Special Representations on $\mathrm{GSp}(4,\mathbb{R})$

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Introduction

In classical terms the *Maaß lift* is a Hecke equivariant map from the space $J_{k,m}$ of Jacobi forms of weight k and index m to the space $M_k(\Gamma_2)$ of Siegel modular forms of weight k and degree 2; see [EZ] for a nice treatment. As usual in such a situation, there should be a group theoretic generalization, meaning a "lifting" of (automorphic) representations of the algebraic groups behind the classical objects. The groups involved in the Maaß lift are the Jacobi group G^J (for Jacobi forms) and the group $G = \mathrm{GSp}(4)$ (for Siegel modular forms).

In [BeS2] we undertook an attempt to construct a *local* Maaß lift, i.e., a prescription of how to associate an irreducible representation σ of $G(F)$ to an irreducible representation π^J of $G^J(F)$, where F is a local field. This should be done in such a way that piecing all the local prescriptions over a number field together should yield a global lifting (preserving the automorphic property) compatible with the classical situation mentioned above.

The method of [BeS2] did not work for all kinds of representations of G^J, but only for those which can be found inside standard induced representations. This is general enough to cover the classical case, but excludes the p-adic supercuspidal representations. On the other hand, over the real numbers, every irreducible representation comes as a constituent of an induced representation, by the subrepresentation theorem. The purpose of the present paper is to explicitly work out the real case, thereby obtaining a complete local Maaß lift over $\mathbb{R}$. The representations in the image of the lift shall be called *special representations*, and might be of some independent interest.

We note that the *holomorphic* case has already been treated in [Be] and [BeS2], but, as we shall see, there are several other cases.

The content of this paper is as follows. The first chapter introduces basic facts and notations. In the second chapter the special representations of G are introduced and investigated. The main point is to exhibit some cases of reducibility (section 2.4). The third chapter gives an overview over the representation theory of $G^J(\mathbb{R})$. Here some care was taken in precisely describing the reducibilities of the induced Jacobi representations. Finally, in chapter four, we bring together the present results with those of [BeS2] to obtain the complete local Maaß lift in the real case. As a by-product we get a complete list and all the reducibilities of the special representations of G.

1 Groups and roots

In this paper we work exclusively over the real number field. Therefore we shall simply write GL(2) for $\mathrm{GL}(2,\mathbb{R})$, and similarly for all the algebraic groups appearing.

1.1 The symplectic group

We shall be concerned with representations of the group

$$G = \mathrm{GSp}(4) = \left\{g \in \mathrm{GL}(4) :\ \exists\, \tilde\mu(g) \in \mathrm{GL}(1) \quad gJ\,{}^tg = \tilde\mu(g)J\right\}, \qquad\qquad J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}.$$

The function $\tilde\mu$ is a character of G, called the multiplier. Its kernel is the symplectic group Sp(4), so that there is an exact sequence

$$1 \longrightarrow \mathrm{Sp}(4) \longrightarrow G \xrightarrow{\tilde\mu} \mathrm{GL}(1) \longrightarrow 1.$$

As a Borel subgroup of G we choose $B = AN$ with the maximal torus

$$A = \left\{t = d(u,a,a') := \mathrm{diag}(a, a', ua^{-1}, ua'^{-1}) :\ u, a, a' \in \mathrm{GL}(1)\right\}$$

and the unipotent radical

$$N = \left\{ n(x,\lambda,\mu,\kappa) := \begin{pmatrix} 1 & & x & \mu \\ & 1 & \mu & \kappa \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ \lambda & 1 & & \\ & & 1 & -\lambda \\ & & & 1 \end{pmatrix} :\ x,\lambda,\mu,\kappa \in \mathbb{G}_a \right\}.$$

The two standard maximal parabolics are then

$$P = MS,$$

where

$$M = \left\{ \begin{pmatrix} A & 0 \\ 0 & u\,{}^tA^{-1} \end{pmatrix} \,:\, A \in \mathrm{GL}(2),\ u \in R^* \right\} \simeq \mathrm{GL}(2) \times \mathrm{GL}(1),$$

$$S = \left\{ \begin{pmatrix} \mathbf{1} & s \\ 0 & \mathbf{1} \end{pmatrix} \,:\, s \text{ symmetric} \right\},$$

and

$$Q = FH,$$

where

$$F = \left\{ \begin{pmatrix} a & & b & \\ & 1 & & \\ c & & d & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & a' & & \\ & & u & \\ & & & ua'^{-1} \end{pmatrix} \,:\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2),\ u, a' \in \mathrm{GL}(1) \right\}, \tag{1}$$

$$H = \left\{ (\lambda, \mu, \kappa) = \begin{pmatrix} 1 & & & \mu \\ \lambda & 1 & \mu & \kappa \\ & & 1 & -\lambda \\ & & & 1 \end{pmatrix} \,:\, \lambda, \mu, \kappa \in \mathbb{G}_a \right\}. \tag{2}$$

The four positive roots are

$$\alpha_1(t) = a^2u^{-1}, \qquad \alpha_2(t) = a'a^{-1},$$
$$\alpha_3(t) = aa'u^{-1}, \qquad \alpha_4(t) = a'^2u^{-1}.$$

(α_1 is the long simple root, α_2 is the short simple root). The Weyl group of G is generated by

$$w_1 = \begin{pmatrix} & & 1 & \\ & 1 & & \\ -1 & & & \\ & & & 1 \end{pmatrix} \in Q, \qquad w_2 = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix} \in P.$$

We have the relation $(w_1w_2)^4 = 1$. The modular factors of the parabolics P, Q and $B = P \cap Q$ are easily computed as

$$\delta_B \begin{pmatrix} a & & & \\ & b & & \\ & & c & \\ & & & d \end{pmatrix} = |a^2bd^{-3}| \qquad (ac = bd), \tag{3}$$

$$\delta_P \begin{pmatrix} A & \\ & u\,{}^tA^{-1} \end{pmatrix} = |u^{-1}\det(A)|^3, \tag{4}$$

$$\delta_Q \begin{pmatrix} a & & b & \\ & 1 & & \\ c & & d & \\ & & & 1 \end{pmatrix} = 1, \qquad \delta_Q \begin{pmatrix} 1 & & & \\ & a' & & \\ & & u & \\ & & & ua'^{-1} \end{pmatrix} = |a'^4u^{-2}|. \tag{5}$$

1.2 The Lie algebra

The Lie algebra of $\mathrm{Sp}(4,\mathbb{R})$ is

$$\mathfrak{sp}(4) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M(4,\mathbb{R}) : \; A = -D^t,\; B = B^t,\; C = C^t \right\}.$$

The Lie algebra $\mathfrak{g}$ of G is then the direct sum of $\mathfrak{sp}(4)$ with the one-dimensional center, consisting of scalar matrices. We shall exclusively deal with representations which are trivial on the center, so the relevant part of $\mathfrak{g}$ is really $\mathfrak{sp}(4)$. A basis of $\mathfrak{sp}(4)$ is given by

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$F = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad G = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad R' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad P' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Q' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Here is the multiplication table for these elements:

	H_1	H_2	F	G	R	R'	P	P'	Q	Q'
H_1	0	0	$2F$	$-2G$	0	0	$-P$	P'	Q	$-Q'$
H_2	0	0	0	0	$2R$	$-2R'$	P	$-P'$	Q	$-Q'$
F	$-2F$	0	0	H_1	0	0	$-Q$	0	0	P'
G	$2G$	0	$-H_1$	0	0	0	0	Q'	$-P$	0
R	0	$-2R$	0	0	0	H_2	0	$-Q$	0	P
R'	0	$2R'$	0	0	$-H_2$	0	Q'	0	$-P'$	0
P	P	$-P$	Q	0	0	$-Q'$	0	$H_2 - H_1$	$2R$	$-2G$
P'	$-P'$	P'	0	$-Q'$	Q	0	$H_1 - H_2$	0	$2F$	$-2R'$
Q	$-Q$	$-Q$	0	P	0	P'	$-2R$	$-2F$	0	$H_1 + H_2$
Q'	Q'	Q'	$-P'$	0	$-P$	0	$2G$	$2R'$	$-H_1 - H_2$	0

A convenient basis for the complexified Lie algebra $\mathfrak{sp}(4,\mathbb{C}) = \mathfrak{sp}(4) \otimes \mathbb{C}$ is as follows.

$$Z = -i\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Z' = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$N_+ = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & -i \\ -1 & 0 & -i & 0 \\ 0 & i & 0 & 1 \\ i & 0 & -1 & 0 \end{pmatrix}, \qquad N_- = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & i \\ -1 & 0 & i & 0 \\ 0 & -i & 0 & 1 \\ -i & 0 & -1 & 0 \end{pmatrix},$$

$$X_+ = \frac{1}{2}\begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad X_- = \frac{1}{2}\begin{pmatrix} 1 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

