Introduction

The Jacobi group G^J is a semi-direct product of SL(2) with a three-dimensional Heisenberg group. Although it is a non-reductive algebraic group, it exhibits several features which are familiar from the theory of reductive groups, especially from the GL(2)-theory. For instance, there is a sort of classical modular forms attached to the Jacobi group, the so-called Jacobi forms, for which one can develop a Hecke theory and a theory of old- and newforms along the lines of the well-known theories for elliptic modular forms; the book [EZ] is the standard reference for classical Jacobi forms.

Just as for GL(2), one can reformulate (parts of) the classical theory of Jacobi forms in terms of local and global (automorphic) representations of the underlying group, which is G^J. In particular, one can associate to a classical Jacobi form f an automorphic representation π_f of the adelized Jacobi group $G^J(\mathbb{A})$, a procedure which is explained in the last chapter of [BeS]. Now π_f can be decomposed into local components π_p,

$$\pi_f = \bigotimes_{p \le \infty} \pi_p$$

(restricted tensor product), and these π_p, which are representations of the local Jacobi groups $G^J(\mathbb{Q}_p)$, can be further analyzed using local methods.

For good reasons one usually only considers Jacobi forms which are invariant under the *full* modular group, and not under some congruence subgroup. As a consequence, for *every* finite p the local component π_p will be *spherical*, meaning it contains a non-zero vector invariant under $G^J(\mathbb{Z}_p)$. This explains the importance of spherical representations in the Jacobi theory. In [Sch1], all the spherical representations in the so-called *good* and *almost good* cases were classified. One purpose of this note is to get insight into all the other cases, which are *bad*.

Our main tools are the *index shifting operators* U_ω and V_ω introduced in [Sch2]. These operators can move a representation from a good to a bad case, or from a bad to an even worse case, and they are invertible. Since a lot is known in the good and almost good case from [Sch1], this will allow for some sort of induction proofs to cover all cases.

Here is a summary of our main results. To every local spherical representation we shall associate a non-negative integer called the *age* of the representation. A *local newform* is then defined to be a spherical representation of age 0. We shall prove the following (Theorem 1.16):

If d is the age of π, then the space of spherical vectors in π has dimension $\left[\frac{d+2}{2}\right]$.

More precisely, if $V^{(0)}$ is the space of spherical vectors in π, then there is a *canonical filtration*

$$V^{(0)} \supset V^{(1)} \supset \ldots \supset V^{(k)} \supset (0), \qquad k = \left[\frac{d}{2}\right],$$

such that the quotient of two consecutive vector spaces is one-dimensional. If π is in particular a local newform, then it contains an essentially *unique* spherical vector.

Having determined the structure of the space of spherical vectors in terms of the age of a representation, we shall compute the age of the *principal series representations* $\pi^J_{\chi,m}$. It turns out that this age only depends on the *conductor* of the inducing character χ.

After the local theory we shall give some global applications. First of all, we will prove that the local operators U_ω and V_ω, when applied to a global object, essentially coincide with the classical operators U_p und V_p defined in §4 of [EZ]. This shows that our theory of local old- and newforms is really the local version of the classical theory of Jacobi old- and newforms in the sense of [EZ]. In particular, f is a classical *new*form if and only if the local components of π_f at all finite places are local newforms. Finally, our local results immediately lead to a direct sum decomposition for spaces of classical Jacobi forms which is stated but not proved in [EZ], and which can otherwise only be obtained with the help of a trace formula (Theorem 2.9).

1 Local Theory

1.1 Notations and basic representation theory of G^J

We set the following notations, which are valid throughout the local part of this paper:

F: a $\mathfrak{p}$-adic field.
$\mathcal{O}$: ring of integers of F.
$\mathfrak{p}$: maximal ideal of $\mathcal{O}$.
ω: a generator of $\mathfrak{p}$.
$q = \#\mathcal{O}/\mathfrak{p}$.
v: the normalized valuation of F, so $v(\omega) = 1$.
$|\cdot|$: the absolute value on F, normalized so that $|\omega| = q^{-1}$.
ψ: a character of F with conductor $\mathcal{O}$.

We let H be the three-dimensional Heisenberg group, consisting of all triples $(\lambda,\mu,\kappa)\in F^3$ with the multiplication law

$$(\lambda,\mu,\kappa)(\lambda',\mu',\kappa') = (\lambda+\lambda',\ \mu+\mu',\ \kappa+\kappa'+\lambda\mu'-\mu\lambda').$$

The group $\mathrm{SL}(2,F)$ acts on H from the right:

$$(\lambda,\mu,\kappa).\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a\lambda+c\mu,\ b\lambda+d\mu,\ \kappa),$$

and the corresponding semidirect product is called the Jacobi group G^J:

$$G^J = \mathrm{SL}(2)\ltimes H.$$

The center of G^J coincides with the center of H, and consists of all elements

$$(0,\,0,\,\kappa),\qquad\qquad \kappa\in F.$$

We will often identify this center with F. The character ψ is then considered a character of the center, and every other such character is of the form ψ^m, where $m\in F$, and $\psi^m(\kappa)=\psi(m\kappa)$.

Every irreducible, admissible representation π of G^J has a central character. If this character is ψ^m for some $m\in F$, then, having fixed ψ once and forever, we will say that π has *index*[1] m. We will concentrate throughout on representations having non-trivial central character, i.e., with index in F^*.

The Stone–von Neumann theorem states that for every $m\in F^*$ there is only one equivalence class of irreducible, admissible representations of G^J with central character ψ^m, the *Schrödinger representation* π_S^m. This representation can be extended to a projective representation of G^J, which is then called the *Schrödinger–Weil representation* π_{SW}^m. The standard model for π_{SW}^m is the Schwartz space $\mathcal{S}(F)$ (locally constant functions on F with compact support) with G^J acting as follows:

$$(\pi_S^m(\lambda,\mu,\kappa)f)(x) = \psi^m(\kappa+(2x+\lambda)\mu)f(x+\lambda), \tag{1}$$

$$\left(\pi_{SW}^m\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}f\right)(x) = \psi^m(bx^2)f(x), \tag{2}$$

$$\left(\pi_{SW}^m\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}f\right)(x) = \delta_m(a)|a|^{1/2}f(ax), \tag{3}$$

$$(\pi_{SW}^m(w)f)(x) = \gamma_m(1)\hat{f}(x) = \gamma_m(1)q^{-v(2m)/2}\int_F \psi(2mxy)f(y)\,dy. \tag{4}$$

Here γ_m is the *Weil constant*, taking values in the eigth roots of unity, and δ_m is the *Weil character*, which is built from γ_m and takes its values in the fourth roots of unity. Further we have $w=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and the measure on F is normalized so that $\mathcal{O}$ gets volume 1; Fourier inversion then holds.

The Schrödinger–Weil representation is fundamental for the representation theory of the Jacobi group, because the following theorem holds (see [BeS] 2.6): *Every irreducible, admissible representation π of G^J with index m is of the form*

$$\pi = \tilde{\pi}\otimes\pi_{SW}^m \tag{5}$$

[1] This comes from the classical theory of Jacobi forms, where we have the underlying global field $\mathbb{Q}$, a global standard character ψ, and $m\in\mathbb{Z}$ really is the index of a classical Jacobi form in the sense of [EZ].

with a uniquely determined irreducible, admissible representation of the metaplectic group Mp. Here Mp is a non-trivial two-fold cover of SL(2). A representation of Mp can be considered as a projective representation of SL(2); the cocycle of this representation cancels with the cocycle of π^m_{SW}, and the result is a representation (not a projective one) of G^J.

1.2 Spherical vectors

Let $(\tilde\pi, \tilde V)$ be an irreducible, admissible representation of $\mathrm{Mp}(F)$. Let π^m_{SW} be the Schrödinger–Weil representation with central character ψ^m. We consider the irreducible, admissible representation of $G^J(F)$

$$\pi = \tilde\pi \otimes \pi^m_{SW} \qquad \text{on the space} \qquad V = \tilde V \otimes \mathcal{S}(F).$$

For every $x \in F$, there is a linear map

$$\begin{aligned} \lambda_x:\ \tilde V \otimes \mathcal{S}(F) &\longrightarrow \tilde V, \\ \varphi \otimes f &\longmapsto f(x)\varphi. \end{aligned} \tag{6}$$

By elementary linear algebra, we have

$$v = 0 \quad \Longleftrightarrow \quad \lambda_x(v) = 0 \text{ for all } x \in F. \tag{7}$$

A vector $v \in V$ is called *spherical*, if it is non-zero and invariant under

$$\pi(K^J), \qquad \text{where } K^J = \mathrm{SL}(2,\mathcal{O}) \ltimes H(\mathcal{O}).$$

If there exists a spherical vector $v \in V$, then π is called spherical. For this to be the case, we must have ψ^m trivial on $\mathcal{O}$. Since ψ was chosen to have conductor $\mathcal{O}$, this is equivalent to $m \in \mathcal{O}$. We let

$$\begin{aligned} n &:= v(m), \\ \tilde n &:= v(2m) = v(2) + n. \end{aligned}$$

These numbers measure the complexity of our situation: As was just seen, if $n < 0$, then there are no spherical vectors. For $n = 0$ and $n = 1$, which are called the *good* resp. *almost good* cases, we have classified all the spherical representations in [Sch1]. It turns out that in these cases the subspace of K^J-invariant vectors is one-dimensional. This may no longer be true for $n \geq 2$, and the main purpose of this note is to get inside into these *bad cases*. If the local field is an extension of $\mathbb{Q}_2$, i.e., if $\tilde n > n$, then there are additional complications.

