Paramodular forms of level 16 and supercuspidal representations

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Abstract

This work bridges the abstract representation theory of GSp(4) with recent computational techniques. We construct four examples of paramodular newforms whose associated automorphic representations have local representations at $p = 2$ that are supercuspidal. We classify all relevant irreducible, admissible, supercuspidal representations of $\text{GSp}(4, \mathbb{Q}_2)$, and show that our examples occur at the lowest possible paramodular level, 16. The required theoretical and computational techniques include paramodular newform theory, Jacobi restriction, bootstrapping and Borcherds products.

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Introduction

This paper consists of a local part and a global part. In the local part we classify irreducible, admissible, supercuspidal representations of $\text{GSp}(4, \mathbb{Q}_2)$ with trivial central character and small conductor. In particular, we prove that there exists a unique such supercuspidal $\mathbf{sc}(16)$ with (the
exponent of the) conductor $a(\text{sc}(16)) = 4$. In the global part we construct Siegel paramodular
cusp forms of weights 6, 11, 13, and 14 and paramodular level 16 generating an automorphic
representation with $\text{sc}(16)$ as its 2-component. To the best of our knowledge, these are the first
examples of Siegel paramodular forms generating automorphic representations with a supercuspidal component. Other types of local representations can be seen in [26].

We give two approaches to the construction of $\text{sc}(16)$. The first approach relies on the
local Langlands correspondence for the groups $\text{GL}(2)$, $\text{GL}(4)$ and $\text{GSp}(4)$. We first construct,
via automorphic induction, a set of six supercuspidals of $\text{GL}(2, E)$, where $E = \mathbb{Q}_2(\sqrt{5})$ is the
unramified quadratic extension of $\mathbb{Q}_2$. Up to unramified twists, these are precisely the depth zero
supercuspidals of $\text{GL}(2, E)$. We automorphically induce again to obtain three supercuspidals
of $\text{GL}(4, \mathbb{Q}_2)$. These are precisely the three depth zero supercuspidals of $\text{GL}(4, \mathbb{Q}_2)$ with trivial
central character. Of these three, exactly one is a transfer from a representation of $\text{GSp}(4, \mathbb{Q}_2)$. This representation of $\text{GSp}(4, \mathbb{Q}_2)$ is the unique generic supercuspidal $\text{sc}(16)$ with trivial central character and conductor 4. As a corollary to our construction, we obtain a complete list of all
supercuspidals of $\text{GSp}(4, F)$ with trivial central character and conductor $\leq 4$; see Table 2. We
also determine, via direct calculation, that the value of the $\varepsilon$-factor at $1/2$ of $\text{sc}(16)$ is $-1$. This sign is important to know for global applications, as it will help us to identify $\text{sc}(16)$ within the
automorphic representations generated by paramodular forms.

Our second approach to $\text{sc}(16)$ is via compact induction. The Langlands parameter $\text{sc}(16)$,
known from the first construction, is of the kind considered in [5]. The results of this paper then
exhibit $\text{sc}(16)$ as being compactly induced from a cuspidal representation $\kappa_0$ of $\text{GSp}(4, \mathbb{Z}_2/2\mathbb{Z}_2)$ (inflated to $\text{GSp}(4, \mathbb{Z}_2)$ and extended trivially to include the center). Since $\text{GSp}(4, \mathbb{Z}/2\mathbb{Z}) \cong S_6$, the irreducible characters of this group are in bijection with the partitions of 6. The representation $\kappa_0$ corresponds to $(2, 2, 1, 1)$ and has dimension 9. It is the unique cuspidal, generic character of $\text{GSp}(4, \mathbb{Z}/2\mathbb{Z})$.

We describe the passage from global paramodular forms to local supercuspidal representations. The automorphic representations studied here are generated by the adelic function canonicallv associated to a paramodular eigenform $f \in S_k(K(N))_{\text{new}}$. The interesting local representations are classified by computing the Hecke eigenvalues of $f$ at primes dividing the level $N$. In order to rigorously compute these eigenvalues, we spanned the Fricke eigenspace containing $f$, $S_k(K(N))^\epsilon$. Accurate upper bounds for the dimension of $S_k(K(N))^\epsilon$ were provided by Jacobi restriction, which classifies all possible Fourier-Jacobi coefficients from $S_k(K(N))^\epsilon$ to some sufficient order. Lower bounds were created by the technique of bootstrapping. Bootstrapping seeds the target space with a Borcherds product, and then generates a subspace that contains the seed and is stable under a good Hecke operator. Bootstrapping is run modulo an auxiliary prime, and the subtle point is that it does not directly compute the action of a good Hecke operator $T(q)$ on $S_k(K(N))^\epsilon$, but rather of a formal Hecke operator $T(q)$ on the Jacobi restriction space of initial Fourier-Jacobi expansions.

Even with the relevant spaces spanned, the eigenvalues at the bad primes resist direct com-putation because they involve Fourier coefficients from more than one 1-dimensional cusp. As in [26], this is overcome using the technique of restriction to a modular curve. We found symmetric $f$ with a supercuspidal local component early on, but only found the antisymmetric example in $S_{14}(K(16))^{-}$ as the computations were becoming prohibitive.

The authors thank ICERM for the June 12-16, 2017, Collaborate@ICERM grant where a substantial part of this work was completed. We thank the Academic Computing Environment
1 Notation

For any commutative ring \( R \), let

\[ \text{GSp}(4, R) = \{ g \in \text{GL}(4, R) : {}^t g J g = \mu(g) J, \text{ for some } \mu(g) \in R^\times \}, \quad J = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}. \]

The kernel of the multiplier homomorphism \( \mu : \text{GSp}(4, R) \to R^\times \) is the group \( \text{Sp}(4, R) \). The \( \mathbb{C} \)-vector space of Siegel modular forms of weight \( k \in \mathbb{Z} \) for a subgroup \( \Gamma \subseteq \text{GSp}(4, \mathbb{R}) \) commensurable with \( \text{Sp}(4, \mathbb{Z}) \) is denoted by \( M_k(\Gamma) \), the subspace of cusp forms by \( S_k(\Gamma) \).

2 Supercuspidal representations of \( \text{GSp}(4, \mathbb{Q}_2) \) of small conductor

Let \( F \) be a non-archimedean local field of characteristic zero. Let \( \mathfrak{o} \) be its ring of integers, \( p \) the maximal ideal of \( \mathfrak{o} \), and \( q \) the cardinality of the residue class field \( \mathfrak{o}/p \). When there is more than one field involved, we sometimes write \( \mathfrak{o}_F, p_F \), and \( q_F \) for clarity.

Let \( W_F \) be the Weil group of \( F \), and \( W_F' \) the Weil-Deligne group. We refer to [33] or [14] for basic facts about the Weil and Weil-Deligne groups and their representations. If \( \phi : W_F' \to \text{GL}(n, \mathbb{C}) \) is a representation of \( W_F' \), then we define the (exponent of the) conductor \( a(\phi) \) of \( \phi \) as in \( \S 10 \) of [33]. If \( \pi \) is an irreducible, admissible representation of \( \text{GL}(n, F) \), then the conductor of \( \pi \) is defined as \( a(\pi) = a(\phi) \), where \( \phi : W_F' \to \text{GL}(n, \mathbb{C}) \) is the Weil-Deligne representation corresponding to \( \pi \) via the local Langlands correspondence.

2.1 Discrete series parameters for \( \text{GSp}(4) \)

The local Langlands correspondence (LLC) for \( \text{GL}(n) \) states that there is a bijection between isomorphism classes of irreducible, admissible representations \( \pi \) of \( \text{GL}(n, F) \) and Langlands parameters, i.e., conjugacy classes of admissible homomorphisms \( \phi : W_F' \to \text{GL}(n, \mathbb{C}) \). This bijection satisfies a number of desirable properties. For example, if \( \pi \) corresponds to \( \phi \), then the central character of \( \pi \) corresponds to \( \det(\phi) \) under the LLC for \( \text{GL}(1) \) (which is essentially the reciprocity law of local class field theory). Another property is that \( \pi \) is an essentially discrete series representation if and only if the image of \( \phi \) is not contained in a proper Levi subgroup; such \( \phi \) are therefore called discrete series parameters. Moreover, supercuspidal \( \pi \) correspond to irreducible \( \phi \).