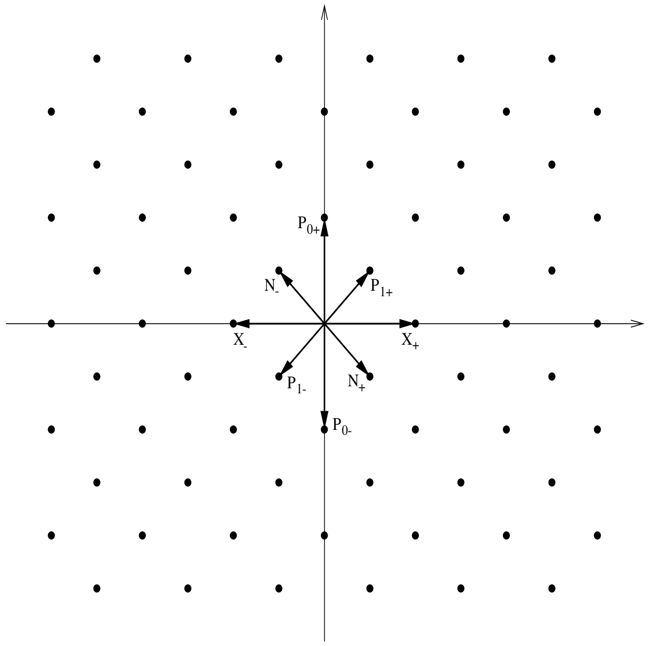
$$P_{1+} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & i \\ 1 & 0 & i & 0 \\ 0 & i & 0 & -1 \\ i & 0 & -1 & 0 \end{pmatrix}, \qquad P_{1-} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & -i \\ 1 & 0 & -i & 0 \\ 0 & -i & 0 & -1 \\ -i & 0 & -1 & 0 \end{pmatrix},$$

$$P_{0+} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & -1 \end{pmatrix}, \qquad P_{0-} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & -1 \end{pmatrix},$$

Multiplication table for this basis:

	Z	Z'	N_+	N_-	X_+	X_-	P_{1+}	P_{1-}	P_{0+}	P_{0-}
Z	0	0	N_+	$-N_-$	$2X_+$	$-2X_-$	P_{1+}	$-P_{1-}$	0	0
Z'	0	0	$-N_+$	N_-	0	0	P_{1+}	$-P_{1-}$	$2P_{0+}$	$-2P_{0-}$
N_+	$-N_+$	N_+	0	$Z'-Z$	0	$-P_{1-}$	$2X_+$	$-2P_{0-}$	P_{1+}	0
N_-	N_-	$-N_-$	$Z-Z'$	0	$-P_{1+}$	0	$-2P_{0+}$	$2X_-$	0	P_{1-}
X_+	$-2X_+$	0	0	P_{1+}	0	Z	0	N_+	0	0
X_-	$2X_-$	0	P_{1-}	0	$-Z$	0	N_-	0	0	0
P_{1+}	$-P_{1+}$	$-P_{1+}$	$-2X_+$	$2P_{0+}$	0	$-N_-$	0	$Z+Z'$	0	N_+
P_{1-}	P_{1-}	P_{1-}	$2P_{0-}$	$-2X_-$	$-N_+$	0	$-Z-Z'$	0	N_-	0
P_{0+}	0	$-2P_{0+}$	$-P_{1+}$	0	0	0	0	$-N_-$	0	Z'
P_{0-}	0	$2P_{0-}$	0	$-P_{1-}$	0	0	$-N_+$	0	$-Z'$	0

If we choose $\langle Z, Z' \rangle$ as a Cartan subalgebra, then we get the following picture of the root system of $\mathfrak{sp}(4, \mathbb{C})$, quite fundamental for this paper.



We say a vector v in a representation of $\mathfrak{g}_{\mathbb{C}}$ has *weight* $(l, l') \in \mathbb{Z}^2$ if

$$Zv = lv \qquad \text{and} \qquad Z'v = l'v.$$

If the representation comes from a representation of G with trivial central character (always the case in this paper), then only weights with l, l' both even or both odd will appear. Thus the dots in the above figure represent the possible weights. The arrows (roots) indicate how the elements of $\mathfrak{g}_{\mathbb{C}}$ change the weights. For example, if v has weight (l, l'), then X_+v has weight $(l+2, l')$.

Let $\mathfrak{k}$ be the 1-eigenspace of the Cartan-involution $\theta : X \mapsto -X^t$, and $\mathfrak{p}$ the (-1)-eigenspace of θ. Then

$$\mathfrak{k}_{\mathbb{C}} = \langle Z, Z', N_+, N_- \rangle,$$

and $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ + \mathfrak{p}_-$, with the maximal abelian algebras

$$\mathfrak{p}_+ = \langle X_+, P_{1+}, P_{0+} \rangle, \qquad \mathfrak{p}_- = \langle X_-, P_{1-}, P_{0-} \rangle.$$

The decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_+ + \mathfrak{p}_-$$

holds, and $\mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_\pm$ is a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

1.3 The maximal compact subgroup

We let K be the standard maximal compact subgroup of G. Its identity component K_0 has index 2 in K, and consists of all elements of K with multiplier 1. It is a maximal compact subgroup of $\mathrm{Sp}(4)$. If

$$\mathbb{H} = \big\{ Z \in M(2, \mathbb{C}) : \ Z \text{ symmetric and } \mathrm{Im}(Z) \text{ positive definite} \big\}$$

is the Siegel upper half plane of degree 2, and if $\mathrm{Sp}(4, \mathbb{R})$ acts on $\mathbb{H}$ by fractional linear transformations,

$$\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, Z \right) \longmapsto (AZ+B)(CZ+D)^{-1},$$

then K_0 is the stabilizer of the special element $I = \begin{pmatrix} i & \\ & i \end{pmatrix} \in \mathbb{H}$. It is easy to check that $K_0 \simeq \mathrm{U}(2)$ via

$$K_0 \ni \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \longmapsto A + iB \in \mathrm{U}(2).$$

If we let $\mathfrak{k} \subset \mathfrak{g}$ be the Lie algebra of K, then $\mathfrak{k}$ coincides with the Lie algebra of the same name mentioned in the previous paragraph. Since $\mathfrak{k}_{\mathbb{C}} = \langle Z, Z', N_+, N_- \rangle$, it follows that N_+, N_- are compact roots; the six other roots are non-compact.

To describe the irreducible representations of K, we first describe those of K_0. The latter are in one-one correspondence with the irreducible representations of $\mathfrak{k}_{\mathbb{C}}$ with integral weights. This Lie algebra is a direct sum

$$\mathfrak{k}_{\mathbb{C}} = \langle Z - Z', N_+, N_- \rangle \oplus \langle Z + Z' \rangle,$$

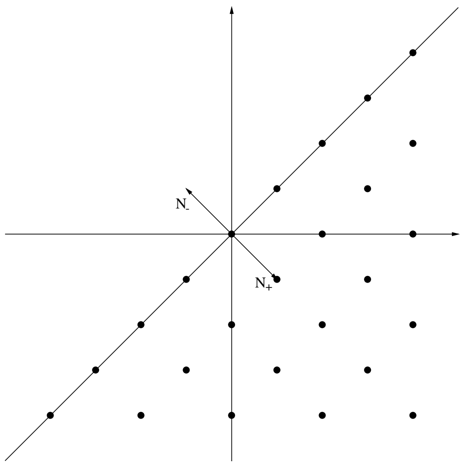
where $Z + Z'$ spans the center of $\mathfrak{k}_{\mathbb{C}}$. Now the subalgebra $\langle Z - Z', N_+, N_- \rangle$ is isomorphic to $\mathfrak{su}(2)$, and therefore its irreducible representations are indexed by non-negative integers indicating the weight of a highest weight vector (a vector annihilated by N_+). We also can prescribe any integer with which $Z+Z'$ is supposed to act, but have to make sure the resulting representation of $\mathfrak{k}_{\mathbb{C}}$ has integral weights. This gives us the following description of the irreducible representations of K_0:

The (isomorphism classes of) irreducible representations of K_0 are in one-one correspondence with the set

$$\{(l,l')\in\mathbb{Z}^2:\ l\geq l'\}.$$

If we let $\rho_{l,l'}$ be the representation corresponding to the pair (l,l'), then $\rho_{l,l'}$ is characterized by the property that it contains a non-zero vector of weight (l,l') that is annihilated by N_+.

As usual we shall only be interested in representations of K_0 with trivial central character. These are the $\rho_{l,l'}$ with l,l' both even or both odd. In our picture, the relevant representations of K_0 correspond to the dots on or below the main diagonal:



The weight structure of $\rho_{l,l'}$ is symmetric to the main diagonal. It contains a highest weight vector of weight (l,l') (annihilated by N_+), and a lowest weight vector of weight (l',l) (annihilated by N_-). It contains the weights "between" these two extremes exactly once. The one-dimensional representations are the $\rho_{l,l}$ with $l\in\mathbb{Z}$. The representation $\rho_{0,0}$ is the trivial one. Obviously the dimension of $\rho_{l,l'}$ is $l-l'+1$. In particular, all our relevant representations are odd-dimensional.

It is now easy to describe the representations of K. They are all obtained by induction from representations of K_0. The induction process has the effect of making the weight structure point symmetric with respect to the origin. More precisely, of (ρ,V) is a representation of K, and if $v\in V$ has weight (l,l'), then $\rho(\mathrm{diag}(1,1,-1,-1))v$ has weight $(-l,-l')$. Thus the weight structure of an irreducible representation of K combines that of $\rho_{l,l'}$ and $\rho_{-l',-l}$, for some pair (l,l').