1.1 Lemma. *A spherical vector $v \in V$ can be uniquely written as*

$$v = \sum_{u \in \mathcal{O}/\omega^{\tilde n}} \varphi_u \otimes \mathbf{1}_{u\omega^{-\tilde n}+\mathcal{O}}, \tag{8}$$

where $\varphi_u \in \tilde V$ and $\mathbf{1}_{u\omega^{-\tilde n}+\mathcal{O}} \in \mathcal{S}(F)$ is the characteristic function of the coset $u\omega^{-\tilde n} + \mathcal{O}$. A vector $v \in V$ of the form (8) is spherical if and only if

i) $\tilde\pi\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\varphi_u = \psi(-mbu^2\omega^{-2\tilde n})\varphi_u \qquad$ *for all $u \in \mathcal{O}/\omega^{\tilde n}$ and all $b \in \mathcal{O}$,* *ii)* $\tilde\pi(w)\varphi_u=\gamma_m(1)^{-1}q^{-\tilde n/2}\sum_{u'\in\mathcal{O}/\omega^{\tilde n}}\psi(-2muu'\omega^{-2\tilde n})\varphi_{u'}$ *for all* $u\in\mathcal{O}/\omega^{\tilde n}$.

If this is the case, then we also have

iii) $\tilde\pi\begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix}\varphi_u=\delta_m(a)^{-1}\varphi_{ua^{-1}}$ *for all* $u\in\mathcal{O}/\omega^{\tilde n}$ *and* $a\in\mathcal{O}^*$.

Proof: It is easily seen from (1) that a function $f\in\mathcal{S}(F)$ is $H(\mathcal{O})$-invariant if and only if it is $\mathcal{O}$-invariant and its support is contained in $\omega^{-\tilde n}\mathcal{O}$. In other words, the $q^{\tilde n}$ functions $\mathbf{1}_{u\omega^{-\tilde n}+\mathcal{O}}$, where u runs through $\mathcal{O}/\omega^{\tilde n}$, constitute a basis for the space of $H(\mathcal{O})$-invariant vectors. Since the Heisenberg group acts trivially on the first factor in the tensor product $\tilde V\otimes\mathcal{S}(F)$, the first assertion follows.

Now we assume that a vector $v\in V$ of the form (8) is given. As we just saw, v is then automatically $H(\mathcal{O})$-invariant. Thus v is spherical if and only if it is $\mathrm{SL}(2,\mathcal{O})$-invariant, and this is the case if and only if

$$\pi\begin{pmatrix}1&b\\0&1\end{pmatrix}v=v\qquad\text{for all } b\in\mathcal{O},\ \text{and}\qquad \pi(w)v=v,\qquad\text{where } w=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

The first of these conditions is equivalent to i), and the second one is equivalent to ii). To see this, we note that

$$\pi_{SW}^m\begin{pmatrix}1&b\\0&1\end{pmatrix}\mathbf{1}_{u\omega^{-\tilde n}+\mathcal{O}}=\psi(mbu^2\omega^{-2\tilde n})\mathbf{1}_{u\omega^{-\tilde n}+\mathcal{O}},$$

as is easily checked using (2). Thus

$$\begin{aligned}\pi\begin{pmatrix}1&b\\0&1\end{pmatrix}v&=\sum_{u\in\mathcal{O}/\omega^{\tilde n}}\tilde\pi\begin{pmatrix}1&b\\0&1\end{pmatrix}\varphi_u\otimes\pi_{SW}^m\begin{pmatrix}1&b\\0&1\end{pmatrix}\mathbf{1}_{u\omega^{-\tilde n}+\mathcal{O}}\\&=\sum_{u\in\mathcal{O}/\omega^{\tilde n}}\psi(mbu^2\omega^{-2\tilde n})\tilde\pi\begin{pmatrix}1&b\\0&1\end{pmatrix}\varphi_u\otimes\mathbf{1}_{u\omega^{-\tilde n}+\mathcal{O}}.\end{aligned}$$

Since the characteristic functions are linearly independent, this vector equals v if and only if condition i) holds. A very similar argument applies to the other condition $\pi(w)v=v$; here one uses the easily proved identity

$$\pi_{SW}^m(w)\mathbf{1}_{u\omega^{-\tilde n}+\mathcal{O}}=\gamma_m(1)q^{-\tilde n/2}\sum_{u'\in\mathcal{O}/\omega^{\tilde n}}\psi(2muu'\omega^{-2\tilde n})\mathbf{1}_{u'\omega^{-\tilde n}+\mathcal{O}}.$$

Finally, statement iii) is proved in the same manner, using (3). ∎

Since G^J is not reductive, K^J is not a *maximal* compact subgroup. More precisely, we have an *ascending* chain of compact subgroups

$$H_k:=\{(\lambda,\mu,\kappa)\in H:\ \lambda,\mu\in\omega^{-k}\mathcal{O},\ \kappa\in\omega^{-2k}\mathcal{O}\}\subset H,\qquad k=0,1,2,\ldots \tag{9}$$

and

$$K_k^J:=\mathrm{SL}(2,\mathcal{O})\ltimes H_k,\qquad k=0,1,2,\ldots \tag{10}$$

Thus $H_0 = H(\mathcal{O})$ and $K_0^J = K^J$. We define the **spherical degree** of a spherical vector v as the non-negative integer k such that

$$v \text{ is } K_k^J\text{-invariant, but not } K_{k+1}^J\text{-invariant.}$$

We denote by $V^{(k)} := V^{\pi(K_k^J)}$ the space of K_k^J-invariant vectors. Thus $V^{(k)}$ consists of all spherical vectors of degree k or higher, and $V^{(0)}$ is the space of all K^J-invariant vectors; we have

$$V^{(0)} \supset V^{(1)} \supset V^{(2)} \supset \ldots .$$

1.2 Lemma. *The spherical degree of a spherical vector in $\pi = \tilde{\pi} \otimes \pi_{SW}^m$ can not be bigger than $\left[\frac{n}{2}\right]$, where $n = v(m)$. In other words, $V^{(k)} = 0$ for $k > \left[\frac{n}{2}\right]$. In particular, for $n = 0$ and $n = 1$, every spherical vector has degree 0.*

Proof: If $v \neq 0$ is invariant under K_k^J, then it is in particular invariant under elements $(0, 0, \kappa\omega^{-2k})$, $\kappa \in \mathcal{O}$, in the Heisenberg group. But this central element acts by multiplication with $\psi^m(\kappa\omega^{-2k})$, and we must therefore have

$$\psi(m\kappa\omega^{-2k}) = 1 \qquad \text{for all } \kappa \in \mathcal{O}.$$

This condition is equivalent with $2k \leq n$. ■

1.3 Lemma. *Let $v \in V$ be a spherical vector. Then the following are equivalent:*

i) v is K_k^J-invariant, i.e., the spherical degree of v is at least k.

ii) $\pi(\lambda\omega^{-k}, 0, 0)v = v$ for all $\lambda \in \mathcal{O}$.

iii) $\pi(0, \mu\omega^{-k}, 0)v = v$ for all $\mu \in \mathcal{O}$.

iv) If v is written in the form (8), then we have

$$\varphi_u = 0 \qquad \text{for all } u \in \mathcal{O}/\omega^{\tilde{n}} \text{ with } v(u) < k.$$

Proof: The group generated by $\mathrm{SL}(2, \mathcal{O})$ and the elements $(\lambda\omega^{-k}, 0, 0)$ (resp. $(0, \mu\omega^{-k}, 0)$) is already K_k^J, which shows the equivalence if i), ii) and iii). As for the last condition, we have

$$\begin{aligned} \pi(0, \mu\omega^{-k}, 0)v &= \sum_{u \in \mathcal{O}/\omega^{\tilde{n}}} \varphi_u \otimes \pi_S^m(0, \mu\omega^{-k}, 0)\mathbf{1}_{u\omega^{-\tilde{n}} + \mathcal{O}} \\ &= \sum_{u \in \mathcal{O}/\omega^{\tilde{n}}} \psi(2mu\mu\omega^{-\tilde{n}-k})\varphi_u \otimes \mathbf{1}_{u\omega^{-\tilde{n}} + \mathcal{O}}. \end{aligned} \tag{11}$$

If $v(u) \geq k$, then $\psi(2mu\mu\omega^{-\tilde{n}-k}) = 1$, so iii) follows from iv).

Assume on the other hand that iii) is fulfilled. Then summing (11) over $\mu \in \mathcal{O}/\omega^k$ yields a factor q^k on the left side, while on the right side only the terms with $v(u) \geq k$ survive. Thus the φ_u with $v(u) < k$ must be zero. ■

1.3 Definition of the U and V operators

For an element $s \in F^*$, consider the automorphisms of $G^J(F)$

$$U_s\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\lambda,\mu,\kappa)\right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(\lambda s,\mu s,\kappa s^2), \tag{12}$$

$$V_s\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\lambda,\mu,\kappa)\right) = \begin{pmatrix} a & bs \\ cs^{-1} & d \end{pmatrix}(\lambda,\mu s,\kappa s). \tag{13}$$

If (π, V) is a representation of G^J of index m as above, then we define new representations on the same space V by

$$U_s\pi := \pi \circ U_s, \qquad\qquad V_s\pi := \pi \circ V_s. \tag{14}$$

Obviously, $U_s\pi$ has index ms^2 and $V_s\pi$ has index ms; this is why we call U_s and V_s *index shifting operators.* See [Sch2] for more motivation and explanation concerning these operators.

Here we are especially interested in the case $s = \omega$. Explicitly we have

$$(U_\omega\pi)\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\lambda,\mu,\kappa)\right) = \pi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\lambda\omega,\mu\omega,\kappa\omega^2)\right), \tag{15}$$

$$(V_\omega\pi)\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\lambda,\mu,\kappa)\right) = \pi\left(\begin{pmatrix} a & b\omega \\ c\omega^{-1} & d \end{pmatrix}(\lambda,\mu\omega,\kappa\omega)\right). \tag{16}$$

1.4 Lemma. *Let $\pi = \tilde{\pi} \otimes \pi_{SW}^m$ on the space $V = \tilde{V} \otimes \mathcal{S}(F)$.*

i) $U_\omega\pi$ is isomorphic to $\tilde{\pi} \otimes \pi_{SW}^{m\omega^2}$, an isomorphism being given by

$$\begin{aligned} V = \tilde{V} \otimes \mathcal{S}(F) &\longrightarrow \tilde{V} \otimes \mathcal{S}(F), \\ \varphi \otimes f &\longmapsto \varphi \otimes f', \end{aligned}$$

where $f'(x) = f(\omega x)$.

ii) Let $\tilde{\pi}'$ be the conjugate representation

$$\tilde{\pi}'(g) = \pi\left(\begin{pmatrix} 1 & \\ & \omega^{-1} \end{pmatrix} g \begin{pmatrix} 1 & \\ & \omega \end{pmatrix}\right)$$

on the same space $\tilde{V}$. Then $V_\omega\pi$ is identical with $\tilde{\pi}' \otimes \pi_{SW}^{m\omega}$.