The local Langlands correspondence is also a theorem for \( \text{GSp}(4) \); see [10]. The Langlands parameters are now admissible homomorphisms \( \phi : W_F' \to \text{GSp}(4, \mathbb{C}) \), taken up to conjugacy by elements of \( \text{GSp}(4, \mathbb{C}) \). A new phenomenon is that to one \( \phi \) there now corresponds either a single representation \( \pi \), as in the \( \text{GL}(n) \) case, or a set of two representations \( \{ \pi_1, \pi_2 \} \). In either case we speak of the \( L \)-packet corresponding to \( \phi \). The size of the \( L \)-packet corresponding to \( \phi \) equals the cardinality of \( S_\phi/S_\phi^0 \mathbb{Z} \), where \( S_\phi \) is the centralizer of the image of \( \phi \), \( S_\phi^0 \) is its identity component, and \( \mathbb{Z} \) is the center of \( \text{GSp}(4, \mathbb{C}) \).

The LLC for \( \text{GSp}(4) \) is such that the central character of the representations in the \( L \)-packet of \( \phi \) corresponds to the multiplier \( \mu \circ \phi \). As in the \( \text{GL}(n) \) case, the \( L \)-packet corresponding to \( \phi \) consists of essentially discrete series representations if and only if the image of \( \phi \) is not contained in a proper Levi subgroup of \( \text{GSp}(4, \mathbb{C}) \). It is also true that irreducible \( \phi : W_F' \to \text{GSp}(4, \mathbb{C}) \)
correspond to singleton supercuspidal $L$-packets. However, there are plenty of supercuspidals whose Langlands parameter is not irreducible.

To better understand $L$-parameters for supercuspidals, we recall some of the discussion of Sect. 7 of [10]. Let $\phi : W'_F \to \text{GSp}(4, \mathbb{C})$ be a discrete series parameter for $\text{GSp}(4)$, meaning the image of $\phi$ is not contained in a proper Levi subgroup of $\text{GSp}(4, \mathbb{C})$. Such parameters are of one of two types (A) or (B).

**Type (A):** Viewed as a four-dimensional representation of $W'_F$, the map $\phi$ decomposes as $\phi_1 \oplus \phi_2$, where $\phi_1$ and $\phi_2$ are inequivalent indecomposable two-dimensional representations of $W'_F$ with $\det(\phi_1) = \det(\phi_2)$. Explicitly, if $\phi_i(w) = \begin{bmatrix} a_i(w) & b_i(w) \\ c_i(w) & d_i(w) \end{bmatrix}$, then

$$\phi(w) = \begin{bmatrix} a_1(w) & b_1(w) \\ c_1(w) & d_1(w) \\ a_2(w) & b_2(w) \\ c_2(w) & d_2(w) \end{bmatrix}.$$  

In this case the packet associated to $\phi$ consists of two elements, a generic representation $\pi^\text{gen}$ and a non-generic $\pi^\text{ng}$. The common central character of these two representations corresponds to $\det(\phi_1) = \det(\phi_2)$. There are three subcases:

- **(A₁):** Both $\phi_1$ and $\phi_2$ are irreducible. In this case $\pi^\text{gen}$ and $\pi^\text{ng}$ are both supercuspidal.

- **(A₂):** One of $\phi_1, \phi_2$ is irreducible, and the other is reducible (but indecomposable). In this case $\pi^\text{gen}$ is a representation of type XIA in the classification of [31]; it sits inside a representation induced from a supercuspidal representation of the Levi component of the Siegel parabolic subgroup. The non-generic $\pi^\text{ng}$ is supercuspidal; it is a representation of type XIA* in the notation of [32].

- **(A₃):** Both $\phi_1$ and $\phi_2$ are reducible (but indecomposable). In this case $\pi^\text{gen}$ is a representation of type Va in the classification of [31]; it sits inside a representation induced from the Borel subgroup. The non-generic $\pi^\text{ng}$ is supercuspidal; it is a representation of type Va* in the notation of [32].

Hence $\pi^\text{ng}$ is always supercuspidal, but $\pi^\text{gen}$ is only supercuspidal for class (A₁). Note that, by Theorem 3.4.3 of [31], non-generic supercuspidals do not contain paramodular vectors of any level. Hence, supercuspidals of the form $\pi^\text{ng}$ cannot occur as local components in automorphic representations attached to paramodular cusp forms.

**Type (B):** Viewed as a four-dimensional representation of $W'_F$, the map $\phi$ is indecomposable. In this case there is a single representation $\pi$ attached to $\phi$, and this $\pi$ is generic. Via the inclusion $\text{GSp}(4, \mathbb{C}) \hookrightarrow \text{GL}(4, \mathbb{C})$ we may view $\phi$ as the Langlands parameter of a discrete series representation $\Pi$ of $\text{GL}(4, F)$. By the definitions involved, $\Pi$ is the image of $\pi$ under the functorial lifting from $\text{GSp}(4)$ to $\text{GL}(4)$ coming from the embedding $\text{GSp}(4, \mathbb{C}) \hookrightarrow \text{GL}(4, \mathbb{C})$ of dual groups. Again there are three subcases:

- **(B₁):** $\phi$ is irreducible as a four-dimensional representation. In this case $\pi$ is supercuspidal.

- **(B₂):** $\phi = \varphi \otimes \text{sp}(2)$ with an irreducible two-dimensional representation $\varphi$ of $W_F$, and $\text{sp}(2)$ being the special indecomposable two-dimensional representation of $W'_F$. In this
case $\pi$ is a representation of type IXa; see Sect. 2.4 of [31]. This $\pi$ sits inside a representation induced from a supercuspidal representation of the Levi component of the Klingeng parabolic subgroup.

- $(B_3)$: $\phi = \xi \otimes \text{sp}(4)$ with a one-dimensional representation $\xi$ of $W_F$. Then $\pi$ is a twist of the Steinberg representation $\text{St}_{GSp(4)}$ (type IVa in the classification of [31]).

Hence $\pi$ is supercuspidal only for class $(B_1)$, i.e., if $\phi$ is irreducible. In this case $\pi$ transfers to a supercuspidal representation $\Pi$ of $GL(4,F)$.

### 2.2 Counting supercuspidals for $GL(2)$ and $GL(4)$

We see from the parameters exhibited in the previous section that, in order to understand supercuspidal representations of $GSp(4,F)$, we need to understand supercuspidal representations of $GL(2,F)$ and $GL(4,F)$, or equivalently, two-dimensional and four-dimensional irreducible representations of $W_F$. In this section we count the number of supercuspidals of $GL(2,F)$ and $GL(4,F)$ with small conductor.

The **conductor** $a(\pi)$ of an irreducible, admissible representation of $GL(n,F)$ is by definition the Artin conductor $a(\phi)$ of its Langlands parameter $\phi$; see §10 of [33]. Here, we always mean the **exponent** of the conductor, so that $a(\pi) = a(\phi)$ is a non-negative integer. Another measure of complexity is the **depth** $d(\pi)$, as defined in [24, 25]. For supercuspidals, there is an easy relationship between depth and conductor, given by

$$d(\pi) = \frac{a(\pi) - n}{n};$$

see Proposition 2.2 of [22]. The set of supercuspidals of a fixed conductor is invariant under unramified twisting.