2 Analyzing certain induced representations on G

2.1 Highest weight vectors on K

Let $K_0 \simeq \mathrm{U}(2)$ be the identity component of K as before. The following coordinates on K_0 will turn out to be convenient. We map

$$\mathbb{R}^4 \ni (\varphi, \varphi', \vartheta, \vartheta') \longmapsto r_3(\varphi) r_4(\varphi') r_1(\vartheta) r_2(\vartheta') \in K_0, \tag{6}$$

where

$$r_1(\vartheta) = \begin{pmatrix} \cos(\vartheta) & & \sin(\vartheta) & \\ & 1 & & \\ -\sin(\vartheta) & & \cos(\vartheta) & \\ & & & 1 \end{pmatrix},$$

$$r_2(\vartheta') = \begin{pmatrix} 1 & & & \\ & \cos(\vartheta') & & \sin(\vartheta') \\ & & 1 & \\ & -\sin(\vartheta') & & \cos(\vartheta') \end{pmatrix},$$

$$r_3(\varphi) = \begin{pmatrix} \cos(\varphi) & \sin(\varphi) & & \\ -\sin(\varphi) & \cos(\varphi) & & \\ & & \cos(\varphi) & \sin(\varphi) \\ & & -\sin(\varphi) & \cos(\varphi) \end{pmatrix},$$

$$r_4(\varphi') = \begin{pmatrix} \cos(\varphi') & & & \sin(\varphi') \\ & \cos(\varphi') & \sin(\varphi') & \\ & -\sin(\varphi') & \cos(\varphi') & \\ -\sin(\varphi') & & & \cos(\varphi') \end{pmatrix}.$$

One easily checks that the differential of this map at $(0,0,0,0)$ is regular. Hence we get reasonable coordinates on K_0 in a neighborhood of the identity. This will suffice: We will only be dealing with K_0-finite functions, which are analytic, and therefore determined on a small neighborhood of the identity of the connected group K_0.

If l, l' are integers, and a function Ψ on K_0 has the property

$$\Psi(g r_1(\vartheta) r_2(\vartheta')) = e^{i(l\vartheta + l'\vartheta')} \Psi(g) \qquad \text{for all } \vartheta, \vartheta' \in \mathbb{R},$$

then we say that Ψ has *weight* (l, l'). Obviously this is equivalent to our previous notion of weight, if Ψ is an element of a function space (e.g. $L^2(K_0)$) on which K_0 acts by right translation. The reason that we put r_1 and r_2 to the right in the coordinates (6) is that we would like to conveniently deal with weights. The reason we put r_3 to the left is that we are mainly interested in functions which are left $r_3(\varphi)$-invariant, as in the following lemma.

2.1 Lemma. *Let $l, l' \in \mathbb{Z}$, both even or both odd, and $l \geq l'$. There exists a non-zero smooth function Ψ on K_0, unique up to scalars, with the following properties:*

i) $\Psi(r_3(\varphi) g) = \Psi(g)$ *for all* $\varphi \in \mathbb{R}$.

ii) Ψ *has weight* (l, l').

iii) $N_+\Psi = 0$.

In a neighborhood of $\mathbf{1}$, *this function is given by*

$$\Psi\big(r_3(\varphi)r_4(\varphi')r_1(\vartheta)r_2(\vartheta')\big) = e^{i(l\vartheta + l'\vartheta')}\cos(2\varphi')^{(l-l')/2}. \tag{7}$$

Proof: As above, let $\rho_{l,l'}$ be the unique representation of K_0 of highest weight (l, l'). It has dimension $l - l' + 1$, which is an odd number by our hypothesis. Conditions ii) and iii) say that Ψ is a highest weight vector in a copy of $\rho_{l,l'}$ appearing in $L^2(K_0)$. If $V \subset L^2(K_0)$ is the subspace of all smooth functions with ii) and iii), then we have

$$\dim_{\mathbb{C}}(V) = l - l' + 1,$$

since, by the Peter-Weyl theorem, the multiplicity of $\rho_{l,l'}$ in $L^2(K_0)$ equals the dimension of $\rho_{l,l'}$, and each copy contains (up to scalars) exactly one highest weight vector. Now consider the action of $\mathrm{O}(2)$ on V by left translation, where $\mathrm{O}(2)$ is embedded in K_0 via

$$A \longmapsto \begin{pmatrix} A & \\ & A \end{pmatrix}.$$

The space V decomposes into irreducible $\mathrm{O}(2)$-representations. The latter are all 2-dimensional, except for the trivial and the signum representation, which are one-dimensional and both trivial on $\mathrm{SO}(2)$. Since the dimension of V is odd, one of the one-dimensional representations must occur. This shows that V contains a left $\mathrm{SO}(2)$-invariant function, and settles the existence part of the lemma.

It remains to prove (7), which will imply uniqueness. If Ψ is a function with properties i), ii), iii), let $\psi(\varphi') := \Psi(r_4(\varphi'))$. Obviously Ψ is determined by ψ. We shall compute

$$N_+\Psi = \frac{1}{2}(N_3\Psi - iN_4\Psi),$$

where

$$N_3 = \begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{pmatrix}, \qquad N_4 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}.$$

Since $N_+\Psi$ is also left $\mathrm{SO}(2)$-invariant, and has a weight (namely $(l+1, l'-1)$), it is enough to compute $(N_+\Psi)(r_4(\varphi'))$. The computation for N_4 is easy in our coordinates:

$$\begin{aligned}(N_4\Psi)(r_4(\varphi')) &= \frac{d}{dt}\Big|_0 \Psi(r_4(\varphi')\exp(tN_4)) = \frac{d}{dt}\Big|_0 \Psi(r_4(\varphi')r_4(t)) \\ &= \frac{d}{dt}\Big|_0 \psi(r_4(\varphi' + t)) = \psi'(\varphi'),\end{aligned}$$

where ψ' denotes the derivative. The computation for N_3 can be done as follows. We have

$$(N_3\Psi)(r_4(\varphi')) = \frac{d}{dt}\Big|_0 \Psi(r_4(\varphi')\exp(tN_3)) = \frac{d}{dt}\Big|_0 \Psi(r_4(\varphi')r_3(t)). \tag{8}$$

Now, at least for small φ' and t, we can express the argument in our coordinates (6):

$$r_4(\varphi')r_3(t) = r_3(\gamma(t))r_4(\delta(t))r_1(\alpha(t))r_2(\beta(t)) \tag{9}$$

with certain functions $\alpha, \beta, \gamma, \delta$ such that

$$\alpha(0) = \beta(0) = \gamma(0) = 0, \qquad \delta(0) = \varphi'.$$

Next we differentiate the matrix equation (9) and put $t = 0$; this results in sixteen linear equations from which the derivatives of $\alpha, \beta, \gamma, \delta$ at 0 can be determined. One finds

$$\alpha'(0) = \frac{-\sin(2\varphi')}{\cos(2\varphi')}, \qquad \beta'(0) = \frac{\sin(2\varphi')}{\cos(2\varphi')}, \qquad \gamma'(0) = \frac{1}{\cos(2\varphi')}, \qquad \delta'(0) = 0.$$

With this the computation (8) continues as

$$\begin{aligned}(N_3\Psi)(r_4(\varphi')) &= \frac{d}{dt}\Big|_0 \Big(e^{i(l\alpha(t)+l'\beta(t))}\psi(\delta(t))\Big)\\ &= i(l\alpha'(0) + l'\beta'(0))\psi(\varphi') + \delta'(0)\psi'(\varphi')\\ &= i(l'-l)\frac{\sin(2\varphi')}{\cos(2\varphi')}\psi(\varphi').\end{aligned}$$

Now it follows that the condition $N_+\Psi = 0$ is equivalent with

$$(l'-l)\frac{\sin(2\varphi')}{\cos(2\varphi')}\psi(\varphi') = \psi'(\varphi'),$$

and this differential equation has the solution $\psi(\varphi') = \cos(2\varphi')^{(l-l')/2}$. ∎

2.2 Special representations on G

We now consider representations of G obtained by parabolic induction from a character of the parabolic P. Relevant are only representations that are trivial on the center, and this can be arranged as follows. For a character χ of $\mathbb{R}^*$ consider the space of all continous functions $f : G \to \mathbb{C}$ with the transformation property

$$f\left(\begin{pmatrix} A & * \\ & u\,{}^tA^{-1}\end{pmatrix} g\right) = \chi(\det(A)u^{-1})\big|\det(A)u^{-1}\big|^{3/2} f(g). \tag{10}$$

This space carries a pre-Hilbert structure in the usual way, and we denote its completion with I_χ. The representation of G on I_χ shall be denoted σ_χ. Thus σ_χ is the representation obtained by familiar (normalized) parabolic induction from the character $\begin{pmatrix} A & * \\ & u\,{}^tA^{-1}\end{pmatrix} \mapsto \chi(\det(A)u^{-1})$ of P. It is obvious that σ_χ is trivial on the center, hence descents to a representation of PGSp(4).

We call an irreducible, admissible representation of G *special*, if it appears as a constituent of σ_χ for some χ. This name was inspired by the paper [PS]. Our goal is to match the special representations with representations of the Jacobi group G^J; this will give us a complete local Maaß lift over the reals (see section 4.2). While doing this we will determine the exact irreducibilities of the representations σ_χ, thus obtaining a list of all special representations of G.

The character χ is of the form

$$\chi(x) = \operatorname{sgn}(x)^\eta |x|^s \qquad (x \in \mathbb{R}^*) \tag{11}$$

with $\eta \in \{0,1\}$ and $s \in \mathbb{C}$. The following lemma gives the K-types of σ_χ.

2.2 Lemma. *Each K_0-type appears in σ_χ with multiplicity at most 1. More precisely, let χ be as in (11), and let $l, l' \in \mathbb{Z}$, both even or both odd, and with $l \geq l'$.*

i) If $\eta = 0$, then the multiplicity of $\rho_{l,l'}$ in σ_χ is

$$\begin{cases} 1 & \text{if } l, l' \text{ are both even,} \\ 0 & \text{if } l, l' \text{ are both odd.} \end{cases}$$

ii) If $\eta = 1$, then the multiplicity of $\rho_{l,l'}$ in σ_χ is

$$\begin{cases} 0 & \text{if } l, l' \text{ are both even,} \\ 1 & \text{if } l, l' \text{ are both odd.} \end{cases}$$

Proof: Let $F \in I_\chi$ be a highest weight vector for $\rho_{l,l'}$. Because of $G = PK_0$, the function F is determined by its restriction to K_0. Now F is K-finite, therefore analytic, and $\Psi := F\big|_{K_0}$ obviously has properties i), ii), iii) from Lemma 2.1. It follows from the uniqueness statement in this lemma that the multiplicity of $\rho_{l,l'}$ in σ_χ can not exceed one.