Proof: Easy exercise, or see [Sch2]. ∎

Now that we have the operators U_ω and V_ω on representations, we are going to define operators of the same name *defined on the space of spherical vectors* in π. More precisely, we shall define linear maps

$$U_\omega : V^{\pi(K^J)} \longrightarrow V^{(U_\omega\pi)(K^J)} \qquad \text{and} \qquad V_\omega : V^{\pi(K^J)} \longrightarrow V^{(V_\omega\pi)(K^J)},$$

beginning with U_ω.

The operator U_ω on spherical vectors

Obviously, if $v \in V$ is a spherical vector for π, then it is also a spherical vector for $U_\omega \pi$. Thus we define

$$U_\omega v := v \qquad \text{for a spherical vector } v \in V. \tag{17}$$

However, if $U_\omega \pi$ is transformed into the standard model for $\tilde{\pi} \otimes \pi_{SW}^m$ with the isomorphism given in Lemma 1.4 i), and if

$$v = \sum_{u \in \mathcal{O}/\omega^{\tilde{n}}} \varphi_u \otimes \mathbf{1}_{u\omega^{-\tilde{n}} + \mathcal{O}},$$

then $U_\omega v$ becomes the vector

$$U_\omega v = \sum_{u \in \mathcal{O}/\omega^{\tilde{n}}} \varphi_u \otimes \mathbf{1}_{u\omega^{-\tilde{n}-1} + \omega^{-1}\mathcal{O}} = \sum_{u' \in \mathcal{O}/\omega^{\tilde{n}+2}} \varphi'_{u'} \otimes \mathbf{1}_{u'\omega^{-\tilde{n}-2} + \mathcal{O}}$$

with

$$\varphi'_{u'} = \begin{cases} 0 \quad , & \text{if } v(u') = 0, \\ \varphi_{u'\omega^{-1}}, & \text{if } v(u') \geq 1. \end{cases}$$

1.5 Lemma. *For every $k \in \mathbb{N}_0$, U_ω induces an isomorphism*

$$V^{\pi(K_k^J)} \xrightarrow{\sim} V^{(U_\omega \pi)(K_{k+1}^J)}$$

In particular, for $v \in V$ a π-spherical vector of spherical degree k, the spherical degree of $U_\omega v$ is $k+1$.

Proof: Clear from (15). ∎

The Operator V_ω on spherical vectors

In contrast to $U_\omega \pi$, a spherical vector $v \in V$ for π is not necessarily spherical for $V_\omega \pi$. In fact, it is invariant under $H(\mathcal{O})$ and under

$$(V_\omega \pi)(\gamma) \qquad \text{for } \gamma \in K_1 \cap K_2.$$

Here

$$K_1 := \mathrm{SL}(2, \mathcal{O}), \qquad K_2 := \begin{pmatrix} & 1 \\ \omega & \end{pmatrix} K_1 \begin{pmatrix} & 1 \\ \omega & \end{pmatrix}^{-1} = V_{\omega^{-1}}(K_1),$$

and

$$K_1 \cap K_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_1 : \ c \in \omega\mathcal{O} \right\}$$

is the Iwahori subgroup. The idea is to restore the full $G^J(\mathcal{O})$-invariance by defining

$$V_\omega v := \sum_{\gamma \in K_1/(K_1 \cap K_2)} \pi(V_\omega(\gamma))v = \sum_{\gamma \in V_\omega(K_1)/(V_\omega(K_1) \cap K_1)} \pi(\gamma)v. \tag{18}$$

This vector is clearly $(V_\omega\pi)(G^J(\mathcal{O}))$-invariant. But the obvious question is: When is $V_\omega v$ non-zero?

Caution: This will not always be the case, as can be seen by the following example. Assume that q is odd and consider the positive Weil representations

$$\sigma^{J+}_{\xi,m} = \pi_W^{-m\xi+} \otimes \pi_{SW}^m$$

in the good case $v(m) = 0$, where $\xi \in F^*/F^{*2}$. By [Sch1] this representation is spherical if and only if $v(\xi)$ is even. Since $\#\mathcal{O}^*/\mathcal{O}^{*2} = 2$, this gives exactly two spherical positive Weil representations in the good case. It is easy to see that

$$V_\omega(\sigma^{J+}_{\xi,m}) = \sigma^{J+}_{\xi,m\omega} = \pi_W^{-m\omega\xi+} \otimes \pi_{SW}^{m\omega}.$$

By [Sch1], this representation is spherical if and only if $\xi \in F^{*2}$. Thus there is only one spherical positive Weil representation in the almost good case. More precisely,

$$V_\omega(\sigma^{J+}_{\xi,m}) \text{ is spherical, if } \xi \in F^{*2},$$
$$V_\omega(\sigma^{J+}_{\xi,m}) \text{ is not spherical, if } v(\xi) \text{ is even, but } \xi \notin F^{*2}.$$

However, we shall see in the next sections that positive Weil representations provide the only counterexamples for $V_\omega v$ to be non-zero. But by a theorem of Waldspurger, these representations are irrelevant for the global theory; see the section on global applications.

1.6 Lemma. *Denote by $M_2(\omega)$ the set of 2-by-2-matrices with coefficients in $\mathcal{O}$ and determinant ω. There is a bijection*

$$\begin{aligned} V_\omega(K_1)/(V_\omega(K_1)\cap K_1) &\stackrel{\sim}{\longrightarrow} M_2(\omega)/K_1, \\ A &\longmapsto \begin{pmatrix} 1 & \\ & \omega \end{pmatrix} A. \end{aligned}$$

Proof: The map is well-defined and obviously injective; the only question is surjectivity. A standard set of representatives for the right hand side is

$$\begin{pmatrix} 1 & \\ & \omega \end{pmatrix}, \quad \begin{pmatrix} \omega & b \\ & 1 \end{pmatrix}, \qquad b \in \mathcal{O}/\omega.$$

Instead of this set, use $\begin{pmatrix} 1 & \\ & \omega \end{pmatrix}$ and

$$\begin{pmatrix} \omega & b \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \begin{pmatrix} -b & \omega \\ -1 & \end{pmatrix}, \qquad b \in \mathcal{O}/\omega.$$

All these matrices lie in the image. ∎

As a consequence of this lemma, we have the formula

$$V_\omega v = v + \sum_{b\in\mathcal{O}/\omega} \pi\begin{pmatrix} \omega & b \\ & \omega^{-1} \end{pmatrix} v \tag{19}$$

(see the defining equation (18)).

The relation between U_ω and V_ω

It is obvious that the automorphisms U_ω and V_ω of $G^J(F)$ commute with each other, see (12) and (13). Therefore

$$U_\omega V_\omega\pi=V_\omega U_\omega\pi.$$

It is also easy to see that the operators U_ω and V_ω on spherical vectors commute. More precisely, we have the following lemma.

1.7 Lemma. *For every $k\in\mathbb{N}_0$, there is a commutative diagram*

$$\begin{array}{ccc}
V^{\pi(K_k^J)} & \xrightarrow[\sim]{U_\omega} & V^{(U_\omega\pi)(K_{k+1}^J)} \\
{\scriptstyle V_\omega}\big\downarrow & & \big\downarrow{\scriptstyle V_\omega} \\
V^{(V_\omega\pi)(K_k^J)} & \xrightarrow[\sim]{U_\omega} & V^{(U_\omega V_\omega\pi)(K_{k+1}^J)}.
\end{array}$$

Proof: As mentioned above, it is easy to see that U_ω and V_ω commute. The only additional information is that if v is $\pi(H_k)$-invariant, then $V_\omega v$ is $(V_\omega\pi)(H_k)$-invariant. This is a straightforward computation (we will prove something stronger in Lemma 1.10). ■

We also note that

$$V_\omega^2(g)=\begin{pmatrix}\omega&\\&\omega^{-1}\end{pmatrix}U_\omega(g)\begin{pmatrix}\omega^{-1}&\\&\omega\end{pmatrix}\qquad\text{for all } g\in G^J(F),\tag{20}$$

and thus

$$V_\omega^2\pi\simeq U_\omega\pi.$$

However, it is definitely not true that $V_\omega^2v=U_\omega v$ for a spherical vector v; this will be seen in section 1.5.

1.4 Spherical representations in the good case

The methods used in the next section to prove our main results are not suitable for treating the good case $n=v(m)=0$. So this has to be done separately, which is the purpose of the present section.

We recall from [BeS] 5.8 the classification of the irreducible, admissible representations of G^J with index m. There are

- The principal series representations $\pi^J_{\chi,m}$, where $\chi:F^*\to\mathbb{C}^*$ is a character such that $\chi^2\neq|\;|^{\pm1}$.
- The special representations $\sigma^J_{\xi,m}$ with $\xi\in F^*/F^{*2}$.
- The positive Weil representations $\sigma^{J+}_{\xi,m}$ with $\xi\in F^*/F^{*2}$.
- The negative Weil representations $\sigma^{J-}_{\xi,m}$ with $\xi\in F^*/F^{*2}$.

- The *strongly supercuspidal* representations.

Here the positive and negative Weil representations are given by

$$\sigma^{J\pm}_{\xi,m} = \pi_W^{-m\xi\pm} \otimes \pi^m_{SW},$$

where $\pi_W^{s\pm}$ is the irreducible subrepresentation of the Weil representation π_W^s consisting of even (+) resp. odd (−) functions in $\mathcal{S}(F)$ (we have $\pi_W^s = \pi_W^{s+} \oplus \pi_W^{s-}$). The principal series representations are induced from a subgroup of G^J (something analogous to a parabolic subgroup). The same induced representation is not irreducible if $\chi^2 = |\ |^{\pm 1}$; in this case it has a unique irreducible quotient, which is either a special representation, or a positive Weil representation.