The smallest conductor that can occur for a supercuspidal representation of $GL(n,F)$ is $a(\pi) = n$. By (1), these are the depth zero supercuspidals. If $\pi$ is one such supercuspidal, and $\chi$ is an unramified character, then the twist $\chi \pi$ is also a depth zero supercuspidal. Let $Z_n$ be the (finite) set of isomorphism classes of depth zero supercuspidals of $GL(n,F)$ up to unramified twists. It is known that $Z_n$ is in bijection with the set of $\text{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ orbits of length $n$ in the group of characters of $\mathbb{F}_q^\times$; see Sect. 8 of [6] and Sect. 6 of [25]. It is an exercise to show that

$$\#Z_2 = \frac{1}{2}q(q - 1), \quad \#Z_4 = \frac{1}{4}q^2(q^2 - 1).$$

Note that if $q = 2$, then every element of $Z_n$ is represented by a unique depth zero supercuspidal with trivial central character. The reason is that depth zero supercuspidals are compactly induced from representations of $ZK$, where the representation on $K = GL(2,\mathfrak{o})$ is inflated from a cuspidal representation of $GL(2,\mathfrak{o}/\mathfrak{p})$. If $\mathfrak{o}/\mathfrak{p}$ has only two elements, then every representation of $K$ thus obtained has trivial central character. In particular, we see from (2) that $GL(2,\mathbb{Q}_2)$ has exactly one depth zero supercuspidal with trivial central character, and $GL(4,\mathbb{Q}_2)$ has exactly three depth zero supercuspidals with trivial central character.

For a unitary character $\omega$ of $F^\times$, let $S_\omega$ be the set of isomorphism classes of depth zero supercuspidals of $GL(2,F)$ with central character $\omega$. By Proposition 3.4 of [37], $\#S_\omega = 0$ if


\[ a(\omega) \geq 2. \] If \( a(\omega) \leq 1, \) then, by (4-1) of [20],

\[
\# S_\omega = \begin{cases} 
\frac{1}{2} (q - 1) & \text{if } q \text{ is odd and } \omega^{(q-1)/2} \text{ is trivial,} \\
\frac{1}{2} (q + 1) & \text{if } q \text{ is odd and } \omega^{(q-1)/2} \text{ is non-trivial,} \\
\frac{1}{2} q & \text{if } q \text{ is even.}
\end{cases}
\]

2.3 Depth zero supercuspidals of \( \text{GL}(2, \mathbb{Q}_2(\sqrt{5})) \)

In this section let \( E = \mathbb{Q}_2(\sqrt{5}) \) be the unramified quadratic extension of \( \mathbb{Q}_2, \) and let \( L \) be the unramified quadratic extension of \( E. \) Note that 2 is a uniformizer both in \( E \) and in \( L. \) Let \( \mathbb{F}_p^n \) be the field with \( p^n \) elements. The residue class field of \( E \) is \( \mathbb{F}_4, \) and the residue class field of \( L \) is \( \mathbb{F}_{16}. \) The polynomial \( X^4 + X + 1 \in \mathbb{F}_2[X] \) is irreducible, so that

\[ \mathbb{F}_{16} \cong \mathbb{F}_2[X]/(X^4 + X + 1). \]

Let \( \bar{y} \) be the image of \( X \) via this isomorphism. Then \( \mathbb{F}_{16} = \mathbb{F}_2(\bar{y}), \) and \( \bar{y} \) satisfies \( \bar{y}^4 = \bar{y} + 1. \) Clearly, the order of \( \bar{y} \) in \( \mathbb{F}_{16}^\times \) is not 3 or 5, so that \( \bar{y} \) is a generator of the cyclic group \( \mathbb{F}_{16}^\times. \) The element \( \bar{y}^3 \) is then a generator of the cyclic group \( \mathbb{F}_4^\times. \) Let \( y \) be an element of \( \mathbb{F}_{16}^\times \) mapping to \( \bar{y} \) under the projection \( \mathbb{F}_{16}^\times \to \mathbb{F}_4^\times. \)

Let \( \bar{\eta} \) be the character of \( \mathbb{F}_{16}^\times \) determined by \( \bar{\eta}(\bar{y}) = e^{2\pi i / 15}. \) For \( r \in \mathbb{Z}/15\mathbb{Z} \) we define a character \( \eta_r \) of \( \mathbb{L}^\times \) by lifting \( \bar{\eta}^r \) to \( \mathbb{F}_{16}^\times \) and setting \( \eta_r(2) = -1. \)

Let \( \theta \) be the generator of \( \text{Gal}(L/\mathbb{Q}_2) \) that induces the map \( x \mapsto x^2 \) on \( \mathbb{F}_{16}. \) Then \( \theta^2 \) generates \( \text{Gal}(L/E). \) We have \( \eta_2^\theta = \eta_2 \). (If \( \sigma \in \text{Gal}(L/E) \) and \( \pi \) is a representation of \( \text{GL}(n, E) \), then \( \pi^\sigma \) is the representation of \( \text{GL}(n, F) \) defined by \( \pi^\sigma(g) = \pi(\sigma(g)). \))

Consider automorphic induction \( AI = AI_{L/E}; \) see [16]. Recall that \( AI \) takes characters \( \xi \) of \( \mathbb{L}^\times \) to irreducible, admissible representations \( \rho \) of \( \text{GL}(2, E). \) By Proposition 4.5 of [16], the central character of \( \rho \) is given by \( \chi_{L/E}(\xi|_{\mathbb{L}^\times}) \), where \( \chi_{L/E} \) is the quadratic character of \( \mathbb{E}^\times \) corresponding to the extension \( L/E. \) On the Galois side, \( AI \) corresponds to induction of parameters, i.e., the parameter of \( \rho \) is

\[ \phi_\rho = \text{ind}_{W_L}^{W_E}(\xi). \]

This parameter is irreducible, i.e., \( \rho \) is supercuspidal, if and only if \( \xi \) is not \( \text{Gal}(L/E) \)-invariant. We have \( a(\rho) = 2a(\xi) \) by the conductor formula (a2) in \( \text{§}10 \) of [33].

We now consider \( AI_{L/E}(\eta_r) \) for \( r \in \{1, \ldots, 15\}. \) This representation is supercuspidal if and only if \( \eta_{4r} \neq \eta_r, \) which translates into \( 5 \nmid r. \) Since \( a(\eta) = 1, \) we have \( a(AI_{L/E}(\eta_r)) = 2 \) for \( r \neq 0. \)

The central character \( \omega \) of \( AI_{L/E}(\eta_r) \) is determined by \( \omega(2) = 1 \) and \( \omega(y^5) = \eta_{5r}(y) = e^{2\pi i r/3}. \) Hence, if we let \( \omega_j \) be the character of \( \mathbb{E}^\times \) which is trivial on \( 1 + \mathbb{p}_E \) and satisfies \( \omega_j(2) = 1 \) and \( \omega_j(y^5) = e^{2\pi i (j-1)/3}, \) then \( \omega_1, \omega_2, \omega_3 \) are the possible central characters of the \( AI_{L/E}(\eta_r). \) We have \( \omega_1^\theta = \omega_1 \) and \( \omega_2^\theta = \omega_2 \) and \( \omega_3^\theta = \omega_3. \) Considering Langlands parameters, it is easy to see that the \( \text{Gal}(E/\mathbb{Q}_2) \)-conjugate of \( AI(\xi) \) is given by \( AI(\xi^\theta) = AI(\xi^\theta), \) and the contragredient is \( AI(\xi)^\lor = AI(\xi^{-1}). \)

Table 1 lists the supercuspidal representations of the form \( AI(\eta_r). \) For each possible central character \( \omega_j, \) there are two supercuspidals, which we denote by \( \rho_{ja} \) and \( \rho_{jb}. \) Note from (2)
Table 1: Representatives for the depth zero supercuspidals of GL(2, E) up to unramified twists. The first column shows Gal(L/E)-orbits of length 2 of the characters ξ = ψ. The ω column shows the central character of the representation AI_{L/E}(ξ). The columns AI_{L/E}(ξ)θ and AI_{L/E}(ξ)∨ show the Gal(E/Q)-conjugate and contragredient of AI_{L/E}(ξ), respectively.