On the other hand, for given l, l' let Ψ be the function from Lemma 2.1. The K_0-type $\rho_{l,l'}$ will be contained in σ_χ if and only if Ψ can be extended to a function on G giving an element of I_χ. For the definition

$$F\left(\begin{pmatrix} A & * \\ & u\,{}^tA^{-1} \end{pmatrix} g\right) := \chi(\det(A)u^{-1})|\det(A)u^{-1}|^{3/2}\Psi(g), \qquad g \in K_0$$

to make sense, we need Ψ to behave from the left as $\chi|\,|^{3/2}$ on $P \cap K_0 = \mathrm{O}(2)$. This is surely true for $\mathrm{SO}(2)$, so we need only check the behavior under the element $h := \operatorname{diag}(1,-1,1,-1)$. It is straightforward to check from Lemma 2.1 that

$$\Psi(hg) = \begin{cases} \Psi(g) & \text{if } l, l' \text{ are even,} \\ -\Psi(g) & \text{if } l, l' \text{ are odd.} \end{cases}$$

On the other hand, the character takes the value $(-1)^\eta$ on h. The assertion follows. ∎

2.3 Differential operators

As a preparation for the next section we will now compute the action of the Lie algebra elements $X_\pm$ etc. on K-finite vectors in I_χ. Throughout we assume Ψ is a smooth function in I_χ of weight (l, l'). We put

$$\psi(\varphi') := \Psi(r_4(\varphi')),$$

noting that ψ determines Ψ. If L denotes one of the operators $N_\pm, X_\pm, P_{0\pm}, P_{1\pm}$, then $L\Psi$ will also be determined by its values on elements $r_4(\varphi')$.

We first have to compute the action of the non-complexified Lie algebra $\mathfrak{g}$. The method used is the one from the proof of Lemma 2.1, but is a little more complicated now, since we are leaving K. If $L \in \mathfrak{g}$, then by definition

$$(L\Psi)(r_4(\varphi')) = \frac{d}{dt}\Big|_0 \Psi\big(r_4(\varphi')\exp(tL)\big).$$

Now we write down an Iwasawa decomposition for the argument,

$$r_4(\varphi')\exp(tL) = n(t)A(t)r_3(\gamma(t))r_4(\delta(t))r_1(\alpha(t))r_2(\beta(t)), \tag{12}$$

where $n(t) \in N$, where

$$A(t) = \begin{pmatrix} a(t) & & & \\ & \tilde{a}(t) & & \\ & & a(t)^{-1} & \\ & & & \tilde{a}(t)^{-1} \end{pmatrix}, \qquad a(t), \tilde{a}(t) > 0,$$

and where $a, \tilde{a}, \alpha, \beta, \gamma, \delta$ are smooth functions in a neighborhood of 0 with

$$a(0) = \tilde{a}(0) = 1, \quad \alpha(0) = \beta(0) = \gamma(0) = 0, \quad \delta(0) = \varphi'.$$

With χ as in (11) it follows that

$$\begin{aligned}(L\Psi)(r_4(\varphi')) &= \frac{d}{dt}\Big|_0 \Psi\big(A(t)r_3(\gamma(t))r_4(\delta(t))r_1(\alpha(t))r_2(\beta(t))\big)\\ &= \frac{d}{dt}\Big|_0 \Big(\big|a(t)\tilde{a}(t)\big|^{s+3/2} e^{i(l\alpha(t)+l'\beta(t))}\psi(\delta(t))\Big)\\ &= \Big(s+\frac{3}{2}\Big)\big(a'(0)+\tilde{a}'(0)\big)\psi(\varphi') + i\big(l\alpha'(0)+l'\beta'(0)\big)\psi(\varphi') + \delta'(0)\psi'(\varphi').\end{aligned}$$

Thus what we need is the derivatives at 0 of the auxiliary functions $a, \tilde{a}, \alpha, \beta, \delta$. To get these, we differentiate the matrix equation (12) and put $t = 0$. This yields sixteen linear equations from which the desired derivatives can be determined. The following table contains the results, depending on L.

L	$a'(0)$	$\tilde{a}'(0)$	$\alpha'(0)$	$\beta'(0)$	$\delta'(0)$
H_1	$\cos(\varphi')^2$	$-\sin(\varphi')^2$	0	0	$\frac{1}{2}\sin(2\varphi')$
H_2	$-\sin(\varphi')^2$	$\cos(\varphi')^2$	0	0	$\frac{1}{2}\sin(2\varphi')$
F	0	0	$\frac{-\sin(\varphi')^4}{\cos(2\varphi')}$	$\frac{\sin(\varphi')^2\cos(\varphi')^2}{\cos(2\varphi')}$	0
G	0	0	$\frac{-\cos(\varphi')^4}{\cos(2\varphi')}$	$\frac{\sin(\varphi')^2\cos(\varphi')^2}{\cos(2\varphi')}$	0
P	0	0	$\frac{2\sin(\varphi')\cos(\varphi')^3}{\cos(2\varphi')}$	$-\frac{2\sin(\varphi')^3\cos(\varphi')}{\cos(2\varphi')}$	0
P'	0	0	$\frac{-2\sin(\varphi')^3\cos(\varphi')}{\cos(2\varphi')}$	$\frac{2\sin(\varphi')\cos(\varphi')^3}{\cos(2\varphi')}$	0
Q	$\frac{1}{2}\sin(2\varphi')$	$\frac{1}{2}\sin(2\varphi')$	0	0	$\sin(\varphi')^2$
Q'	$\frac{1}{2}\sin(2\varphi')$	$\frac{1}{2}\sin(2\varphi')$	0	0	$-\cos(\varphi')^2$
R	0	0	$\frac{\sin(\varphi')^2\cos(\varphi')^2}{\cos(2\varphi')}$	$\frac{-\sin(\varphi')^4}{\cos(2\varphi')}$	0
R'	0	0	$\frac{\sin(\varphi')^2\cos(\varphi')^2}{\cos(2\varphi')}$	$\frac{-\cos(\varphi')^4}{\cos(2\varphi')}$	0

The action of $\mathfrak{g}$ extends to $\mathfrak{g}_{\mathbb{C}}$ by linearity. Plugging in the values from the table, one obtains the following formulas, where we put $k := s + 3/2$.

$$(N_+\Psi)(r_4(\varphi')) = \frac{i}{2}(l'-l)\frac{\sin(2\varphi')}{\cos(2\varphi')}\psi(\varphi') - \frac{i}{2}\psi'(\varphi').$$

$$(N_-\Psi)(r_4(\varphi')) = \frac{i}{2}(l'-l)\frac{\sin(2\varphi')}{\cos(2\varphi')}\psi(\varphi') + \frac{i}{2}\psi'(\varphi').$$

$$(X_+\Psi)(r_4(\varphi')) = \frac{1}{2\cos(2\varphi')}\Big(k\cos(2\varphi')^2 + l - \frac{1}{2}(l+l')\sin(2\varphi')^2\Big)\psi(\varphi') + \frac{1}{4}\sin(2\varphi')\psi'(\varphi').$$

$$(X_-\Psi)(r_4(\varphi')) = \frac{1}{2\cos(2\varphi')}\Big(k\cos(2\varphi')^2 - l + \frac{1}{2}(l+l')\sin(2\varphi')^2\Big)\psi(\varphi') + \frac{1}{4}\sin(2\varphi')\psi'(\varphi').$$

$$(P_{0+}\Psi)(r_4(\varphi')) = \frac{1}{2\cos(2\varphi')}\Big(k\cos(2\varphi')^2 + l' - \frac{1}{2}(l+l')\sin(2\varphi')^2\Big)\psi(\varphi') + \frac{1}{4}\sin(2\varphi')\psi'(\varphi').$$

$$(P_{0-}\Psi)(r_4(\varphi')) = \frac{1}{2\cos(2\varphi')}\Big(k\cos(2\varphi')^2 - l' + \frac{1}{2}(l+l')\sin(2\varphi')^2\Big)\psi(\varphi') + \frac{1}{4}\sin(2\varphi')\psi'(\varphi').$$

$$(P_{1+}\Psi)(r_4(\varphi')) = \frac{i}{2}(l+l'+2k)\sin(2\varphi')\psi(\varphi') - \frac{i}{2}\cos(2\varphi')\psi'(\varphi').$$

$$(P_{1-}\Psi)(r_4(\varphi')) = \frac{i}{2}(l+l'-2k)\sin(2\varphi')\psi(\varphi') + \frac{i}{2}\cos(2\varphi')\psi'(\varphi').$$

2.4 Cases of reducibility

Let the character χ be of the form (11) again. We shall work with the number $k = s + 3/2$ instead of s, so that the transformation law (10) becomes

$$f\left(\begin{pmatrix} A & * \\ & u\,{}^tA^{-1}\end{pmatrix} g\right) = \operatorname{sgn}(\det(A)u^{-1})^\eta \left|\det(A)u^{-1}\right|^k f(g). \tag{13}$$

In this section we shall identify certain subrepresentations of σ_χ in case k is an integer ≥ 2, what we assume from now on.