1.8 Theorem. *Assume we are in the good case $n = v(m) = 0$. Then the following is a complete list of all spherical representations π of G^J with index m.*

i) The principal series representations $\pi^J_{\chi,m}$ with an unramified character χ and $\chi^2 \neq |\ |^{\pm 1}$.

ii) If F is not an extension of $\mathbb{Q}_2$: The positive Weil representations

$$\sigma^{J+}_{\xi,m} \qquad \text{with } \xi \in \mathcal{O}^*/\mathcal{O}^{*2}.$$

If $F = \mathbb{Q}_2$: The positive Weil representations

$$\sigma^{J+}_{\xi,m} \qquad \text{with } \xi \in (1+4\mathbb{Z}_2)/\mathbb{Z}_2^{*2} = (1+4\mathbb{Z}_2)/(1+8\mathbb{Z}_2).$$

This theorem was proved in [Sch1]. We briefly recall the arguments. To begin with, it is straightforward to write down a spherical vector in the induced model for the principal series representations, showing that the representations mentioned in i) are indeed spherical. One can also write down a spherical vector for the representations in ii). So all the representations mentioned in Theorem 1.8 are indeed spherical. To show that there are no further spherical representations, one can utilize the spherical Hecke algebra of G^J, which in the good case is just a polynomial ring in the standard Hecke operator $T^J(\omega)$. For general reasons, the characters of the Hecke algebra are in one-one correspondence with the spherical representations. After computing the Hecke eigenvalues of the spherical representations already found (see [Sch1] 4.2), it turns out that every Hecke algebra character is associated with one of these representations. Thus there can be no further spherical representations, because there are simply no characters left.

1.9 Lemma. *If we are in the good case $n = 0$, and π is one of the spherical principal series representations mentioned in Theorem 1.8 i), then*

$$V_\omega v \neq 0 \qquad \text{for a spherical vector } v \text{ in } \pi.$$

Proof: We note that there is up to scalars only one spherical vector v in π, since the Hecke algebra is commutative (see [BeS] 6.2). The induced model is given by functions $\Phi:\ G^J \to \mathbb{C}$ such that

$$\Phi\left(\begin{pmatrix} a & x \\ & a^{-1} \end{pmatrix}(0,\mu,\kappa)g\right) = \chi(a)\psi^m(\kappa)|a|^{3/2}\Phi(g) \tag{21}$$

for all $a \in F^*$, $x, \mu, \kappa \in F$, and $g \in G^J$. The action of G^J on this space is by right translation. Now if such a Φ is spherical, meaning right K^J-invariant, then it is obviously determined by its values on elements $(\lambda, 0, 0)$, $\lambda \in F$. It is not hard to see, and we will prove it in section 1.6, that the spherical vector is given by

$$\Phi(\lambda, 0, 0) = \begin{cases} 0, & \text{if } \lambda \notin \mathcal{O}, \\ 1, & \text{if } \lambda \in \mathcal{O}. \end{cases} \tag{22}$$

Now by formula (19), we have

$$\begin{aligned}(V_\omega \Phi)(\lambda, 0, 0) &= \Phi(\lambda, 0, 0) + \sum_{b \in \mathcal{O}/\omega} \Phi\left((\lambda, 0, 0)\begin{pmatrix} \omega & b \\ & \omega^{-1} \end{pmatrix}\right) \\ &= \Phi(\lambda, 0, 0) + \sum_{b \in \mathcal{O}/\omega} \Phi\left(\begin{pmatrix} \omega & b \\ & \omega^{-1} \end{pmatrix}(0, \lambda b, \lambda^2 b\omega)(\lambda\omega, 0, 0)\right) \\ &= \Phi(\lambda, 0, 0) + \chi(\omega)|\omega|^{3/2}\Bigg(\sum_{b \in \mathcal{O}/\omega} \psi^m(\lambda^2 b\omega)\Bigg)\Phi(\lambda\omega, 0, 0).\end{aligned}$$

This expression is zero if $\lambda \notin \mathcal{O}$. But for $\lambda \in \mathcal{O}$ we get

$$(V_\omega \Phi)(\lambda, 0, 0) = 1 + \chi(\omega)|\omega|^{3/2} q = 1 + \chi(\omega) q^{-1/2}.$$

If this were zero, then we would have $\chi(\omega) = -q^{1/2}$, and thus $\chi^2 = |\ |^{-1}$. This is not the case by hypothesis. ∎

1.5 The main results

In Theorem 1.8 a restriction was made on the field F: If F has even residue characteristic, then only $F = \mathbb{Q}_2$ is allowed. Since our results in this section build upon Theorem 1.8, we have to impose this restriction throughout this section. Fortunately, this will not affect any of the global applications later on.

We fix a representation π of index m, a positive integer d, and look at all representations of index $m\omega^d$ that can be obtained by repeatedly applying U_ω and V_ω. That is, we consider the representations

$$U_\omega^r V_\omega^s \pi, \qquad r, s \geq 0,\ 2r + s = d.$$

By the earlier observation $V_\omega^2 \pi \simeq U_\omega \pi$ (see (20)), all these representations are isomorphic (namely, isomorphic to $V_\omega^d \pi$). We choose some fixed model W for all these representations.

Now let v be a spherical vector for π. Then the vectors

$$U_\omega^r V_\omega^s v, \qquad r, s \geq 0,\ 2r + s = d,$$

can all be considered elements of W, and our goal is to show that they are linearly independent. The idea is that U_ω increases the spherical degree, while V_ω does not.

1.10 Lemma. *Assume that π is not a positive Weil representation. If v has spherical degree k, then $V_\omega v$ has also. In particular, $V_\omega v \neq 0$.*

Proof: First we consider the good case $n = 0$. By Theorem 1.8, π must be a principal series representation. Then, by Lemma 1.9, we have $V_\omega v \neq 0$. Lemma 1.2 now shows that both v and $V_\omega v$ have spherical degree 0.

Now let $n > 1$. It is easy to see that $V_\omega v$ is $(V_\omega \pi)(H_k)$-invariant if v is $\pi(H_k)$-invariant. Thus the spherical degree k of v can at most increase. We show by induction on k that it does not. Using the commutative diagram from Lemma 1.7, one shows easily that the assertion for $k+1$ follows from the assertion for k. Thus we need only consider the case $k = 0$.

Let

$$v = \sum_{u \in \mathcal{O}/\omega^{\tilde{n}}} \varphi_u \otimes \mathbf{1}_{u\omega^{-\tilde{n}}+\mathcal{O}}$$

as in Lemma 1.1. Since v has spherical degree 0, we have

$$\varphi_u \neq 0 \qquad \text{for at least one } u \in \mathcal{O}/\omega^{\tilde{n}} \text{ with } v(u) = 0 \tag{23}$$

by Lemma 1.3 (in fact $\varphi_u \neq 0$ for all such u, by Lemma 1.1 iii)). A straightforward computation shows

$$\pi^m_{SW}\begin{pmatrix} \omega & b \\ & \omega^{-1} \end{pmatrix} \mathbf{1}_{u\omega^{-\tilde{n}}+\mathcal{O}} = \delta_m(\omega) q^{-1/2} \sum_{c \in \mathcal{O}/\omega} \psi(mb\omega(u + c\omega^{\tilde{n}})^2 \omega^{-2\tilde{n}-2}) \mathbf{1}_{(u+c\omega^{\tilde{n}})\omega^{-\tilde{n}-1}+\mathcal{O}},$$

and thus, by the formula (19),

$$V_\omega v = \sum_{u \in \mathcal{O}/\omega^{\tilde{n}}} \varphi_u \otimes \mathbf{1}_{u\omega^{-\tilde{n}}+\mathcal{O}} + \delta_m(\omega) q^{-1/2} \sum_{b \in \mathcal{O}/\omega} \sum_{u \in \mathcal{O}/\omega^{\tilde{n}}} \sum_{c \in \mathcal{O}/\omega}$$
$$\psi(mb\omega(u + c\omega^{\tilde{n}})^2 \omega^{-2\tilde{n}-2}) \tilde{\pi}\begin{pmatrix} \omega & b \\ & \omega^{-1} \end{pmatrix} \varphi_u \otimes \mathbf{1}_{(u+c\omega^{\tilde{n}})\omega^{-\tilde{n}-1}+\mathcal{O}}. \tag{24}$$

Now we apply the operator λ_x (see (6)) with $x = (u+c\omega^{\tilde{n}})\omega^{-\tilde{n}-1}$ for some $c \in \mathcal{O}$ and some $u \in \mathcal{O}$ with $v(u) = 0$. Assuming in contrast to the assertion that $V_\omega v$ has spherical degree at least 1, by Lemma 1.3 we get

$$\sum_{b \in \mathcal{O}/\omega} \psi(mb\omega(u + c\omega^{\tilde{n}})^2 \omega^{-2\tilde{n}-2}) \tilde{\pi}\begin{pmatrix} \omega & b \\ & \omega^{-1} \end{pmatrix} \varphi_u = 0$$

(now b runs over a *fixed* system of representatives for $\mathcal{O}/\omega$). Now

$$\psi(mb\omega(u + c\omega^{\tilde{n}})^2 \omega^{-2\tilde{n}-2}) = \psi(mbu^2 \omega^{-2\tilde{n}-1}) \psi(2mbuc\omega^{-\tilde{n}-1})$$

(here we used $n \geq 1$), and if b is a unit, then $2mbu\omega^{-\tilde{n}-1} \notin \mathcal{O}$. Hence after summation over $c \in \mathcal{O}/\omega$ only the term with $b = 0$ survives. This yields $\varphi_u = 0$, in contradiction to (23). ∎

1.11 Lemma. *Let $v_1, \ldots, v_l$ be (non-zero) spherical vectors for π, pairwise of different spherical degree. Then $v_1, \ldots, v_l$ are linearly independent.*

Proof: Let $\sum a_i v_i = 0$ be a linear combination. Assume that v_1 has the lowest spherical degree. Using Lemma 1.3, one easily sees that a_1 must be zero. One continues in this manner, proving that all a_i are zero. ∎

1.12 Theorem. *Assume that π is not a positive Weil representation. Let v be a spherical vector for π, and $d \in \mathbb{N}$. Then the vectors*

$$U_\omega^r V_\omega^s v, \qquad r, s \geq 0,\ 2r + s = d,$$

are linearly independent.