<table>
<thead>
<tr>
<th>ξ</th>
<th>AI(ξ)</th>
<th>ω</th>
<th>AI(ξ)θ</th>
<th>AI(ξ)∨</th>
</tr>
</thead>
<tbody>
<tr>
<td>η_3 or η_12</td>
<td>ρ_{1a}</td>
<td>ω_1</td>
<td>ρ_{1b}</td>
<td>ρ_{1a}</td>
</tr>
<tr>
<td>η_6 or η_9</td>
<td>ρ_{1b}</td>
<td>ω_1</td>
<td>ρ_{1a}</td>
<td>ρ_{1b}</td>
</tr>
<tr>
<td>η_1 or η_4</td>
<td>ρ_{2a}</td>
<td>ω_2</td>
<td>ρ_{3a}</td>
<td>ρ_{3b}</td>
</tr>
<tr>
<td>η_7 or η_13</td>
<td>ρ_{2b}</td>
<td>ω_2</td>
<td>ρ_{3b}</td>
<td>ρ_{3a}</td>
</tr>
<tr>
<td>η_2 or η_8</td>
<td>ρ_{3a}</td>
<td>ω_3</td>
<td>ρ_{2a}</td>
<td>ρ_{2b}</td>
</tr>
<tr>
<td>η_11 or η_14</td>
<td>ρ_{3b}</td>
<td>ω_3</td>
<td>ρ_{2b}</td>
<td>ρ_{2a}</td>
</tr>
</tbody>
</table>

that there are exactly six depth zero supercuspidals of GL(2, E) up to unramified twists. The following lemma implies that the six representations \{ρ_{1a}, ρ_{1b}, ρ_{2a}, ρ_{2b}, ρ_{3a}, ρ_{3b}\} represent these six classes of depth zero supercuspidals up to unramified twists. Note that having exactly two depth zero supercuspidals for a given central character \(\omega_j\) is consistent with (3).

2.3.1 Lemma. Let \(j \in \{1, 2, 3\}\).

i) The representation \(ρ_{ja}\) is not a twist of \(ρ_{jb}\).

ii) Let \(ρ = ρ_{ja}\) or \(ρ = ρ_{jb}\). Then \(ρ^θ\) is not isomorphic to a twist of \(ρ^∨\).

iii) Let \(ρ, ρ' \in \{ρ_{1a}, ρ_{1b}, ρ_{2a}, ρ_{2b}, ρ_{3a}, ρ_{3b}\}\). Then \(ρ\) is not an unramified twist of \(ρ'\), unless \(ρ = ρ'\).

Proof. i) Assume that \(ρ_{ja} = χ \otimes ρ_{jb}\) for some character \(χ\) of \(E^×\); we will obtain a contradiction. Taking central characters on both sides, we see that \(χ^2 = 1\). We have \(a(χ) \leq 1\) by Proposition 3.4 of [37].

Assume that \(a(χ) = 0\). Then \(χ\) is either the trivial character, or \(χ = χ_{L/E}\), the unique non-trivial, unramified, quadratic character of \(E^×\). In either case \(χ \otimes ρ_{jb} = ρ_{jb}\), a contradiction.

Assume that \(a(χ) = 1\). Then \(χ\) induces a non-trivial character of \(σ_E^×/(1 + p_E)\). In particular, the image of \(χ|_{σ_E^×}\) consists of the third roots of unity, contradicting \(χ^2 = 1\).

ii) follows from i) and Table 1.

iii) Assume that \(ρ\) is an unramified twist of \(ρ'\). Then the restrictions of the central characters of \(ρ\) and \(ρ'\) to \(σ_E^×\) coincide. Hence \(ρ = ρ_{js}\) and \(ρ' = ρ_{js}\) with the same \(j\). By i), we conclude \(ρ = ρ'\).

2.3.2 Lemma. Let \(L\) be the unramified extension of degree 4 over \(Q_2\). Let the characters \(η_r\) of \(L^×\) be defined as above. Then, for \(ξ \in \{η_3, η_6, η_9, η_12\}\),

\[ ε(1/2, ξ, ψ_L) = -1. \] (4)
Here, \( \psi_L = \psi \circ \text{tr}_{L/Q_2} \), where \( \psi \) is a character of \( Q_2 \) that is trivial on \( \mathbb{Z}_2 \) but not on \( 2^{-1}\mathbb{Z}_2 \).

**Proof.** Let \( F_1 = \mathbb{F}_2[X]/(X^4 + X + 1) \), and \( \bar{y} \) be the element corresponding to \( X \), as at the beginning of this section. The Frobenius of the extension \( F_{16}/F_2 \) is given by squaring, so that
\[
\text{tr}_{F_{16}/F_2}(x) = x + x^2 + x^4 + x^8
\]
for any \( x \in F_{16} \). Using this formula and \( \bar{y}^4 = \bar{y} + 1 \), it is easy to calculate the trace of any element of \( F_{16} \). The results are as follows,

\[
\begin{array}{c|cccccccccccccc}
i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
\hline
\text{tr}_{F_{16}/F_2}(\bar{y}^i) & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
\end{array}
\]

Let \( \xi \in \{ \eta_3, \eta_6, \eta_9, \eta_{12} \} \). By the formula (63) in §11 of [33],
\[
\varepsilon(1/2, \xi, \psi_L) = q_L^{-a(\xi)/2} \int_{\omega_L^{-a(\xi)}\mathbb{Z}_L} \xi^{-1}(x)\psi_L(x) \, dx.
\]

For this formula to hold, it is important that \( \psi_L \) has conductor \( \mathfrak{o}_L \), which is the case for our additive character. The element \( \omega_L \) is a uniformizer; in our case we may take \( \omega_L = 2 \). We further have \( a(\xi) = 1 \) and \( q_L = 16 \), so that
\[
\varepsilon(1/2, \xi, \psi_L) = \frac{1}{4} \int_{\mathfrak{o}_L^{2^{-1}}} \xi^{-1}(x)\psi_L(x) \, dx = \frac{1}{4} |2^{-1}|_L \int_{\mathfrak{o}_L^{2^{-1}}} \xi^{-1}(2^{-1}x)\psi_L(2^{-1}x) \, dx
\]
\[
= -4 \int_{\mathfrak{o}_L^{2^{-1}}} \xi^{-1}(x)\psi_L(2^{-1}x) \, dx = -4 \text{vol}(1 + \mathfrak{p}_L) \sum_{x \in \mathfrak{o}_L^{2^{-1}}/(1 + \mathfrak{p}_L)} \xi^{-1}(x)\psi_L(2^{-1}x)
\]
\[
= -\frac{1}{4} \sum_{x \in \mathfrak{o}_L^{2^{-1}}/(1 + \mathfrak{p}_L)} \xi^{-1}(x)\psi(2^{-1}\text{tr}_{L/Q_2}(x)).
\]

We have
\[
\psi(2^{-1}\text{tr}_{L/Q_2}(x)) = \begin{cases} 1 & \text{if } \text{tr}_{L/Q_2}(x) \in 2\mathbb{Z}_2, \\
-1 & \text{if } \text{tr}_{L/Q_2}(x) \in 2\mathbb{Z}_2^\times. \end{cases}
\]