The cases k even, $\eta = 0$, and k odd, $\eta = 1$

In these cases σ_χ contains the K_0-types (k, k) and $(-k, -k)$, by Lemma 2.2. Let $\Psi \in I_\chi$ be a vector spanning the one-dimensional representation $\rho_{k,k}$ of K_0. Of course we have

$$N_+\Psi = N_-\Psi = 0.$$

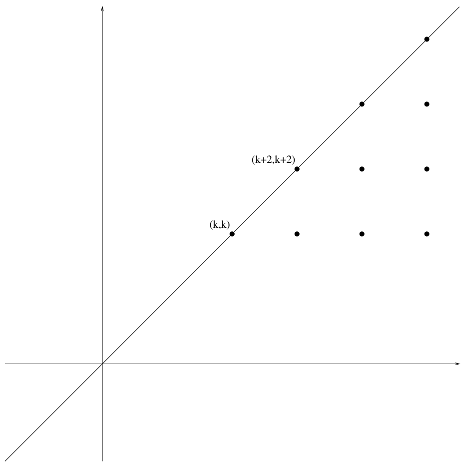
By Lemma 2.1, the function $\psi(\varphi') := \Psi(r_4(\varphi'))$ is a non-zero constant. It is therefore immediate from the formulas in the previous section that

$$X_-\Psi = P_{0-}\Psi = P_{1-}\Psi = 0.$$

As a consequence, the $\mathfrak{g}_\mathbb{C}$-invariant subspace

$$V_k^+ := \mathcal{U}(\mathfrak{g}_\mathbb{C})\Psi = \mathcal{U}(\mathfrak{p}_+)\Psi = \sum_{\alpha,\beta,\gamma\geq 0} \mathbb{C}X_+^\alpha P_{1+}^\beta P_{0+}^\gamma \Psi$$

is a *proper* subspace of σ_χ. Indeed, the only possible K_0-types appearing in V_k^+ are the ones indicated in the following picture:



To find out whether these K_0-types are really all contained in V_k^+, we introduce the elements

$$L_+ := 4X_+P_{0+} - P_{1+}^2, \qquad L_- := 4X_-P_{0-} - P_{1-}^2$$

of $\mathcal{U}(\mathfrak{g}_\mathbb{C})$. It is immediately checked that

$$[L_+, N_\pm] = 0, \qquad [L_-, N_\pm] = 0.$$

2.3 Lemma. *Let $l, l' \in \mathbb{Z}$, $l \geq l'$, and let v be a highest weight vector for the K_0-type $\rho_{l,l'}$ appearing in some representation π of $\mathfrak{g}_\mathbb{C}$.*

i) If L_+v (resp. L_-v) is non-zero, then it is a highest weight vector for $\rho_{l+2,l'+2}$ (resp. $\rho_{l-2,l'-2}$). In particular, this K_0-type appears in π.

ii) If $X_+v \neq 0$, then it is a highest weight vector for $\rho_{l+2,l'}$, which therefore appears in π.

iii) If $P_{0-}v \neq 0$, then it is a highest weight vector for $\rho_{l,l'-2}$, which therefore appears in π.

Proof: This follows simply from the fact that the operators $L_\pm, X_+, P_{0-}$ all commute with N_+. ∎

Because of this lemma, to check that all the possible K_0-types appear in V_k^+, all we have to do is check that L_+ and X_+ do not annihilate any of the highest weight vectors in question. This is easily done using the explicit form of these highest weight vectors from Lemma 2.1, and the formulas of the previous paragraph.

2.4 Lemma. *The $\mathfrak{g}_{\mathbb{C}}$-representation V_k^+ contains precisely the K_0-types $\rho_{k+l,k+l'}$, each with multiplicity one, where l, l' are even integers with $l \geq l' \geq 0$. The subspace of V_k^+ consisting of vectors of weight $(k+l, k+l')$ has dimension $1 + [\min(l,l')/2]$ ($l, l' \geq 0$ and even).*

Proof: The first assertion was just proved. The second assertion then follows by easy combinatorics. ∎

Note that V_k^+ is only "half" of the (K-finite vectors of the) G-representation generated by Ψ. The reason is that the element $\mathrm{diag}(1,1,-1,-1) \in G$ interchanges the weights (l,l') and $(-l,-l')$, thus making the weight structure symmetric with respect to the origin. But still the G-representation generated by Ψ remains a proper subrepresentation of σ_χ, a fact which will be used in subsequent sections.

The cases k even, $\eta = 1$, and k odd, $\eta = 0$

In these cases, by Lemma 2.2, σ_χ does not contain the K_0-types (k,k) and $(-k,-k)$, but it contains the type $(k-1, 1-k)$. Let $\Psi \in I_\chi$ be a highest weight vector for $\rho_{k-1,1-k}$. We shall show now that Ψ generates a proper subrepresentation.

2.5 Lemma. *The vector Ψ is annihilated by the operators*

$$kP_{1-} - N_-P_{0-}, \qquad 2k(2k-1)X_- - N_-^2P_{0-}, \qquad \textit{and} \qquad (k-1)N_-P_{0-} - kP_{0-}N_-.$$

Proof: By Lemma 2.1 we have (after suitable scaling)

$$\psi(\varphi') := \Psi(r_4(\varphi')) = \cos(2\varphi')^{k-1},$$

at least for small φ'. Now it is just a simple calculation using the formulas at the end of section 2.3. ∎

We shall first investigate the K_0-types of the space $\mathcal{U}(\mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_-)\Psi$. The idea is to show that only the K_0-types $(k-1, 1-k-l')$ with even $l' \geq 0$ appear, by showing that $\mathcal{U}(\mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_-)\Psi$ is spanned by elements $N_-^\alpha P_{0-}^\beta \Psi$. This is a purely algebraic problem using the relations from Lemma 2.5. To begin with, it is clear that all these K_0-types appear by Lemma 2.3 iii) (a direct calculation shows that the non-zero hypothesis in this Lemma is fulfilled).

2.6 Lemma. *For each $\beta \geq 0$,*

$$P_{0-}N_-P_{0-}^{\beta}\Psi = \frac{k+\beta-1}{k+\beta}N_-P_{0-}^{\beta+1}\Psi.$$

Proof: The case $\beta = 0$ is contained in Lemma 2.5. Now use induction:

$$\begin{aligned} P_{0-}N_-P_{0-}^{\beta}\Psi &= (N_-P_{0-} - P_{1-})P_{0-}^{\beta}\Psi \\ &\overset{2.5}{=} N_-P_{0-}^{\beta+1}\Psi - \frac{1}{k}P_{0-}^{\beta}N_-P_{0-}\Psi \\ &\overset{\text{I.H.}}{=} N_-P_{0-}^{\beta+1}\Psi - \frac{1}{k}\frac{k}{k+1}\cdots\frac{k+\beta-2}{k+\beta-1}P_{0-}N_-P_{0-}^{\beta}\Psi. \end{aligned}$$

Now take the right-most term to the other side, and the assertion follows. ■

2.7 Lemma. *For each $\beta \geq 0$ there is a (rational) number $c(\beta)$ such that*

$$P_{0-}N_-^2P_{0-}^{\beta}\Psi = c(\beta)N_-^2P_{0-}^{\beta+1}\Psi.$$

Proof: For $\beta = 0$ one computes $c(0) = \frac{(k-1)(2k-3)}{k(2k-1)}$ using the relations in Lemma 2.5. For $\beta > 0$ one uses the commutation relations, the previous lemmas, and the induction hypothesis to compute

$$\begin{aligned} P_{0-}N_-^2P_{0-}^{\beta}\Psi &= \left(N_-^2P_{0-}^{\beta+1} - 2N_-P_{0-}^{\beta}P_{1-} + 2P_{0-}^{\beta}X_-\right)\Psi \\ &\overset{2.5}{=} \left(N_-^2P_{0-}^{\beta+1} - \frac{2}{k}N_-P_{0-}^{\beta}N_-P_{0-} + \frac{1}{k(2k-1)}P_{0-}^{\beta}N_-^2P_{0-}\right)\Psi \\ &\overset{2.6,\,\text{I.H.}}{=} \left(N_-^2P_{0-}^{\beta+1} - \frac{2}{k+\beta}N_-^2P_{0-}^{\beta+1} + \frac{1}{k(2k-1)}c(1)\ldots c(\beta-1)P_{0-}N_-^2P_{0-}^{\beta}\right)\Psi. \end{aligned}$$

It follows that

$$cP_{0-}N_-^2P_{0-}^{\beta}\Psi = \frac{k+\beta-2}{k+\beta}N_-^2P_{0-}^{\beta+1}\Psi$$

with some number c. Now $P_{0-}^{\beta+1}\Psi$ is non-zero, as already noted. Then so is $N_-^2P_{0-}^{\beta+1}\Psi$, because we are just moving inside a K_0-type. Thus $c \neq 0$, and so one can divide by c to get the assertion. ■

2.8 Lemma. *For each $\alpha, \beta \geq 0$ there are (rational) numbers $d(\alpha,\beta), e(\alpha,\beta), f(\alpha,\beta)$ such that*

i) $X_-N_-^{\alpha}P_{0-}^{\beta}\Psi = d(\alpha,\beta)N_-^{\alpha+2}P_{0-}^{\beta+1}\Psi.$

ii) $P_{1-}N_-^{\alpha}P_{0-}^{\beta}\Psi = e(\alpha,\beta)N_-^{\alpha+1}P_{0-}^{\beta+1}\Psi.$

iii) $P_{0-}N_-^{\alpha}P_{0-}^{\beta}\Psi = f(\alpha,\beta)N_-^{\alpha}P_{0-}^{\beta+1}\Psi.$

Proof: i) X_- commutes with N_- and P_{0-}. Then use Lemmas 2.5 and 2.7.

ii) $P_{1-}N_-^{\alpha}P_{0-}^{\beta}\Psi = (N_-P_{1-} - 2X_-)N_-^{\alpha-1}P_{0-}^{\beta}\Psi$. Now use i) and induction.

iii) $P_{0-}N_-^{\alpha}P_{0-}^{\beta}\Psi = (N_-P_{o-} - P_{1-})N_-^{\alpha-1}P_{0-}^{\beta}\Psi$. Now use ii) and induction. ■

It is now immediate from Lemma 2.8 that

$$\mathcal{U}(\mathfrak{k}_{\mathbb{C}}+\mathfrak{p}_-)\Psi=\sum_{\alpha,\beta,\gamma,\delta\geq 0}\mathbb{C}X_-^{\alpha}P_{1-}^{\beta}P_{0-}^{\gamma}N_-^{\delta}\Psi=\sum_{\alpha,\beta\geq 0}\mathbb{C}N_-^{\alpha}P_{0-}^{\beta}\Psi.$$

Thus the weight structure of $\mathcal{U}(\mathfrak{k}_{\mathbb{C}}+\mathfrak{p}_-)\Psi$ is clear: The vector $P_{0-}^{\beta}\Psi$ is a highest weight vector in $\rho_{k-1,1-k-2\beta}$, and with N_- we just move along inside this K_0-type.