Proof: As explained at the beginning of this section, these vectors are spherical vectors in some model W for the representation $V_\omega^d \pi$ (of index $m\omega^d$). By Lemma 1.10, the operater V_ω leaves the spherical degree unchanged, while U_ω increases the spherical degree by one. Thus the vectors in question all have different spherical degrees. By the previous lemma, they are linearly independent. ∎

1.13 Definition. *Let π be a spherical representation of $G^J(F)$.*

i) π is called a **(local) newform** *if it is not of the form $V_\omega^d \pi'$ for some $d \geq 1$ and some spherical π'.*

ii) π is called an **almost-newform** *if it is of the form $V_\omega \pi'$ for some newform π'.*

iii) If $\pi = V_\omega^d \pi'$ for a newform π', then π is said to have **age** *d.*

Our goal is to determine the dimension of the space of spherical vectors for a spherical representation of age d.

1.14 Lemma. *Let (π, V) be a spherical representation such that the space $V^{(0)}$ of spherical vectors has at least dimension 2. Then π is of the form $U_\omega \pi'$ for some spherical π'.*

Proof: Let v_1 and v_2 be linearly independent spherical vectors. A suitable linear combination of v_1 and v_2 yields a spherical vector

$$v = \sum_{u \in \mathcal{O}/\omega^{\tilde{n}}} \varphi_u \otimes \mathbf{1}_{u\omega^{-\tilde{n}} + \mathcal{O}} \quad \text{such that } \varphi_u = 0 \text{ for some } u \in \mathcal{O}/\omega^{\tilde{n}} \text{ with } v(u) = 0.$$

By Lemma 1.1 iii), we then have $\varphi_u = 0$ for *all* $u \in \mathcal{O}/\omega^{\tilde{n}}$ with $v(u) = 0$. But by Lemma 1.3, this means that v has spherical degree at least one, and so the representation $\pi' := U_{\omega^{-1}} \pi$ is spherical. ∎

1.15 Proposition. *If π is a newform or an almost-newform, then the dimension of its space of spherical vectors is 1.*

Proof: Immediate from Lemma 1.14 (recall that $U_\omega \pi = V_\omega^2 \pi$). ∎

1.16 Theorem. *Let (π, V) be a spherical representation of age d, and $\mathcal{V}^{(0)} \subset \mathcal{V}$ its space of spherical vectors. Then*

$$\dim(V^{(0)}) = \left[\frac{d+2}{2}\right] = \begin{cases} \dfrac{d+2}{2}, & \text{if } d \text{ is even,} \\ \dfrac{d+1}{2}, & \text{if } d \text{ is odd.} \end{cases}$$

More precisely, if $V^{(k)}$ denotes the space of K_k^J-invariant vectors, then we have

$$V^{(0)} \supset V^{(1)} \supset \ldots \supset V^{([d/2])} \supset 0,$$

and the quotient of two consecutive vector spaces in this filtration is one-dimensional.

Proof: First we proof the dimension formula. We know that "$\geq$" holds by Theorem 1.12. For the other direction we do induction on d. The cases $d = 0$ and $d = 1$ are taken care of by Proposition 1.15. Thus let $d \geq 2$.

Let $k = (d+2)/2$ if d is even, and $k = (d+1)/2$ if d is odd. Suppose we had $k+1$ linearly independent vectors in $V^{(0)}$. By forming suitable linear combinations, we can arrange that at least k of these vectors have spherical degree greater than one (see the reasoning in the proof of 1.14). But then the representation $U_{\omega^{-1}}\pi$, which has age $d-2$, would have at least k linearly independent spherical vectors, in contradiction to the induction hypothesis.

Let π' be a newform such that $V_\omega^d \pi' = \pi$, and let v be a spherical vector for π'. We now know that the vectors

$$U_\omega^k V_\omega^l v, \qquad 2k + l = d,$$

form a basis for $V^{(0)}$. We further know that the spherical degree of $U_\omega^k V_\omega^l v$ is k. From these facts the last assertion follows. ∎

Our results can also be summarized in the following table, which is self-explanatory.

representation	index	age	spherical vectors of degree 0	1	2	…	$\dim V^{(0)}$
⋮	⋮	⋮	⋮	⋮	⋮		⋮
$V_\omega^4\pi$	$m\omega^4$	4	$V_\omega^4 v$	$V_\omega^2 U_\omega v$	$U_\omega^2 v$		3
$V_\omega^3\pi$	$m\omega^3$	3	$V_\omega^3 v$	$V_\omega U_\omega v$			2
$V_\omega^2\pi$	$m\omega^2$	2	$V_\omega^2 v$	$U_\omega v$			2
$V_\omega\pi$	$m\omega$	1	$V_\omega v$				1
π (newform)	m	0	v				1

1.6 Principal series representations

Now we determine the age of the principal series representations $\pi^J_{\chi,m}$. In particular, we determine all the newforms amongst these, and the dimension of the spaces of spherical vectors.

So let χ be a character of F^*. By [BeS] 5.5, the principal series representation $\pi^J_{\chi,m}$ is an induced representation, its space $\mathcal{B}^J_{\chi,m}$ consisting of smooth functions $\Phi:\ G^J \to \mathbb{C}$ such that

$$\Phi\left(\begin{pmatrix} a & x \\ & a^{-1}\end{pmatrix}(0,\mu,\kappa)g\right) = \chi(a)\psi^m(\kappa)|a|^{3/2}\Phi(g) \tag{25}$$

for all $a \in F^*$, $x,\mu,\kappa \in F$, and $g \in G^J$. The action of G^J on this space is by right translation. Now if such a Φ is a *spherical* vector, it is determined by its values on

$$A^JN^J = \left\{\begin{pmatrix} a & x \\ & a^{-1}\end{pmatrix}(\lambda,\mu,\kappa):\ a \in F^*,\ x,\lambda,\mu,\kappa \in F\right\}.$$

Conversely, a function on this group can be uniquely extended to a K^J-rightinvarant function on G^J if and only if it is right-invariant under

$$A^JN^J \cap K^J = \left\{\begin{pmatrix} a & x \\ & a^{-1}\end{pmatrix}(\lambda,\mu,\kappa):\ a \in \mathcal{O}^*,\ x,\lambda,\mu,\kappa \in \mathcal{O}\right\}. \tag{26}$$

1.17 Lemma. *Let Φ be a function on A^JN^J transforming as in (25). Then Φ is right-invariant under $A^JN^J \cap K^J$ if and only if the following holds:*

i) $\Phi(\lambda,0,0) = \Phi(\lambda',0,0)$ *if* $\lambda - \lambda' \in \mathcal{O}$.

ii) $\chi(a)\Phi(a\lambda,0,0) = \Phi(\lambda,0,0)$ *for all* $a \in \mathcal{O}^*$ *and* $\lambda \in F$.

iii) If $\Phi(\lambda,0,0) \neq 0$*, then*

$$v(\lambda) \geq \begin{cases} -n/2, & \text{if } n \text{ is even,} \\ -(n-1)/2, & \text{if } n \text{ is odd.}\end{cases}$$

(Here $n = v(m)$ as usual.)

Proof: Φ is obviously determined by the values $\Phi(\lambda,0,0)$, $\lambda \in F$. Writing down the condition

$$\Phi((\lambda,0,0)k) = \Phi(\lambda,0,0) \qquad \text{for all } \lambda \in F,\ k \in A^JN^J \cap K^J,$$

one quickly arrives at i), as well as at the condition

$$\chi(a)\psi^m(2\lambda a\mu + \lambda^2 ax)\Phi(a\lambda,0,0) = \Phi(\lambda,0,0) \qquad \text{for all } \lambda \in F,\ a \in \mathcal{O}^*,\ x,\mu \in \mathcal{O}. \tag{27}$$

For $x = \mu = 0$ one obtains the condition ii). For $a = 1$ and $\mu = 0$ one obtains condition iii), because ψ has conductor $\mathcal{O}$. Conversely, (27) follows from i), ii) and iii), as is easy to see. ∎

Let χ have conductor $\omega^l\mathcal{O}$, $l \geq 0$. For $l = 0$ this means that χ is unramified, whereas for $l \geq 1$ it means that χ is trivial on $1 + \omega^l\mathcal{O}$, but not on $(1 + \omega^{l-1}\mathcal{O}) \cap \mathcal{O}^*$.

1.18 Lemma. *Let Φ be a function on A^JN^J as in the previous Lemma, i.e., Φ transforms as in (25), and is right-invariant under $A^JN^J \cap K^J$. Let χ have conductor $\omega^l\mathcal{O}$ with $l \geq 1$. Then, for all $\lambda \in F$,*

$$\Phi(\lambda, 0, 0) \neq 0 \quad \Longrightarrow \quad v(\lambda) \leq -l.$$

Proof: If $a \in 1 + \lambda^{-1}\mathcal{O}$, then $\lambda - a\lambda \in \mathcal{O}$, and so

$$\Phi(a\lambda, 0, 0) = \Phi(\lambda, 0, 0)$$

by condition i) of the previous lemma. Thus

$$\chi(a) = 1 \qquad \text{for all } (a \in 1 + \lambda^{-1}\mathcal{O}) \cap \mathcal{O}^*.$$

by condition ii). It follows that $\omega^l\mathcal{O} \supset \lambda^{-1}\mathcal{O}$, which is equivalent to $v(\lambda) \leq -l$. ∎

1.19 Proposition. *Let $\omega^l\mathcal{O}$ be the conductor of χ.*

i) Let n be even. If $l = \frac{n}{2}$, then $\pi^J_{\chi,m}$ is spherical, and the dimension of its space of K^J-invariant vectors is 1.

ii) If n is even and $l > \frac{n}{2}$, then $\pi^J_{\chi,m}$ is not spherical.

iii) If n is odd and $l \geq \frac{n+1}{2}$, then $\pi^J_{\chi,m}$ is not spherical.

Proof: i) First we treat the case $n = 0$. Then χ is unramified, and it follows from Lemma 1.17 that there is (up to scalars) a unique spherical vector, determined by

$$\Phi(\lambda, 0, 0) = \begin{cases} 0, & \text{if } \lambda \notin \mathcal{O}, \\ 1, & \text{if } \lambda \in \mathcal{O}. \end{cases} \tag{28}$$

(This was already used in the proof of Lemma 1.9.)