Hence, using (5),
\[
\varepsilon(1/2, \xi, \psi_L) = -\frac{1}{4} \left( \sum_{i \in \{1,2,4,5,8,10,15\}} \xi^{-1}(y^i) - \sum_{i \in \{3,6,7,9,11,12,13,14\}} \xi^{-1}(y^i) \right)
\]
\[
= -\frac{1}{4} (\zeta + \zeta^2 + \zeta^4 + \zeta^5 + \zeta^8 + \zeta^{10} + \zeta^{15} - \zeta^3 - \zeta^6 - \zeta^7 - \zeta^9 - \zeta^{11} - \zeta^{12} - \zeta^{13} - \zeta^{14}),
\]
where \( \zeta = \xi^{-1}(y) \), a primitive fifth root of unity. Using \( \zeta^5 = 1 \) and \( 1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 = 0 \), this simplifies to
\[
\varepsilon(1/2, \xi, \psi_L) = -\frac{1}{4} (\zeta + \zeta^2 + \zeta^4 + 1 + \zeta^3 + 1 + 1 - \zeta^3 - \zeta - \zeta^2 - \zeta^4 - \zeta - \zeta^2 - \zeta^3 - \zeta^4)
\]
$= -\frac{1}{4}(3 - \zeta - \zeta^2 - \zeta^3 - \zeta^4) = -1.$

This concludes the proof. \hfill \blacksquare

### 2.4 Supercuspidals of $GSp(4, \mathbb{Q}_2)$ with small conductor

As in the previous section, let $E$ be the unramified quadratic extension of $\mathbb{Q}_2$. Let $\theta$ be the non-trivial element of $Gal(E/\mathbb{Q}_2)$. We now consider automorphic induction $AI = AI_{E/\mathbb{Q}_2}$. Recall that $AI$ takes irreducible, admissible representations $\rho$ of $GL(2, E)$ to irreducible, admissible representations $\pi$ of $GL(4, \mathbb{Q}_2)$. By Proposition 4.5 of [16], the central characters $\omega_\rho$ and $\omega_\pi$ are related by $\omega_\pi = \omega_\rho|_{\mathbb{Q}_2^\times}$. If $\phi_\rho$ is the parameter of $\rho$, then the parameter of $\pi$ is

$$\phi_\pi = \text{ind}_{W_E}^{W_{E/\mathbb{Q}_2}}(\phi_\rho).$$

Assume that $\rho$ is supercuspidal, or equivalently, that $\phi_\rho$ is irreducible. Then $\pi$ is supercuspidal if and only if $\rho \neq \rho^\theta$, where the Galois conjugate $\rho^\theta$ is defined by $\rho^\theta(g) = \rho(g^\theta)$ for $g \in GL(2, E)$. In other words, $\phi_\pi$ is irreducible if and only if $\phi_\rho \neq \phi_\rho^\theta$, where $\phi_\rho^\theta(w) = \phi_\rho(\theta w \theta^{-1})$ for $w \in W_E$ (here we think of $\theta$ as an element of $W_{\mathbb{Q}_2}$ that is not in $W_E$). Also, we have $AI(\rho) = AI(\rho^\theta)$.

We apply $AI = AI_{E/\mathbb{Q}_2}$ to the supercuspidal representations of $GL(2, E)$ listed in Table 1. It follows from this table that

$$AI(\rho_{1a}) = AI(\rho_{1b}), \quad AI(\rho_{2a}) = AI(\rho_{3a}), \quad AI(\rho_{2b}) = AI(\rho_{3b}), \quad (7)$$

and these are supercuspidal representations of $GL(4, \mathbb{Q}_2)$. They all have trivial central character. By the conductor formula for induced representations of the Weil group, see (a2) in §10 of [33], they have conductor 4. It follows that the representations in (7) are precisely the three depth zero supercuspidals of $GL(4, \mathbb{Q}_2)$ with trivial central character; see Sect. 2.2.

We will next determine which of the three supercuspidals in (7) are transfers from $GSp(4)$. An irreducible, admissible representation $\pi$ of $GL(4, F)$ is a transfer from $GSp(4, F)$ if and only if its parameter $\phi_\pi : W_F^\vee \to GL(4, \mathbb{C})$, after suitable conjugation, has image in $GSp(4, \mathbb{C})$. Assume this is the case, and consider the exterior square map $\wedge^2 : GL(4, \mathbb{C}) \to GL(6, \mathbb{C})$. Since the composition of $\wedge^2$ with the inclusion $GSp(4, \mathbb{C}) \hookrightarrow GL(4, \mathbb{C})$ decomposes as the direct sum of a five-dimensional and a one-dimensional representation of $GSp(4, \mathbb{C})$, it follows that $\wedge^2 \circ \phi_\pi$ contains a one-dimensional representation of $W_F$.

The following lemma was spelled out in a preprint version of [10] but not in the published version. We include a proof here.

#### 2.4.1 Lemma. Let $E/F$ be a quadratic extension. Let $\theta$ be an element of $W_F$ that is not in $W_E$. Let $(\phi, V)$ be an irreducible two-dimensional representation of $W_E$, and let $\phi^\theta(w) = \phi(\theta w \theta^{-1})$ for $w \in W_E$. Then

$$\wedge^2(\text{ind}_{W_E}^{W_F}(\phi)) = U \oplus \text{ind}_{W_E}^{W_F}(\det(\phi)),$$

where $U$ is a 4-dimensional representation of $W_F$ whose restriction to $W_E$ is isomorphic to $\phi \otimes \phi^\theta$.  

Proof. As a model for \( \phi := \text{ind}_{\mathcal{W}_E}^L (\phi) \), we may take \( V \oplus V \), with action
\[
\phi(w)(v_1 \oplus v_2) = \phi(w)v_1 \oplus \phi (w) v_2 \quad (w \in \mathcal{W}_E), \quad \phi(\theta)(v_1 \oplus v_2) = v_2 \oplus \phi (\theta^2)v_1.
\] (8)

If spaces \( V_1 \) and \( V_2 \) carry an action of a group \( G \), then
\[
\bigwedge^2 (V_1 \oplus V_2) \cong \bigwedge^2 V_1 \oplus (V_1 \otimes V_2) \oplus \bigwedge^2 V_2
\]
as \( G \)-spaces. It follows that, as a \( \mathcal{W}_E \)-representation,
\[
\bigwedge^2 (\text{ind}_{\mathcal{W}_E}^L (\phi)) = \text{det}(\phi) \otimes (\phi \otimes \phi^\theta) \oplus \text{det}(\phi)^\theta,
\]
It is easy to see that \( \text{det}(\phi) \otimes \text{det}(\phi)^\theta \) is invariant under the action of \( \theta \), and that in fact this two-dimensional space is isomorphic to \( \text{ind}_{\mathcal{W}_E}^L (\text{det}(\phi)) \) as a \( \mathcal{W}_E \)-representation. The space \( U \) realizing \( \phi \otimes \phi^\theta \) is also invariant under \( \theta \).

2.4.2 Lemma. The representations \( A_{I E/Q_2}(\rho_{2a}) \) and \( A_{I E/Q_2}(\rho_{2b}) \) appearing in (7) are not transfers from \( \text{GSp}(4,F) \).

Proof. Let \( \rho = \rho_{2a} \) or \( \rho_{2b} \). Let \( \phi : \mathcal{W}_E \to \text{GL}(2, \mathbb{C}) \) be the parameter of \( \rho \). Then the parameter of \( A_{I E/Q_2}(\rho) \) is \( \text{ind}_{\mathcal{W}_E}^{W_{Q_2}} (\phi) \). By Lemma 2.4.1,
\[
\bigwedge^2 (\text{ind}_{\mathcal{W}_E}^{W_{Q_2}} (\phi)) = U \oplus \text{ind}_{\mathcal{W}_E}^{W_{Q_2}} (\text{det}(\phi)),
\]
where \( U \) is isomorphic to \( \phi \otimes \phi^\theta \) as a \( \mathcal{W}_E \)-representation. By Lemma 2.3.1 ii), the space \( U \) is irreducible, even as a \( \mathcal{W}_E \)-representation. Since \( \text{det}(\phi) = \omega_2 \) is not \( \text{Gal}(E/Q_2) \)-invariant, the two-dimensional \( \text{ind}_{\mathcal{W}_E}^{W_{Q_2}} (\text{det}(\phi)) \) is irreducible as a \( W_{Q_2} \)-representation. Hence \( \bigwedge^2 (\text{ind}_{\mathcal{W}_E}^{W_{Q_2}} (\phi)) \) does not contain any one-dimensional component. By our remarks above, \( A_{I E/Q_2}(\rho) \) cannot be a transfer from \( \text{GSp}(4,F) \).