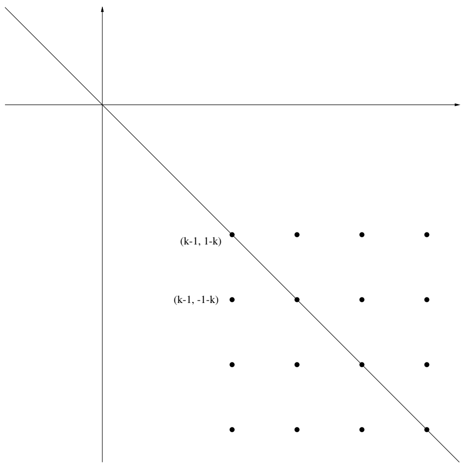
2.9 Lemma. *If v is an element of a K_0-type $\rho_{l,l'}$ that appears in some $\mathfrak{g}_{\mathbb{C}}$-representation π, then the space $\mathcal{U}(\mathfrak{p}_+)v$ is entirely contained in the sum of K_0-types $\rho_{j,j'}$ with $j\geq l$.*

Proof: This is completely clear if v is the highest weight vector in $\rho_{l,l'}$, since by applying $\mathfrak{p}_+$ we can never get "to the left". But then it is also true for $N_-^{\alpha}v$, since N_- normalizes $\mathfrak{p}_+$. ■

Let

$$V_k^- := \mathcal{U}(\mathfrak{g}_{\mathbb{C}})\Psi=\mathcal{U}(\mathfrak{p}_+)\mathcal{U}(\mathfrak{k}_{\mathbb{C}}+\mathfrak{p}_-)\Psi.$$

By Lemma 2.9 and our result about the weight structure of $\mathcal{U}(\mathfrak{k}_{\mathbb{C}}+\mathfrak{p}_-)\Psi$, the representation V_k^- contains no K_0-types $\rho_{l,l'}$ with $l<k-1$. By a symmetric argument, V_k^- cannot contain K_0-types $\rho_{l,l'}$ with $l'>1-k$. We have shown that the only possible K_0-types appearing in V_k^- are the ones indicated in the following picture.



To prove that each of these K_0-types really does appear, one can use Lemma 2.3 again. Thus we get the following.

2.10 Lemma. *The $\mathfrak{g}_{\mathbb{C}}$-representation V_k^- contains precisely the K_0-types $\rho_{k-1+l,1-k+l'}$, each with multiplicity one, where l,l' are even non-negative integers.*

Note that, in contrast to V_k^+, each weight space in V_k^- is infinite-dimensional. Note also that the weight structure of V_k^- is already point symmetric with respect to the origin. Thus V_k^- is the space of K-finite vectors of the G-subrepresentation of σ_χ generated by Ψ.

Summary

In case the character χ is of the form

$$\chi(x) = \mathrm{sgn}(x)^\eta |x|^{k-3/2}, \qquad \eta \in \{0,1\},\ k \in \mathbb{Z},\ k \geq 2,$$

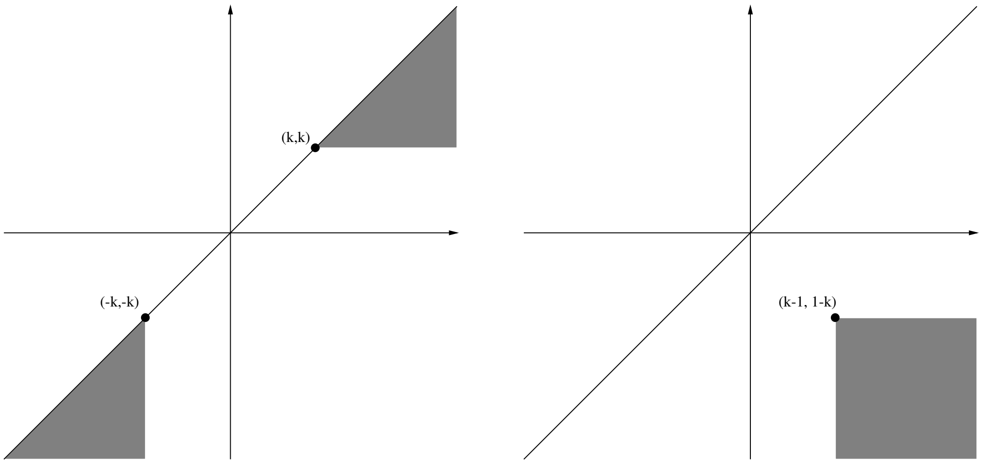
the representation σ_χ defined in 2.2 is not irreducible. We call the cases

k even, $\eta = 0$, or k odd, $\eta = 1$: The *even weight* cases.
k even, $\eta = 1$, or k odd, $\eta = 0$: The *odd weight* cases.

In each case we found a proper G-subrepresentation generated by a vector Ψ. We name this subrepresentation

σ_k^+ in the even weight case,
σ_k^- in the odd weight case.

The shaded regions in the following diagrams indicate the K-types contained in σ_k^+ resp. σ_k^-:



We showed that $\sigma_k^\pm$ are irreducible G-representations. It will turn out in section 4 that the quotients

$$\sigma_{3-k}^- := \sigma_\chi/\sigma_k^+ \qquad \text{resp.} \qquad \sigma_{3-k}^+ := \sigma_\chi/\sigma_k^- \tag{14}$$

are also irreducible. Note there is no conflict of notations, since the integer k is assumed to be ≥ 2 throughout (the symbols $\sigma_l^\pm$ are now defined for each integer l). The K-structure of the quotients is obtained by taking away the shaded regions in the above diagrams. Each K-type appears with multiplicity at most 1.

It is almost obvious that σ_k^+ is a holomorphic discrete series representation, since Ψ is a lowest weight vector; see [BeS2] for some more details on this case. The representation σ_k^-, on the other hand, is not in the discrete series, since no possible Harish-Chandra parameter yields these K_0-types (see the Blattner formula in [HS]). The quotients $\sigma_{-k}^{\pm}$ are Langlands quotients, coming from a tempered (one-dimensional) representation on

$$M^0 = \left\{ \begin{pmatrix} A & \\ & {}^tA^{-1} \end{pmatrix} :\ A \in \mathrm{GL}(2),\ \det(A) \in \{\pm 1\} \right\}.$$

3 The Jacobi group and its representations

3.1 Induced representations on G^J

Let SL(2) be embedded in G via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & & b & \\ & 1 & & \\ c & & d & \\ & & & 1 \end{pmatrix},$$

and let $H \subset G$ be the Heisenberg group, as defined in section 1.1. Elements of H will simply be written as triples (λ, μ, κ), as in (2). The semidirect product

$$G^J := \mathrm{SL}(2) \ltimes H$$

is called the *Jacobi group*. This is a non-reductive group, but it has something like a "Borel subgroup", namely

$$B^J = \left\{ \begin{pmatrix} a & x \\ & a^{-1} \end{pmatrix} (0, \mu, \kappa) :\ a \in \mathbb{R}^*,\ x, \mu, \kappa \in \mathbb{R} \right\}.$$

The center of G^J consists of all the elements $(0, 0, \kappa) \in H$, is thus isomorphic to $\mathbb{R}$. We shall now recall from [BeS1] the basic representation theory of G^J, together with some complemental facts that will be needed in subsequent sections. We fix the additive character

$$\psi(x) = e^{2\pi i x}$$

of $\mathbb{R}$, and write $\psi^m(x) = e^{2\pi i m x}$ for $m \in \mathbb{R}^*$. For a character χ of $\mathbb{R}^*$, and an element $m \in \mathbb{R}^*$, we define a space $\mathcal{B}_{\chi,m}^J$ consisting of functions $f :\ G^J \to \mathbb{C}$ with the property

$$f\left(\begin{pmatrix} a & x \\ & a^{-1} \end{pmatrix} (0, \mu, \kappa) \right) = \chi(a)|a|^{3/2} \psi^m(\kappa) f(g), \qquad a \in \mathbb{R}^*,\ x, \mu, \kappa \in \mathbb{R},\ g \in G^J,$$

and regularity conditions formulated in [BeS1] 3.3. Let $\pi_{\chi,m}^J$ be the representation of G^J on $\mathcal{B}_{\chi,m}^J$ by right translation. In other words,

$$\pi_{\chi,m}^J = \mathrm{Ind}_{B^J}^{G^J}(\chi\psi^m),$$

the representation induced from the character

$$\begin{pmatrix} a & x \\ & a^{-1} \end{pmatrix} (0, \mu, \kappa) \longmapsto \chi(a)\psi^m(\kappa)$$

of B^J. Our goal is to give the precise decomposition of $\pi_{\chi,m}^J$ into irreducible representations. We shall reduce this problem to the analogous one for the metaplectic group, and then use the results of [Wa1] and [Wa2].