Now assume that $n \geq 2$, and that Φ is a spherical vector in the induced model. By the previous lemmas, we can have $\Phi(\lambda, 0, 0) \neq 0$ only for $v(\lambda) = -\frac{n}{2}$. Then condition ii) in Lemma 1.17 forces Φ to be the function determined by

$$\Phi(\lambda, 0, 0) = \begin{cases} 0, & \text{if } v(\lambda) \neq -\frac{n}{2}, \\ c\,\chi(a)^{-1}, & \text{if } \lambda = a\omega^{-n/2},\ a \in \mathcal{O}^*, \end{cases} \tag{29}$$

where $c = \Phi(\omega^{-n/2}, 0, 0)$. Thus, up to scalar multiples, there can be at most one spherical vector. On the other hand, one checks easily that Φ as in (29) (say with $c = 1$) has the three properties listed in Lemma 1.17. It can thus be extended to a function on A^JN^J right-invariant under $A^JN^J \cap K^J$, and can then be further extended to a spherical vector in the induced model.

ii) and iii) In these cases, Lemma 1.17 iii) and Lemma 1.18 show that there can be no spherical vectors. ∎

1.20 Theorem. *The newforms amongst the principal series representations are exactly the representations*

$$\pi^J_{\chi,m}, \qquad \text{where } n = v(m) \text{ is even, and the conductor of } \chi \text{ is } \omega^{n/2}\mathcal{O}.$$

Proof: Let n be odd, and assume $\pi^J_{\chi,m}$ is a spherical representation. By iii) in Proposition 1.19, we must have χ trivial on $1+\omega^{(n-1)/2}\mathcal{O}$. But then by i) in the same proposition, $\pi^J_{\chi,m\omega^{-1}}$ is spherical. Thus

$$\pi^J_{\chi,m} = V_\omega \pi^J_{\chi,m\omega^{-1}}$$

is not a newform. So newforms can only occur for even n.

Let $n \geq 2$ be even, and consider $\pi = \pi^J_{\chi,m}$, where χ has conductor $\omega^{n/2}\mathcal{O}$. By i) in the Proposition, π has an essentially unique spherical vector. Thus π can not have age $d \geq 2$, because then it would have more spherical vectors by Theorem 1.16. But π can also not have age 1, because then we would have a newform in the odd case, which we just saw is not possible. The conclusion is that π must itself be a newform.

If n is even and the conductor is $\omega^l\mathcal{O}$ with $l > \frac{n}{2}$, then $\pi^J_{\chi,m}$ is not spherical by Proposition 1.19 ii). Finally assume that n is even and $\pi = \pi^J_{\chi,m}$ is spherical with χ being trivial on $(1+\omega^{n/2-1}\mathcal{O}) \cap \mathcal{O}^* = (1+\omega^{(n-2)/2}\mathcal{O}) \cap \mathcal{O}^*$. Then

$$\pi' = \pi^J_{\chi,m\omega^{-2}}$$

is spherical by the proposition, and thus $\pi = U_\omega \pi'$ is not a newform. ∎

1.21 Corollary. *Let $\pi = \pi^J_{\chi,m}$ and $\omega^l\mathcal{O}$ the conductor of χ. Then π is spherical if and only if $l \leq \frac{n}{2}$. In this case the age of π is $n-2l$. In particular, if $V^{(0)}$ is the space of K^J-invariant vectors of π, then*

$$\dim(V^{(0)}) = \begin{cases} \dfrac{n}{2}+1-l & \text{if } n \text{ is even,} \\ \dfrac{n+1}{2}-l & \text{if } n \text{ is odd.} \end{cases}$$

Proof: Let n be even. By the theorem,

$$\pi' := U_{\omega^{l-n/2}}\,\pi = \pi^J_{\chi,m\omega^{2l-n}}$$

is a newform. Thus $\pi = V_\omega^{n-2l}\,\pi'$ has age $n-2l$.

Let n be odd. Then π is not a newform by the theorem. Thus $\pi' := V_{\omega^{-1}}\pi$ is spherical, and of age $n-1-2l$ by what we just saw. Consequently $\pi = V_\omega\pi'$ is of age $n-2l$.

The dimension formula follows from Theorem 1.16. ∎

2 Global applications

2.1 The U and V operators on classical Jacobi forms

For positive integers k, m let $J_{k,m}$ be the space of classical Jacobi forms of weight k and index m in the sense of [EZ]. Any $f \in J_{k,m}$ is then a function $\mathbb{H} \times \mathbb{C} \to \mathbb{C}$, where $\mathbb{H}$ is the upper complex half-plane, and we denote by $J^{\text{cusp}}_{k,m}$ the subspace of cusp forms.

The real Jacobi group $G^J(\mathbb{R})$ acts on $\mathbb{H}\times\mathbb{C}$ by

$$g.(\tau,z)=\left(\frac{a\tau+b}{c\tau+d},\frac{z+\lambda\tau+\mu}{c\tau+d}\right),$$

where

$$g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}(\lambda,\mu,\kappa)\in G^J(\mathbb{R}),\qquad\text{and}\qquad(\tau,z)\in\mathbb{H}\times\mathbb{C}.$$

The *automorphic factor* of weight k and index m is

$$j_{k,m}(g,(\tau,z))=(c\tau+d)^{-k}\exp\left(2\pi im\left(-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d}+\lambda^2\tau+2\lambda z+\lambda\mu+\kappa\right)\right),$$

with g and (τ,z) as above.

We consider the adelized Jacobi group $G^J(\mathbb{A})$, where $\mathbb{A}$ is the adele ring of $\mathbb{Q}$. To $f\in J_{k,m}$, there is associated a function ϕ_f on $G^J(\mathbb{A})$ in the following way. Every element of $G^J(\mathbb{A})$ can be written (in a non-unique way) as

$$g=\gamma g_\infty k_0\qquad\text{with }\gamma\in G^J(\mathbb{Q}),\ g_\infty\in G^J(\mathbb{R}),\ k_0\in\prod_{p<\infty}G^J(\mathbb{Z}_p).$$

Then ϕ_f is well-defined by

$$\phi_f(g)=j_{k,m}(g_\infty,(i,0))f(g_\infty(i,0))\qquad\text{with }g=\gamma g_\infty k_0\text{ as above.}$$

From [BeS] 4.1.2 and 7.4.5 we get the following result.

2.1 Proposition. *The map $f\mapsto\phi_f$ establishes an isomorphism between $J_{k,m}^{\rm cusp}$ and the space $\mathcal{A}_{k,m}^{\rm cusp}$ of functions $\phi:\ G^J(\mathbb{A})\to\mathbb{C}$ sharing the following properties.*

i) $\phi(\gamma g)=\phi(g)$ *for all* $g\in G^J(\mathbb{A})$, $\gamma\in G^J(\mathbb{Q})$.

ii) $\phi(g(0,0,\kappa))=\psi^m(\kappa)\phi(g)$ *for all* $g\in G^J(\mathbb{A})$, $\kappa\in\mathbb{A}$.

iii) $\phi(gr(\theta))=e^{ik\theta}\phi(g)$ *for all* $g\in G^J(\mathbb{A})$, $r(\theta)=\begin{pmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{pmatrix}$.

iv) $\phi(gk_0)=\phi(g)$ *for all* $g\in G^J(\mathbb{A})$, $k_0\in\prod_{p<\infty}G^J(\mathbb{Z}_p)$.

v) ϕ *is smooth as a function on* $G^J(\mathbb{R})$*, and as such is annihilated by the differential operators* $\mathcal{L}_{X_-}$ *and* $\mathcal{L}_{Y_-}$.

vi) ϕ *is bounded.*

Some comments are in order. First of all, the function $\psi:\ \mathbb{A}\to\mathbb{C}^*$ appearing in ii) is the "standard character" described in [BeS] 2.2; it is a character of $\mathbb{A}/\mathbb{Q}$ with the property that its infinite component coincides with the character $x\mapsto e^{2\pi ix}$ of $\mathbb{R}$.

Second, the differential operators in v) come from elements X_- and Y_- in the complexified Lie algebra of $G^J(\mathbb{R})$. Specifically, X_- involves only the $\mathfrak{sl}(2)$-part of the Lie-algebra, and is the usual "annihilation operator"; see [BeS] 1.4 for more details. The element Y_- is in the complexified Lie algebra of the Heisenberg group and equals $\frac{1}{2}(P - iQ)$, where

$$P = (1,0,0) \qquad \text{and} \qquad Q = (1,0,0)$$

are elements of the Heisenberg Lie algebra. For later use note that

$$\exp(tP) = (t,0,0) \qquad \text{and} \qquad \exp(tQ) = (0,t,0). \tag{30}$$

If we start with an *eigenform* $f \in J_{k,m}^{\rm cusp}$, then it turns out that ϕ_f generates an irreducible subrepresentation π_f of a canonically defined space $L^2(G^J(\mathbb{Q})\backslash G^J(\mathbb{A}))_m$. This is the *automorphic representation* of $G^J(\mathbb{A})$ associated to f. It is clear from ii) in Proposition 2.1 that π_f has central character ψ^m; this now explains the notion of *index* we introduced in 1.1 for local representations.

As any global representation π_f may be decomposed into local components:

$$\pi_f = \bigotimes_{p\le\infty} \pi_p, \qquad \pi_p \text{ a representation of } G^J(\mathbb{Q}_p) \tag{31}$$

(restricted tensor product). Here at the archimedean component we have $\pi_\infty = \pi_{m,k}^{J+}$, a discrete series representation of $G^J(\mathbb{R})$ (see [BeS] 3.1).

In the GL(2)-theory, our *index* m corresponds to the *level* of a classical modular form, and only those π_p where $p \nmid m$ would be spherical. However, in the Jacobi theory, the index m manifestes itself differently (namely as the central character of π_f), and *all* the local components π_p for $p < \infty$ in (31) turn out to be spherical. This explains the importance of spherical representations in the theory of Jacobi forms.