2.4.3 Theorem. The group \( \text{GSp}(4, \mathbb{Q}_2) \) admits a unique generic supercuspidal representation \( \text{sc}(16) \) with conductor \( a(\text{sc}(16)) = 4 \) and trivial central character. As a four-dimensional representation of \( W_{Q_2} \), the Langlands parameter of \( \text{sc}(16) \) is
\[
\phi_{\text{sc}(16)} = \text{ind}_{\mathcal{W}_L}^{W_{Q_2}} (\xi),
\]
where \( L \) is the unramified extension of \( \mathbb{Q}_2 \) of degree 4, and \( \xi \) is any character of \( L^\times \) with the following properties: \( \xi \) is trivial on \( 1 + p_L \); the values of the restriction of \( \xi \) to \( \phi_L^\times \) are the fifth roots of unity; \( \xi(2) = -1 \). We have \( \varepsilon(1/2, \text{sc}(16), \psi) = -1 \), where \( \psi \) is a character of \( \mathbb{Q}_2 \) which is trivial on \( \mathbb{Z}_2 \) but not on \( 2^{-1}\mathbb{Z}_2 \).

Proof. Let \( \pi \) be a generic supercuspidal representation of \( \text{GSp}(4, \mathbb{Q}_2) \) with \( a(\pi) = 4 \) and trivial central character. The requirement that \( \pi \) be generic excludes supercuspidals of type \( Va^* \) and \( XIa^* \); these are the ones with parameters of type \( (A_2) \) and \( (A_3) \), as defined in Sect. 2.1. Assume that \( \pi \) has a parameter of type \( (A_1) \); we will obtain a contradiction. Parameters of type \( (A_1) \) are
Table 2: The supercuspidals \( \pi \) of \( \text{GSp}(4, \mathbb{Q}_2) \) with conductor \( a(\pi) \leq 4 \) and trivial central character. The character \( \xi \) is the unique non-trivial, unramified, quadratic character of \( \mathbb{Q}_2^* \). The representation \( \tau_2 \) is the unique supercuspidal of \( \text{GL}(2, \mathbb{Q}_2) \) with trivial central character and conductor 2. The representation \( \tau_3 \) (resp. \( \xi_3 \)) is the unique supercuspidal of \( \text{GL}(2, \mathbb{Q}_2) \) with trivial central character, conductor 3 and root number 1 (resp. \(-1\)). The representation \( \text{sc}(16) \) is the one from Theorem 2.4.3.

<table>
<thead>
<tr>
<th>( a(\pi) )</th>
<th>( \pi )</th>
<th>type</th>
<th>generic</th>
<th>( \varepsilon(1/2, \pi) )</th>
<th>( L(s, \pi)^{-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( \delta^*(\nu^{1/2}\tau_2, \nu^{-1/2}) )</td>
<td>XLa*</td>
<td>no</td>
<td>(-1)</td>
<td>( 1 - q^{2s-1} )</td>
</tr>
<tr>
<td>3</td>
<td>( \delta^*(\nu^{1/2}\tau_2, \nu^{-1/2}) )</td>
<td>XLa*</td>
<td>no</td>
<td>(-1)</td>
<td>( 1 + q^{s-1/2} )</td>
</tr>
<tr>
<td>4</td>
<td>( \delta^*(\nu^{1/2}\tau_3, \nu^{-1/2}) )</td>
<td>XLa*</td>
<td>no</td>
<td>(-1)</td>
<td>( 1 + q^{-s-1/2} )</td>
</tr>
<tr>
<td>( \text{sc}(16) )</td>
<td>yes</td>
<td></td>
<td></td>
<td>(-1)</td>
<td>1</td>
</tr>
</tbody>
</table>

of the form \( \phi_1 \oplus \phi_2 \), where \( \phi_1, \phi_2 \) are inequivalent irreducible, two-dimensional representations of \( W_F \) with \( \det(\phi_1) = \det(\phi_2) = 1 \). Since \( a(\pi) = 4 \), we must have \( a(\phi_1) = a(\phi_2) = 2 \). Hence \( \phi_1 \) and \( \phi_2 \) correspond to supercuspidals of \( \text{GL}(2, \mathbb{Q}_2) \) with conductor 2 and trivial central character. By (3), there exists only one such supercuspidal. Hence \( \phi_1 \cong \phi_2 \), a contradiction.

By our considerations in Sect. 2.1, the parameter of \( \pi \) is of type \((B_1)\), i.e., irreducible as a four-dimensional representation. Hence \( \pi \) transfers to a supercuspidal representation \( \pi' \) on \( \text{GL}(4, \mathbb{Q}_2) \) with trivial central character and \( a(\pi') = 4 \). It follows that \( \pi' \) is one of the representations in (7). By Lemma 2.4.2 we must have \( \pi' = AI_{E/\mathbb{Q}_2}(\rho_{1a}) = AI_{E/\mathbb{Q}_2}(\rho_{1b}) \), where \( E \) is the unramified quadratic extension of \( \mathbb{Q}_2 \). This shows that, as a four-dimensional representation, the parameter of \( \pi \) is

\[
\text{ind}_{W_E^2}(\phi_a) = \text{ind}_{W_E^2}(\phi_b),
\]

where \( \phi_a \) is the parameter of \( \rho_{1a} \). By the considerations on p. 284/285 of [29], there exists a unique symplectic structure on the space of \( \text{ind}_{W_E^2}(\phi_a) \) for which \( W_Q \) acts with trivial similitude. We proved that the parameter of \( \pi \) is uniquely determined. The uniqueness and existence of \( \pi \) now follows from the local Langlands correspondence for \( \text{GSp}(4, \mathbb{Q}_2) \).

Let \( \eta_j \) be as in Table 1. Then \( \eta_3, \eta_6, \eta_9, \eta_{12} \) are precisely the characters \( \xi \) of \( L^X \) with \( \xi(2) = -1 \), trivial on \( 1 + \mathfrak{p}_L \), and such that the values of the restriction of \( \xi \) to \( \mathcal{O}_F^* \) are the fifth roots of unity. Inducing \( \eta_3 \) or \( \eta_{12} \) to \( W_E \) gives the parameter \( \phi_a \) of \( \rho_{1a} \), and inducing \( \eta_6 \) or \( \eta_9 \) to \( W_E \) gives the parameter \( \phi_b \) of \( \rho_{1b} \). Hence (9) follows by transitivity of induction.

We have \( \varepsilon(1/2, \pi, \psi) = \varepsilon(1/2, \xi, \psi_L) \) by Corollary 4 to Theorem 5.6 of [16], or by \( \epsilon(2) \) in §11 of [33]. Hence the assertion about \( \varepsilon(1/2, \pi, \psi) \) follows from Lemma 2.3.2.
2.4.4 Corollary. Table 2 contains a complete list of all the irreducible, admissible, supercuspidal representations $\pi$ of $GSp(4, \mathbb{Q}_2)$ with trivial central character and conductor $a(\pi) \leq 4$.