3.2 Induced representations on Mp

Let Mp be the real metaplectic group, the non-trivial two-fold cover of $\mathrm{SL}(2,\mathbb{R})$. Again let χ be a character of $\mathbb{R}^*$, and $m \in \mathbb{R}^*$. In [Wa2], p. 225, Waldspurger defines a space

$$\tilde{\mathcal{B}}_{\chi,m}$$

of functions on Mp, on which Mp acts by right translation. This is $\tilde{\mathcal{B}}(\chi)$ in the notation of [Wa2], with reference character $e^{2\pi imx}$, but we prefer to keep the m (i.e., the character) in the notation. Denote by $\tilde{\pi}_{\chi,m}$ the representation of Mp on $\tilde{\mathcal{B}}_{\chi,m}$.

A complete list of irreducible, admissible representations of Mp was given in [Wa1]. There are the *principal series representations*

$$\tilde{\pi}_{s,\nu}, \qquad s \in \mathbb{C} \setminus \left(\frac{1}{2} + \mathbb{Z}\right),\ \nu = \pm\frac{1}{2}, \tag{15}$$

whose $\tilde{K}$-types are $\nu + 2\mathbb{Z}$. There are further the *positive discrete series representations*

$$\tilde{\pi}_n^+, \qquad n \in \frac{1}{2} + \mathbb{Z}, \tag{16}$$

which are lowest weight representations with weight structure

$$n,\ n+2,\ n+4,\ \ldots .$$

And there are finally the *negative discrete series representations*

$$\tilde{\pi}_n^+, \qquad n \in \frac{1}{2} + \mathbb{Z}, \tag{17}$$

which are highest weight representations with weight structure

$$\ldots,\ -n-4,\ -n-2,\ -n.$$

Note that in contrast to $\tilde{\pi}_{\chi,m}$, the irreducible representations (15), (16) and (17) do not depend on the choice of a character. Thus there is no m in the notation.

The following two theorems, one for each sign of m, fully answer the reducibility question for the induced representations. They are implicit in [Wa1] and [Wa2].

3.1 Theorem. *The character χ is of the form*

$$\chi(x) = (\mathrm{sgn}(x))^\eta\, |x|^s, \qquad \eta \in \{0,1\},\ s \in \mathbb{C}.$$

Assume $\underline{m < 0}$, and put $\nu = -\frac{1}{2}\chi(-1)$.

i) If $s \notin \frac{1}{2} + \mathbb{Z}$, then $\tilde{\mathcal{B}}_{\chi,m}$ is irreducible. Its $\tilde{K}$-types are

$$\nu + 2n, \qquad n \in \mathbb{Z}.$$

In fact, $\tilde{\pi}_{\chi,m} \simeq \tilde{\pi}_{s,\nu}$

ii) Assume $s = k - \frac{3}{2}$, k even.

a) Assume $\eta = 0$. Then $\tilde{\pi}_{\chi,m}$ has a unique subrepresentation isomorphic to $\tilde{\pi}^+_{k-1/2}$. The quotient is isomorphic to $\tilde{\pi}^-_{5/2-k}$.

b) Assume $\eta = 1$. Then $\tilde{\pi}_{\chi,m}$ has a unique subrepresentation isomorphic to $\tilde{\pi}^-_{k-1/2}$. The quotient is isomorphic to $\tilde{\pi}^+_{5/2-k}$.

iii) Assume $s = k - \frac{3}{2}$, k odd.

a) Assume $\eta = 0$. Then $\tilde{\pi}_{\chi,m}$ has a unique subrepresentation isomorphic to $\tilde{\pi}^-_{k-1/2}$. The quotient is isomorphic to $\tilde{\pi}^+_{5/2-k}$.

b) Assume $\eta = 1$. Then $\tilde{\pi}_{\chi,m}$ has a unique subrepresentation isomorphic to $\tilde{\pi}^+_{k-1/2}$. The quotient is isomorphic to $\tilde{\pi}^-_{5/2-k}$.

3.2 Theorem. *The character χ is of the form*

$$\chi(x) = (\mathrm{sgn}(x))^\eta \, |x|^s, \qquad \eta \in \{0,1\}, \ s \in \mathbb{C}.$$

Assume $\underline{m > 0}$, and put $\nu = \frac{1}{2}\chi(-1)$.

i) If $s \notin \frac{1}{2} + \mathbb{Z}$, then $\tilde{\mathcal{B}}_{\chi,m}$ is irreducible. Its $\tilde{K}$-types are

$$\nu + 2n, \qquad n \in \mathbb{Z}.$$

In fact, $\tilde{\pi}_{\chi,m} \simeq \tilde{\pi}_{s,\nu}$

ii) Assume $s = k - \frac{3}{2}$, k even.

a) Assume $\eta = 0$. Then $\tilde{\pi}_{\chi,m}$ has a unique subrepresentation isomorphic to $\tilde{\pi}^-_{k-1/2}$. The quotient is isomorphic to $\tilde{\pi}^+_{5/2-k}$.

b) Assume $\eta = 1$. Then $\tilde{\pi}_{\chi,m}$ has a unique subrepresentation isomorphic to $\tilde{\pi}^+_{k-1/2}$. The quotient is isomorphic to $\tilde{\pi}^-_{5/2-k}$.

iii) Assume $s = k - \frac{3}{2}$, k odd.

a) Assume $\eta = 0$. Then $\tilde{\pi}_{\chi,m}$ has a unique subrepresentation isomorphic to $\tilde{\pi}^+_{k-1/2}$. The quotient is isomorphic to $\tilde{\pi}^-_{5/2-k}$.

b) Assume $\eta = 1$. Then $\tilde{\pi}_{\chi,m}$ has a unique subrepresentation isomorphic to $\tilde{\pi}^-_{k-1/2}$. The quotient is isomorphic to $\tilde{\pi}^+_{5/2-k}$.

3.3 Remark. Theorems 3.1 and 3.2 are equivalent, because by the remark ii) in [Wa2], p. 225, we have

$$\tilde{\pi}_{\chi,-m} = \tilde{\pi}_{\mathrm{sgn}\,\chi,m}. \tag{18}$$

3.3 Reducibility of Jacobi representations

There is a canonical isomorphism

$$\mathcal{B}^J_{\chi,m} \simeq \tilde{\mathcal{B}}_{\chi,-m} \otimes \mathcal{S}, \tag{19}$$

where $\mathcal{S}$ is the Schwartz space, carrying the (projective) Schrödinger–Weil representation π^m_{SW} of G^J. This can be proved as in the $\mathfrak{p}$-adic case, following the arguments of [BeS1] 5.4.2. It is important to note here the minus sign in $\tilde{\mathcal{B}}_{\chi,-m}$. Via the isomorphism (19), the reducibility question for $\mathcal{B}^J_{\chi,m}$ is reduced to that for $\tilde{\mathcal{B}}_{\chi,-m}$, and this was answered in the previous section.

To state the results, we define the Jacobi *principal series representations*

$$\pi^J_{s,\nu,m} := \tilde{\pi}_{s,\nu} \otimes \pi^m_{SW}, \qquad s \in \mathbb{C} \setminus \Big(\frac{1}{2} + \mathbb{Z}\Big),\ \nu = \pm\frac{1}{2},$$

the *positive discrete series representations*

$$\pi^{J+}_{k,m} := \tilde{\pi}^+_{k-1/2} \otimes \pi^m_{SW}, \qquad k \in \mathbb{Z},$$

and the *negative discrete series representations*

$$\pi^{J-}_{k,m} := \tilde{\pi}^-_{k-1/2} \otimes \pi^m_{SW}, \qquad k \in \mathbb{Z}.$$

Via (19), the theorems of the previous section immediately give the following results.

3.4 Theorem. *The character χ is of the form*

$$\chi(x) = (\mathrm{sgn}(x))^\eta\, |x|^s, \qquad \eta \in \{0,1\},\ s \in \mathbb{C}.$$

Assume $\underline{m > 0}$, and put $\nu = -\frac{1}{2}\chi(-1)$.

i) If $s \notin \frac{1}{2} + \mathbb{Z}$, then $\mathcal{B}^J_{\chi,m}$ is irreducible. Its K-types are

$$\frac{1}{2} + \nu + 2n, \qquad n \in \mathbb{Z}.$$

In fact, $\pi^J_{\chi,m} \simeq \pi^J_{s,\nu,m}$

ii) Assume $s = k - \frac{3}{2}$, k even.

a) Assume $\eta = 0$. Then $\pi^J_{\chi,m}$ has a unique subrepresentation, isomorphic to $\pi^{J+}_{k,m}$. The quotient is isomorphic to $\pi^{J-}_{3-k,m}$.

b) Assume $\eta = 1$. Then $\pi^J_{\chi,m}$ has a unique subrepresentation, isomorphic to $\pi^{J-}_{k,m}$. The quotient is isomorphic to $\pi^{J+}_{3-k,m}$.

iii) Assume $s = k - \frac{3}{2}$, k odd.

a) Assume $\eta = 0$. Then $\pi^J_{\chi,m}$ has a unique subrepresentation, isomorphic to $\pi^{J-}_{k,m}$. The quotient is isomorphic to $\pi^{J+}_{3-k,m}$.

b) *Assume* $\eta = 1$. *Then* $\pi^J_{\chi,m}$ *has a unique subrepresentation, isomorphic to* $\pi^{J+}_{k,m}$. *The quotient is isomorphic to* $\pi^{J-}_{3-k,m}$.