In (14) we introduced local index shifting operators on representations of the Jacobi group. Now we define the analogous *global* index shifting operators by the same formula, where s is now an idele. It is clear that if $s = (s_p)_p$, then

$$\pi = \otimes \pi_p \qquad \Longrightarrow \qquad U_s\pi = \otimes U_{s_p}\pi_p \qquad \text{and} \qquad V_s\pi = \otimes V_{s_p}\pi_p.$$

It is further clear that if $s \in \mathbb{Q}^*$, then U_s and V_s take automorphic representations to automorphic representations. We will be interested in the operators U_p and V_p for a prime number $p \in \mathbb{Q}^*$. For some more information on local and global index shifting see [Sch2].

Now we start with an eigenform $f \in J_{k,m}^{\rm cusp}$ and let $\mathcal{V}$ be the space of π_f. For a fixed prime p we consider the representation $U_p\pi_f$, which is again automorphic. Its space is again $\mathcal{V}$, but $G^J(\mathbb{A})$ no longer acts by right translation. But it is obvious how to obtain a model for $U_p\pi_f$ where $G^J(\mathbb{A})$ acts by right translation; in this model, ϕ_f becomes the function

$$(U_p\phi_f)(g) := \phi_f(U_p(g)), \qquad g \in G^J(\mathbb{A}). \tag{32}$$

We check that $U_p\phi_f$ is again a Jacobi form:

2.2 Lemma. *The function $U_p\phi_f$ has the properties i) through vi) from Proposition 2.1, with m replaced by mp^2. It thus corresponds to a classical Jacobi form of index mp^2.*

Proof: Everything is clear except maybe v). As mentioned above, X_- is an element in the complexification of the Lie-algebra of SL(2). Since the automorphism U_p leaves the SL(2)-part unaffected, we have

$$\mathcal{L}_{X_-}(U_p\phi_f) = U_p(\mathcal{L}_{X_-}\phi_f) = 0.$$

As for Y_-, we compute

$$\begin{aligned}\mathcal{L}_P(U_p\phi_f)(g) &= \frac{d}{dt}\Big|_0 (U_p\phi_f)(g\exp(tP)) = \frac{d}{dt}\Big|_0 \phi_f(U_p(g(t,0,0))) \\ &= \frac{d}{dt}\Big|_0 \phi_f(U_p(g)(tp,0,0)) = p\,\frac{d}{dt}\Big|_0 \phi_f(U_p(g)(t,0,0)),\end{aligned}$$

and thus $\mathcal{L}_P(U_p\phi_f) = p\,U_p(\mathcal{L}_P\phi_f)$. Similarly, $\mathcal{L}_Q(U_p\phi_f) = p\,U_p(\mathcal{L}_Q\phi_f)$, and therefore

$$\mathcal{L}_{Y_-}(U_p\phi_f) = p\,U_p(\mathcal{L}_{Y_-}\phi_f) = 0.$$

∎

Now we try a similar construction with the operator V_p. Consider the automorphic representation $V_p\pi_f$ with index mp on the same space $\mathcal{V}$. We move the vector ϕ_f to another model where $G^J(\mathbb{A})$ acts by right translation, obtaining the function

$$\phi_1(g) = \phi(V_p(g)).$$

In contrast to the situation with U_p, this function does not necessarily fulfill condition iv) in Proposition 2.1. For every finite $q \neq p$, the invariance is clear, but for p we only have invariance under $H(\mathbb{Z}_p)$ and under

$$\mathrm{SL}(2,\mathbb{Z}_p) \cap V_{p^{-1}}(\mathrm{SL}(2,\mathbb{Z}_p)) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}_p) :\ p|c \right\} =: K_0(p).$$

We can however make this function right $G^J(\mathbb{Z}_p)$-invariant by defining

$$\phi_2(g) := \sum_{\gamma\in\mathrm{SL}(2,\mathbb{Z}_p)/K_0(p)} \phi_1(g\gamma).$$

This function now has property iv), but is still not the one we are heading for. Instead we go to yet another model and define

$$(V_p\phi_f)(g) := \phi_2\left(g\begin{pmatrix} \sqrt{p} & \\ & \sqrt{p}^{-1} \end{pmatrix}\right) = \sum_{\gamma\in\mathrm{SL}(2,\mathbb{Z}_p)/K_0(p)} \phi_f\left(V_p(g\gamma)\begin{pmatrix} \sqrt{p} & \\ & \sqrt{p}^{-1} \end{pmatrix}\right). \tag{33}$$

where $\begin{pmatrix} \sqrt{p} & \\ & \sqrt{p}^{-1} \end{pmatrix}$ is of course an element of $G^J(\mathbb{R})$.

2.3 Lemma. *The function $V_p\phi_f$ has the properties i) through vi) from Proposition 2.1, with m replaced by mp. It thus corresponds to a classical Jacobi form of index mp.*

Proof: Again everything is obvious except v). The arguments for this are similar as in Lemma 2.2; in doing the computation it becomes clear why we introduced the matrix $\begin{pmatrix} \sqrt{p} & \\ & \sqrt{p}^{-1} \end{pmatrix}$ in the definition of $V_p\phi_f$. ∎

For the next Lemma recall our definitions (17) and (18) of the operators U_ω and V_ω *on spherical vectors* of a local representation.

2.4 Lemma. *Let $\pi_f = \otimes\pi_q$, and $\mathcal{V}_q$ some model for the local component π_q. Let $\phi_f = \sum_i \otimes_q v_{i,q}$ via an isomorphism $\mathcal{V} = \otimes\mathcal{V}_q$, where at the archimedean place the $v_{i,\infty}$ are lowest weight vectors in $\pi_\infty = \pi^{J+}_{k,m}$. Then we have*

$$U_p\phi_f = \sum_i \otimes_q (U_p v_{i,q})$$

and

$$V_p\phi_f = \sum_i \otimes_q (V_p v_{i,q})$$

where $U_p v_{i,\infty}$ is a lowest weight vector in $U_p\pi_\infty = \pi^{J+}_{mp^2,k}$ and $V_p v_{i,\infty}$ is a lowest weight vector in $V_p\pi_\infty = \pi^{J+}_{mp,k}$.

Proof: The claim about the finite places is obvious by construction. At the archimedean place, since we can work with square roots, there are isomorphisms $U_p\pi^{J+}_{m,k} = \pi^{J+}_{mp^2,k}$ and $V_p\pi^{J+}_{m,k} = \pi^{J+}_{mp,k}$ (see [Sch2] Proposition 2.2; introducing the matrix $\mathrm{diag}(\sqrt{p}, \sqrt{p}^{-1})$ in (33) has precisely the effect of establishing this second isomorphism). To be a lowest weight vector means having the property v) from Proposition 2.1, which we already saw. ∎

We finally identify our global operators U_p and V_p with well-known operators of the same name from the classical theory of Jacobi forms.

2.5 Proposition. *The operators U_p and V_p defined by (32) resp. (33) coincide with the classical operators U_p and V_p defined in §4 of [EZ]. More precisely, for $f \in J^{\mathrm{cusp}}_{k,m}$ we have*

$$U_p\phi_f = \phi_{U_pf} \qquad \text{and} \qquad V_p\phi_f = p^{1-k/2}\,\phi_{V_pf}.$$

Proof: We begin with $U_p\phi_f$. Let $F \in J^{\mathrm{cusp}}_{k,mp^2}$ be the classical Jacobi form corresponding to this function. Let $(\tau, z) \in \mathbb{H} \times \mathbb{C}$, and

$$g = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & \\ & \sqrt{y}^{-1} \end{pmatrix} (\lambda, \mu, 0) \in G^J(\mathbb{R})$$

be such that $g.(i,0) = (\tau, z)$. Then by definition we have

$$\begin{aligned} F(\tau,z) &= j_{k,mp^2}(g,(i,0))^{-1}(U_p\phi_f)(g) \\ &= y^{-k/2}e^{2\pi i m p^2(\lambda^2 i+\lambda\mu)}\phi_f\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}\sqrt{y} & \\ & \sqrt{y}^{-1}\end{pmatrix}(\lambda p,\mu p,0)\right) \\ &= y^{-k/2}e^{2\pi i m((\lambda p)^2 i+(\lambda p)(\mu p))}j_{k,m}(U_p(g),(i,0))\,f(U_p(g)(i,0)) \\ &= f(\tau,pz). \end{aligned}$$

But this is exactly the definition of $U_p f$ on page 41 of [EZ].

Now let $F \in J^{\text{cusp}}_{k,mp}$ be the classical Jacobi form corresponding to $V_p\phi_f$. Using Lemma 2.6 below, we have

$$\begin{aligned} F(\tau,z) &= j_{k,mp}(g,(i,0))^{-1}(V_p\phi_f)(g) \\ &= y^{-k/2}e^{-2\pi i m p(\lambda^2 i+\lambda\mu)}\sum_{\gamma\in V_p\Gamma_p/(V_p\Gamma_p\cap\Gamma_p)}\phi\left(V_p(g)\begin{pmatrix}\sqrt{p} & \\ & \sqrt{p^{-1}}\end{pmatrix}\gamma\right) \\ &= y^{-k/2}e^{-2\pi i m p(\lambda^2 i+\lambda\mu)}\sum_{\gamma\in M_2(p)/\Gamma}\phi\left(V_p(g)\begin{pmatrix}\sqrt{p} & \\ & \sqrt{p^{-1}}\end{pmatrix}\begin{pmatrix}1 & \\ & p^{-1}\end{pmatrix}\gamma\right) \qquad (34) \\ &= y^{-k/2}e^{-2\pi i m p(\lambda^2 i+\lambda\mu)}\sum_{\gamma\in M_2(p)/\Gamma}\phi\left(\gamma^{-1}\begin{pmatrix}1 & \\ & p\end{pmatrix}V_p(g)\begin{pmatrix}\sqrt{p} & \\ & \sqrt{p^{-1}}\end{pmatrix}\right) \qquad (35) \\ &= y^{-k/2}e^{-2\pi i m p(\lambda^2 i+\lambda\mu)}\sum_{\gamma\in \Gamma\backslash M_2(p)}\phi\left(\gamma\begin{pmatrix}p^{-1} & \\ & 1\end{pmatrix}V_p(g)\begin{pmatrix}\sqrt{p} & \\ & \sqrt{p^{-1}}\end{pmatrix}\right). \qquad (36) \end{aligned}$$