Proof. Let $\pi$ be an irreducible, admissible, supercuspidal representations of $GSp(4, \mathbb{Q}_2)$ with trivial central character and conductor $a(\pi) \leq 4$. Assume first that $\pi$ is generic. Then $\pi$ cannot be of type $V_a^*$ or $X_a^*$. Equivalently, the Langlands parameter $\phi$ of $\pi$ cannot be of type $(A_2)$ or $(A_3)$. Assume that $\phi$ is of type $(A_1)$, so that $\phi = \phi_1 \oplus \phi_2$ with irreducible, two-dimensional, inequivalent representations $\phi_1, \phi_2$ of $W_{\mathbb{Q}_2}$ for which $det(\phi_1) = det(\phi_2) = 1$. Since $a(\phi_1), a(\phi_2) \geq 2$ and $a(\phi) \leq 4$, we have $a(\phi_1) = a(\phi_2) = 2$ and $a(\phi) = 4$. It follows that $\pi$ must be the representation $sc(16)$ of Theorem 2.4.3. But then $\pi$ transfers to a supercuspidal on $GL(4, \mathbb{Q}_2)$, contradicting the reducibility of $\phi$. This contradiction shows that $\phi$ cannot be of type $(A_1)$. Alternatively, one can argue that, by (3), there is only one supercuspidal $\tau_2$ of $GL(2, \mathbb{Q}_2)$ with conductor 2 and trivial central character, contradicting the inequivalence of $\phi_1$ and $\phi_2$.

We proved that a generic supercuspidal $\pi$ of $GSp(4, \mathbb{Q}_2)$ with trivial central character and conductor $a(\pi) \leq 4$ must have a parameter $\phi$ of type $(B_1)$. Hence $\pi$ transfers to a supercuspidal on $GL(4, \mathbb{Q}_2)$ and must have $a(\pi) = 4$. Thus $\pi$ is the representation $sc(16)$ of Theorem 2.4.3.

Next assume that $\pi$ is a non-generic supercuspidal of $GSp(4, \mathbb{Q}_2)$ with trivial central character and conductor $a(\pi) \leq 4$. Then $\pi$ must have a parameter $\phi$ of type $(A)$. Since $a(\phi) \leq 4$, the argument above shows that $\phi$ cannot be of type $(A_1)$, so that $\phi$ is of type $(A_2)$ or $(A_3)$.

Assume that $\phi$ is of type $(A_3)$. By definition, $\phi = \phi_1 \oplus \phi_2$, where $\phi_1, \phi_2$ are reducible but indecomposable, inequivalent, and satisfy $det(\phi_1) = det(\phi_2) = 1$. Hence $\phi_i$ is the parameter of $\sigma_i St_{GL(2)}$ for distinct quadratic characters $\sigma_1, \sigma_2$ of $\mathbb{Q}_2^\times$. The restrictions on the conductors imply that $\sigma_1$ and $\sigma_2$ must both be unramified; see the proposition in §10 of [33]. Hence one of $\sigma_1, \sigma_2$ is trivial, and the other is the unique non-trivial, unramified, quadratic character $\xi$ of $\mathbb{Q}_2^\times$.

(This $\xi$ is given by the local Hilbert symbol $\langle \cdot, 5 \rangle$.) The corresponding $\pi$ is the representation $\delta^*(\xi, v\xi, \nu^{-1/2})$ of type $Va^*$.

Assume that $\phi$ is of type $(A_2)$. By definition, $\phi = \phi_1 \oplus \phi_2$, where $\phi_1$ is irreducible and $\phi_2$ is the parameter of $\sigma St_{GL(2)}$ for some character $\sigma$ of $\mathbb{Q}_2^\times$. Moreover $\det(\phi_1) = \det(\phi_2) = 1$. Since $a(\phi) \leq 4$, the character $\sigma$ must be unramified, so that either $\sigma = 1$ or $\sigma = \xi$. In both cases $a(\phi_2) = 1$, which implies $a(\phi_1) \in \{2, 3\}$. There is only one possible $\phi_1$ with $a(\phi_1) = 2$, namely the parameter of $\tau_2$, the unique supercuspidal of $GL(2, \mathbb{Q}_2)$ with trivial central character and conductor 2; see (3). From this $\phi_1$ we therefore obtain two supercuspidals $\pi$ with $a(\pi) = 3$. Using the notation of [32], these are the representations $\delta^*(\nu^{1/2}\tau_2, \nu^{-1/2})$ and $\delta^*(\nu^{1/2}\tau_2, \xi\nu^{-1/2})$ of type $X_a^*$.

Finally, consider the case $a(\phi_1) = 3$. By Theorem 3.9 of [37], there are exactly two possibilities for $\phi_1$. One corresponds to a supercuspidal representation $\tau_3$ of $GL(2, \mathbb{Q}_2)$ with trivial central character, $a(\tau_3) = 3$ and $\varepsilon(1/2, \tau_3) = 1$. The other corresponds to the twist $\xi\tau_3$, which is distinguished from $\tau_3$ by the value of the $\varepsilon$-factor $\varepsilon(1/2, \xi\tau_3) = -1$. The two possibilities of $\phi_1$, together with the two possibilities for $\sigma$, lead to four supercuspidals $\pi$ of type $X_a^*$.

For the non-generic representations, the values of the $L$- and $\varepsilon$-factors in Table 2 can be read off Tables A.8 and A.9 of [31]. Note that $Va^*$ has the same factors as $Va$, since they constitute a two-element $L$-packet; similarly for $X_a$ and $X_a^*$. The $\varepsilon$-factor for $sc(16)$ is given in Theorem 2.4.3. The $L$-factor for $sc(16)$ is 1, since the parameter of $sc(16)$ is irreducible.
2.5 The representation sc(16) via compact induction

We give an alternative construction of the supercuspidal representation sc(16) by employing compact induction. Consider the Langlands parameter $\phi_{sc}(16)$ of sc(16) given in (9). After choosing a suitable basis of $\text{ind}_{W_{Q_2}}^{W(Q_2)}(\xi)$ we may think of $\phi_{sc}(16)$ as a map $W_{Q_2} \to \text{GSp}(4, \mathbb{C})$. The image lies in fact in $\text{Sp}(4, \mathbb{C})$, the dual group of $G = \text{SO}(5) \cong \text{PGSp}(4)$, so that, if we wish, we may work in a semisimple context.

In this section we consider the Vogan L-packet of $\phi_{sc}(16)$. Recall that a Vogan L-packet may contain representations across all pure inner forms of a group; see [38] or the overview in Sect. 3 of [15]. As explained in Sect. 8 of [14], the split group $\text{SO}(2n + 1)$ has a unique non-split pure inner form $\text{SO}^*(2n + 1)$. We will see that the L-packet of $\phi_{sc}(16)$ has two elements, one being a representation of $\text{SO}(5, \mathbb{Q}_2) \cong \text{PGSp}(4, \mathbb{Q}_2)$ (this is our $\text{sc}(16)$), the other one a representation of $\text{SO}^*(5, \mathbb{Q}_2)$.

The parameter $\phi_{sc}(16) : W_{Q_2} \to \text{Sp}(4, \mathbb{C})$ is discrete in the sense that its image has finite centralizer. It is tame in the sense that the image of wild inertia is trivial; this is because the character $\xi : L^\times \to \mathbb{C}^\times$ is trivial on $1 + p_L$. Moreover, $\phi_{sc}(16)$ is in general position, meaning the image of tame inertia is generated by a regular, semisimple element. Hence $\phi_{sc}(16)$ is among the Langlands parameters considered in [5]. The construction in [5] attaches a Vogan L-packet to each tame, discrete Langlands parameter in general position. In the context of $\text{GSp}(4)$, the paper [23] assures that the packets thus obtained coincide with the L-packets defined in [10] and [11].

The centralizer $C_\phi$ of the image of $\phi_{sc}(16) : W_{Q_2} \to \text{Sp}(4, \mathbb{C})$ is precisely the center $\pm I_4$ of $\text{Sp}(4, \mathbb{C})$. The work [5] attaches to each irreducible character $\rho$ of $C_\phi$ a depth-zero supercuspidal representation on a pure inner form of the group under consideration. In our case, going through the definitions shows that the trivial character of $C_\phi$ gives rise to a representation of $\text{SO}(5, \mathbb{Q}_2)$, and the non-trivial character to a representation of $\text{SO}^*(5, \mathbb{Q}_2)$. We will concentrate on the former, since (by [23]) this is our supercuspidal $\text{sc}(16)$.