3.5 Theorem. *The character* χ *is of the form*

$$\chi(x) = (\mathrm{sgn}(x))^\eta\, |x|^s, \qquad \eta \in \{0,1\},\ s \in \mathbb{C}.$$

Assume $\underline{m < 0}$, *and put* $\nu = \frac{1}{2}\chi(-1)$.

i) *If* $s \notin \frac{1}{2} + \mathbb{Z}$, *then* $\mathcal{B}^J_{\chi,m}$ *is irreducible. Its* K*-types are*

$$-\frac{1}{2} + \nu + 2n, \qquad n \in \mathbb{Z}.$$

In fact, $\pi^J_{\chi,m} \simeq \pi^J_{s,\nu,m}$

ii) *Assume* $s = k - \frac{3}{2}$, k *even.*

a) *Assume* $\eta = 0$. *Then* $\pi^J_{\chi,m}$ *has a unique subrepresentation isomorphic to* $\pi^{J-}_{k,m}$. *The quotient is isomorphic to* $\pi^{J+}_{3-k,m}$.

b) *Assume* $\eta = 1$. *Then* $\pi^J_{\chi,m}$ *has a unique subrepresentation isomorphic to* $\pi^{J+}_{k,m}$. *The quotient is isomorphic to* $\pi^{J-}_{3-k,m}$.

iii) *Assume* $s = k - \frac{3}{2}$, k *odd.*

a) *Assume* $\eta = 0$. *Then* $\pi^J_{\chi,m}$ *has a unique subrepresentation isomorphic to* $\pi^{J+}_{k,m}$. *The quotient is isomorphic to* $\pi^{J-}_{3-k,m}$.

b) *Assume* $\eta = 1$. *Then* $\pi^J_{\chi,m}$ *has a unique subrepresentation isomorphic to* $\pi^{J-}_{k,m}$. *The quotient is isomorphic to* $\pi^{J+}_{3-k,m}$.

3.6 Remark. In contrast to (18), it is not true that $\pi^J_{\chi,-m}$ is isomorphic to $\pi^J_{\mathrm{sgn}\,\chi,m}$, since the central characters are different.

3.4 Representations of Q

The parabolic subgroup $Q \subset G$ introduced in 1.1 obviously contains the Jacobi group G^J. More precisely, if

$$T_1 := \{\mathrm{diag}(1,1,u,u) :\ u \in \mathbb{R}^*\},$$

and if C denotes the center of G, then

$$Q = C \times (T_1 \ltimes G^J).$$

Now let π^J be any representation of G^J. As in [BeS2] we shall write

$$C - \mathrm{Ind}^Q_{G^J}(\pi^J)$$

for the representation of Q obtained by inducing π^J to $T_1 \ltimes G^J$, and extending this representation to Q by making C act trivially. This is the same thing as inducing all the way from G^J to Q, and then taking the C-invariant vectors.

Note that if π^J is irreducible, it is not fixed under the conjugation action by any non-trivial element of T_1, since $T_1 \simeq \mathbb{R}^*$ acts by multiplication on the center of G^J, and thus changes the *index* (central character) of π^J. It follows by Mackey's theory that the representation $C - \mathrm{Ind}_{G^J}^Q(\pi^J)$ is irreducible on Q. In general, if we take π^J to be a Jacobi representation of finite length, then $C - \mathrm{Ind}_{G^J}^Q(\pi^J)$ has the same irreducibilities as π^J (a composition series of π^J induces to a composition series on Q). In particular, the constituents exhibited in Theorems 3.4 and 3.5 give constituents on Q.

4 Maaß lifts

4.1 The intertwining operator

We now fix any *index* $m \in \mathbb{R}^*$. Let Q be the parabolic subgroup of G containing G^J, as before. It was shown in [BeS2] that for χ of the form (11) with $\mathrm{Re}(s) > -\frac{1}{4}$, there is a Q-intertwining operator

$$i_m:\ \sigma_\chi \longrightarrow C - \mathrm{Ind}_{G^J}^Q(\pi^J_{\chi,m}) = C - \mathrm{Ind}_{B^J}^Q(\chi\psi^m). \tag{20}$$

Here $\pi^J_{\chi,m}$ is the induced Jacobi representation as defined in section 3.1. Explicitly, on the standard spaces of the induced representations, i_m is given by

$$(i_m\Psi)(g) = \int\limits_{\mathbb{R}} \Psi(w_1 w_2 n_\kappa g)\psi^{-m}(\kappa)\, d\kappa \qquad (g \in Q,\ \Psi \in I_\chi),$$

where

$$w_1 = \begin{pmatrix} & & 1 & \\ & 1 & & \\ -1 & & & \\ & & & 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}, \quad n_\kappa = \begin{pmatrix} 1 & & & \\ & 1 & & \kappa \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Moreover, it was shown in [BeS2] that i_m is *injective.*

4.1 Theorem. *Consider the representation σ_χ of G defined in 2.2, with χ as in (11) and* $\mathrm{Re}(s) > -\frac{1}{4}$.

i) If $s \notin \frac{1}{2} + \mathbb{Z}$, then σ_χ is irreducible.

ii) If $s = k - \frac{3}{2}$ with $k \in \mathbb{Z}$, $k \geq 2$, then σ_χ has length 2, meaning there exists a unique proper, non-trivial subrepresentation. More precisely, we have the composition series

$\sigma_\chi \supset \sigma_k^+ \supset 0$ *in the holomorphic case,*

$\sigma_\chi \supset \sigma_k^- \supset 0$ *in the skew-holomorphic case*

(see section 2.4 for these notations).

Proof: As mentioned above, we have an injective Q-intertwining operator

$$i_m : \sigma_\chi \longrightarrow C-\mathrm{Ind}_{G^J}^{Q}(\pi^J_{\chi,m}).$$

Thus the G-length of the left hand side can be no greater than the Q-length of the right hand side. But we know the reducibilities on the right: By Theorems 3.4 and 3.5, and the remarks in section 3.4, we have irreducibility if $s \notin \frac{1}{2}+\mathbb{Z}$. This proves i).

Now assume $s = k - \frac{3}{2}$ with $k \in \mathbb{Z}$, $k \geq 2$. Then, by Theorems 3.4 and 3.5, the right side has Q-length two. By the results of section 2.4, the left side has G-length at least two, with $\sigma_k^\pm$ being a proper subrepresentation. From this ii) follows. ■

4.2 Corollary. *The intertwining operator i_m in (20) is an isomorphism of Q-representations. The constituents of σ_χ are not only G-irreducible, but also Q-irreducible.*

4.2 Maaß lifts

We call a representation σ of G with trivial central character a *Maaß lift* of the representation π^J of G^J, if

$$\sigma \simeq C-\mathrm{Ind}_{G^J}^{Q}(\pi^J) \qquad \text{as } Q\text{-representations.}$$

Obviously Maaß lifts of irreducible Jacobi representations are not only irreducible as G-representations, but even after restriction to Q. Frobenius reciprocity yields

$$\mathrm{Hom}_Q\Big(\sigma\big|_Q, C-\mathrm{Ind}_{G^J}^{Q}(\pi^J)\Big) \simeq \mathrm{Hom}_{G^J}\Big(\sigma\big|_{G^J}, \pi^J\Big).$$

Thus Maaß lifts have what is called a Fourier-Jacobi model. In this context cf. [Hi] and [BR] (the latter for the p-adic case).

By Corollary 4.2 the intertwining operator i_m produces a lot of Maaß lifts. By matching the constituents of σ_χ (section 2.4) with those of $C-\mathrm{Ind}_{G^J}^{Q}(\pi^J_{\chi,m})$ (Theorems 3.4 and 3.5), one obtains the following list.

index	parameters	G^J – representation	G – representation
$m>0$	$s \notin \frac{1}{2}+\mathbb{Z},\ \nu=-\frac{1}{2}\chi(-1)$	$\pi^J_{s,\nu,m}$	$\sigma_\chi,\ \chi(x)=\mathrm{sgn}(x)^\eta \lvert x\rvert^s$
$m<0$	$s \notin \frac{1}{2}+\mathbb{Z},\ \nu=\frac{1}{2}\chi(-1)$	$\pi^J_{s,\nu,m}$	$\sigma_\chi,\ \chi(x)=\mathrm{sgn}(x)^\eta \lvert x\rvert^s$
$m>0$	even weights	$\pi^{J+}_{k,m}$ $\pi^{J-}_{3-k,m}$	σ^+_k σ^-_{3-k}
	odd weights	$\pi^{J-}_{k,m}$ $\pi^{J+}_{3-k,m}$	σ^-_k σ^+_{3-k}
$m<0$	even weights	$\pi^{J-}_{k,m}$ $\pi^{J+}_{3-k,m}$	σ^+_k σ^-_{3-k}
	odd weights	$\pi^{J+}_{k,m}$ $\pi^{J-}_{3-k,m}$	σ^-_k σ^+_{3-k}

In this table k is an integer ≥ 2, and we mean with

$$\begin{array}{ll} \text{even weights :} & k \text{ even}, \eta=0, \ \underline{\text{or}}\ \ k \text{ odd}, \eta=1, \\ \text{odd weights :} & k \text{ even}, \eta=1, \ \underline{\text{or}}\ \ k \text{ odd}, \eta=0. \end{array}$$

For reasons of convergence of the intertwining operator i_m we have to assume $s>-\frac{1}{4}$. But this is enough, since one can show

$$\tilde{\pi}_{s,\nu} \simeq \tilde{\pi}_{-s,\nu}, \qquad \text{and therefore} \qquad \pi^J_{s,\nu,m} \simeq \pi^J_{-s,\nu,m}$$

(cf. [Wa2] I.4, [BeS1] 3.1.9). Consequently the third column in the table lists each equivalence class of irreducible G^J-representations at least once. Thus we get a list of Maaß lifts for *all* Jacobi representations.

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