In line (34), the element $\begin{pmatrix}1 & \\ & p^{-1}\end{pmatrix}\gamma$ lies in $G^J_{\mathbb{Q}_p}$. Then we have used the left-invariance of ϕ under rational matrices. The element $\gamma^{-1}\begin{pmatrix}1 & \\ & p\end{pmatrix}$ in line (35) lies in $G^J_{\mathbb{R}}$. Now for an arbitrary element $\gamma = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in M_2(p)$ one can compute

$$\begin{aligned} &\phi\left(\gamma\begin{pmatrix}p^{-1} & \\ & 1\end{pmatrix}V_p(g)\begin{pmatrix}\sqrt{p} & \\ & \sqrt{p^{-1}}\end{pmatrix}\right) \\ &= \phi\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}p^{-1} & \\ & 1\end{pmatrix}V_p\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}\sqrt{y} & \\ & \sqrt{y}^{-1}\end{pmatrix}\right)(\lambda,\mu p,0)\begin{pmatrix}\sqrt{p} & \\ & \sqrt{p^{-1}}\end{pmatrix}\right) \\ &= \phi\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}p^{-1} & \\ & 1\end{pmatrix}V_p\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}\sqrt{y} & \\ & \sqrt{y}^{-1}\end{pmatrix}\right)\begin{pmatrix}\sqrt{p} & \\ & \sqrt{p^{-1}}\end{pmatrix}(\lambda\sqrt{p},\mu\sqrt{p},0)\right) \\ &= \phi\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\sqrt{p^{-1}} & \\ & \sqrt{p^{-1}}\end{pmatrix}\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}\sqrt{y} & \\ & \sqrt{y}^{-1}\end{pmatrix}(\lambda\sqrt{p},\mu\sqrt{p},0)\right) \end{aligned}$$

Let g' denote this last argument of ϕ. Then

$$g'(i,0) = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\sqrt{p^{-1}} & \\ & \sqrt{p^{-1}}\end{pmatrix}(\tau, z\sqrt{p}) = \left(\frac{a\tau+b}{c\tau+d}, \frac{pz}{c\tau+d}\right)$$

and

$$\begin{aligned}
j_{k,m}(g',(i,0)) &= j_{k,m}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \sqrt{p^{-1}} & \\ & \sqrt{p^{-1}} \end{pmatrix},(\tau,z\sqrt{p})\right) \\
&\qquad \cdot j_{k,m}\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} \sqrt{y} & \\ & \sqrt{y}^{-1} \end{pmatrix}(\lambda\sqrt{p},\mu\sqrt{p},0),(i,0)\right) \\
&= p^{k/2}(c\tau+d)^{-k}e^{mp}\left(\frac{-cz^2}{c\tau+d}\right)y^{k/2}e^{2\pi imp(\lambda^2 i+\lambda\mu)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\phi(g') &= j_{k,m}(g',(i,0))f(g'(i,0)) \\
&= p^{k/2}(c\tau+d)^{-k}e^{mp}\left(\frac{-cz^2}{c\tau+d}\right)y^{k/2}e^{2\pi imp(\lambda^2 i+\lambda\mu)}f\left(\frac{a\tau+b}{c\tau+d},\frac{pz}{c\tau+d}\right).
\end{aligned}$$

Plugging all that into (36), one obtains

$$F(\tau,z) = \sum_{\gamma\in\Gamma\backslash M_2(p)} p^{k/2}(c\tau+d)^{-k}e^{mp}\left(\frac{-cz^2}{c\tau+d}\right)f\left(\frac{a\tau+b}{c\tau+d},\frac{pz}{c\tau+d}\right).$$

Up to a factor $p^{1-k/2}$, the function on the right is exactly the function V_pf defined on page 41 of [EZ].■

2.6 Lemma. *Let* $M_2(p)=\{A\in \mathrm{SL}(2,\mathbb{Z}):\ \det(A)=p\}$ *and* $M_{2,p}(p)=\{A\in\mathrm{SL}(2,\mathbb{Z}_p):\ \det(A)=p\}$*. With* $\Gamma=\mathrm{SL}(2,\mathbb{Z})$ *and* $\Gamma_p=\mathrm{SL}(2,\mathbb{Z}_p)$ *there are bijections*

$$V_p\Gamma_p/(V_p\Gamma_p\cap\Gamma_p)\xrightarrow{\sim} M_{2,p}(p)/\Gamma_p\xrightarrow{\sim} M_2(p)/\Gamma.$$

The first bijection is induced by the map $A\mapsto\begin{pmatrix}1&0\\0&p\end{pmatrix}A$*.*

Proof: The second bijection is induced by the inclusion $M_2(p)\hookrightarrow M_{2,p}(p)$. It is easy to see that this inclusion induces an injective map on the cosets. To see that it is a bijection, it is enough to show that both coset spaces have the same number of elements. But a standard set of representatives for both coset spaces is provided by

$$\begin{pmatrix} a & b \\ & d \end{pmatrix},\qquad a,b,d\in\mathbb{Z},\ ad=p,\ b\,(\mathrm{mod}\ a). \tag{37}$$

As for the first map, see Lemma 1.6. ■

2.2 Consequences

To apply our local theory to the global situation, we need an additional global ingredient. In our main result Theorem 1.16 we had to exclude the positive Weil representations of the Jacobi group, because for these it may happen that the operator V_ω on spherical vectors is zero. Luckily, *positive Weil representations do not appear as local components of cuspidal automorphic representations of the*

Jacobi group. This follows from the corresponding result for a cuspidal automorphic representation $\tilde{\pi}$ of the metaplectic group (see [Wa] Proposition 23), and the fundamental relation

$$\pi = \tilde{\pi} \otimes \pi_{SW}^m,$$

which also holds globally (cp. with (5)). Thus, in the cuspidal situation, our local results apply to *every* spherical local component. We mention that the fact that $\pi = \otimes \pi_p$ does not contain a positive Weil representation at the *archimedean* place is responsible for there being no Jacobi cusp forms of weight 1; see [BeS] 7.5.7.

Let $\pi = \otimes \pi_p$ be a cuspidal automorphic representation of $G^J(\mathbb{A})$ such that π_p is spherical for every finite p. A *global spherical vector* in π is a linear combination of tensors $\otimes v_p$ such that v_p is a spherical vector in π_p for every finite p. Due to the non-commutativity of the local Jacobi Hecke algebras, the space of global spherical vectors may be quite big. However, in view of Theorem 1.16, we immediately get the following.

2.7 Proposition. *The cuspidal automorphic representation $\pi = \otimes \pi_p$ as above contains a unique Jacobi form (up to scalars) if and only if π_p is a newform or an almost-newform for every finite p.*

We recall the Definition of Jacobi new- and oldforms in the classical context (see [EZ] §4). An *oldform* in $J_{k,m}$ is one which is obtained as a linear combination of elements of the form $V_p f'$ and $U_p f''$, where f' and f'' are Jacobi forms of smaller index (and p varies). The *newforms* in $J_{k,m}$ are those which are orthogonal to the space of oldforms with respect to the Petersson scalar product.

2.8 Proposition. *Let $f \in J_{k,m}^{\text{cusp}}$ be an eigenform, and let $\pi_f = \otimes \pi_p$ be the associated automorphic representation of $G^J(\mathbb{A})$. Then the following are equivalent:*

i) f is a newform in the sense of [EZ].

ii) For every finite p, the component π_p is a local newform in the sense of Definition 1.13.

Proof: We saw in the previous section that if $f \in \otimes_q \pi_q$, then

$$V_p f \in \otimes_q V_p \pi_q \qquad \text{and} \qquad U_p f \in \otimes_q U_p \pi_q.$$

So if all the π_p are newforms, then f must be a newform.

Now assume that there is a p such that π_p is not a newform. Then $\pi' := V_{p^{-1}} \pi_p$ is spherical, and if π_p is not even an almost-newform, then $\pi'' := U_{p^{-1}} \pi_p$ is also spherical. It follows that the *global* representations

$$\pi' := V_{p^{-1}} \pi_f \qquad \text{and, if necessary,} \qquad \pi'' := U_{p^{-1}} \pi_f$$

each contain classical Jacobi forms (meaning adelic functions with the properties from Proposition 2.1). We further know from the local theory that every spherical vector in π_p can be obtained as a linear combination of vectors of the form $V_p v'$ and $U_p v''$, where v' and v'' are spherical vectors in π' and π'', resp. It follows that ϕ_f is a linear combination of vectors of the form $V_p \phi'$ and $U_p \phi''$, where ϕ' and ϕ'' are Jacobi forms in π' resp. π''. By Proposition 2.5, f must be an oldform. ∎

As a consequence of the last two propositions we can state that if $\pi = \otimes\pi_p$ is a cuspidal automorphic representation, and if π_p is a local newform for every finite p, then π contains a unique classical Jacobi-newform.

We finally note that from Theorems 1.12 and 1.16 (together with the formulas in [EZ] Theorem 4.2) one immediately gets the following.

2.9 Theorem.

$$J_{k,m}^{\rm cusp} = \bigoplus_{\substack{l,l' \\ l^2 l' | m}} U_l V_{l'} J_{k,m/l^2 l'}^{\rm cusp,new}.$$

We have thus proved this result using essentially local methods, whereas the usual proof utilizes a trace formula; see [EZ] p. 49.

References

[BeS] Berndt, R., Schmidt, R.: *Elements of the Representation Theory of the Jacobi Group.* Birkhäuser, PM **163**, Basel 1998

[EZ] Eichler, M., Zagier, D.: *The Theory of Jacobi Forms.* Birkhäuser, PM **55**, Basel 1985

[Sch1] Schmidt, R.: *Spherical Representations of the Jacobi Group.* Abh. Math. Sem. Univ. Hamburg **68** (1998), 273-296

[Sch2] Schmidt, R.: *On Old and New Jacobi Forms.* To appear in J. of Number Theory.

[Wa] Waldspurger, J.-L.: *Correspondance de Shimura.* J. Math pures et appl. **59** (1980), 1-133