As explained in Sect. 4.4 of [5], each irreducible character $\rho$ of $C_\phi$ gives rise to an orbit of vertices in the Bruhat-Tits building of $G = \text{PGSp}(4)$ over $\mathbb{Q}_2$. By Lemma 6.2.1 of [5], these vertices are hyperspecial if and only if $\rho$ is trivial. It is exactly the hyperspecial vertices that lead to generic depth-zero supercuspidals, consistent with the fact that $\text{sc}(16)$ is generic.

We may work with the hyperspecial vertex $x_0$ whose associated parahoric subgroup is $p(K)$, where $K = \text{GSp}(4, \mathbb{Z}_2)$ and $p : \text{GSp}(4, \mathbb{Q}_2) \to G(\mathbb{Q}_2)$ is the projection. Let $G_0$ be the reductive group over the residue class field $f = \mathbb{F}_2$ attached to $x_0$, so that $G_0(f) \cong p(K)/p(K)^+$, where $p(K)^+$ is the pro-unipotent radical of $p(K)$. In our case $p(K)^+$ is a principal congruence subgroup, and $G_0 = \text{Sp}(4)$. The construction of $\text{sc}(16)$ is then as follows. The parameter $\phi_{sc}(16)$ determines an $f$-minisotropic maximal torus $T_0$ in $G_0$. The restriction of $\phi_{sc}(16)$ to tame inertia defines a character $\theta$ of $T_0(f)$ via the tame local Langlands correspondence for tori. Since $\phi_{sc}(16)$ is in general position, the character $\theta$ will be in general position in the sense of Definition 5.15.
3 PARAMODULAR CUSP FORMS OF WEIGHT $k \leq 14$ AND LEVEL $N = 16$

of [6]. Deligne-Lusztig induction therefore yields an irreducible, cuspidal character

$$\kappa_0 = \pm R_{T, \theta}$$

of $G_0(f) \cong \text{Sp}(4, f)$. Let $\kappa$ be the inflation of $\kappa_0$ to $p(K)$ via $G_0(f) \cong p(K)^\perp$. Then

$$\text{sc}(16) = c\text{-Ind}_{p(K)}^G(\kappa),$$

where we identify representations of $G(Q_2)$ with representations of $\text{GSp}(4, Q_2)$ with trivial central character. Alternatively, we can first pull back $\kappa$ to a character of $K$, extend it trivially to $Z_K$, where $Z$ is the center of $\text{GSp}(4, Q_2)$, and compactly induce to $\text{GSp}(4, Q_2)$. By Proposition 6.6 of [25], the induced representation in (12) is irreducible and supercuspidal.

Making things explicit, one finds that $T_0$ is the maximal torus corresponding to the conjugacy class consisting of length 2 elements in the 8-element Weyl group of $G_0$; see Sect. 3.3 of [4] for the correspondence between conjugacy classes in the Weyl group and maximal tori. The group $T_0(f)$ is cyclic of order 5. The characters $\theta$ of $T_0(f)$ in general position are precisely the isomorphisms of this group with the fifth roots of unity. By Corollary 7.2 of [6], the character $\kappa_0$ in (11) has degree 9.

It is an exercise in elementary character theory to show that $\text{Sp}(4, f)$ has exactly one irreducible, cuspidal representation $\kappa_0$ of dimension 9, and that this representation is generic; see [8] for information on the characters of $\text{Sp}(4, F_2^n)$. This $\kappa_0$ corresponds to the irreducible character with Young diagram

$$\begin{array}{c|c|c|c}
\hline
& * & * & * \\
\hline
& * & * & * \\
\hline
& * & * & * \\
\hline
\end{array}$$

under the isomorphism of $\text{Sp}(4, f)$ with the symmetric group $S_6$ described in Sect. 3.5.2 of [39]. There is in fact only one other irreducible, cuspidal character of $\text{Sp}(4, f)$, namely the one-dimensional sign character under the isomorphism $\text{Sp}(4, f) \cong S_6$.

To summarize, $\text{sc}(16)$ is a depth-zero supercuspidal representation of $\text{GSp}(4, Q_2)$ which may be constructed as follows. Take the unique irreducible, cuspidal character $\kappa_0$ of $\text{Sp}(4, f)$ that is not one-dimensional; it has dimension 9 and is generic. Inflate $\kappa_0$ to a representation $\kappa$ of $K = \text{GSp}(4, Z_2)$ and extend it to $ZK$ by making it trivial on the center $Z$ of $\text{GSp}(4, Q_2)$. Then we have $\text{sc}(16) = c\text{-Ind}_{ZK}^{\text{GSp}(4, Q_2)}(\kappa)$. The Vogan $L$-packet of $\text{sc}(16)$ contains an additional representation which lives on the non-split inner form of $\text{GSp}(4)$.

3 Paramodular cusp forms of weight $k \leq 14$ and level $N = 16$

A good reference for the notation in this section and hereafter is [26]. For each $N \in \mathbb{N}$, the paramodular group, $K(N)$, and its normalizing Fricke involution, $\mu_N$, are defined by

$$K(N) = \left[\begin{array}{cccc}
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
N^* & N^* & N^* & * \\
\end{array}\right] \cap \text{Sp}(4, Q), \ * \in \mathbb{Z}; \ \ \mu_N = \frac{1}{\sqrt{N}} \left[\begin{array}{ccc}
0 & -N & 0 \\
1 & 0 & 0 \\
0 & 0 & -1 \\
0 & 0 & N \\
\end{array}\right].$$

Let $S_k(K(N))^\epsilon$ for $\epsilon = \pm$ denote the Fricke eigenspace of $S_k(K(N))$ with eigenvalue $\pm 1$, so that we have the decomposition $S_k(K(N)) = S_k(K(N))^+ \oplus S_k(K(N))^-$.

Let $S_k(K(N))^\epsilon$ for $\epsilon = \pm$ denote the Fricke eigenspace of $S_k(K(N))$ with eigenvalue $\pm 1$, so that we have the decomposition $S_k(K(N)) = S_k(K(N))^+ \oplus S_k(K(N))^-$.
a power of a prime, the Fricke sign is also the Atkin-Lehner sign at that prime. The Gritsenko
lift is an injective linear map from \( J_{k,N}^{\text{cusp}} \) to \( S_k(K(N))' \) for \( \epsilon = (-1)^k \). Paramodular forms that
are not Gritsenko lifts will be called nonlifts.

We are searching for a supercuspidal paramodular form, i.e., a newform \( f \in S_k(K(N)) \) whose
associated adelic representation has a supercuspidal local component. Since non-generic super-
cuspidals do not admit non-zero paramodular vectors by Theorem 3.4.3 of [31], a supercuspidal
coming from a paramodular newform \( f \) is necessarily generic. In particular, \( f \) must be a nonlift.
By Table 2, among 2-powers, the smallest \( N \) for which \( f \) can be supercuspidal is \( N = 16 \). By
Corollary 7.5.5 of [31], the value of the \( \varepsilon \)-factor at 1/2 of an irreducible, admissible, generic
representation coincides with the eigenvalue of the Atkin-Lehner involution on the newform. It
therefore follows from Table 2 that if \( S_k(K(16)) \) contains a supercuspidal form, it must occur
in \( S_k(K(16))^- \). Hence, we pay special attention to these spaces.

Our first goal is to find all the nonlift newforms in \( S_k(K(16))^\pm \) for \( k \leq 14 \). In order to
separate the nonlift newforms from the nonlift oldforms, we also find all the nonlift eigenforms in
\( S_k(K(N)) \) for \( k \leq 14 \) and \( N \in \{1, 2, 4, 8\} \); we separate these eigenforms into their Fricke
eigenspaces as well. The dimensions of \( S_k(K(N)) \) are known for \( N \in \{1, 2, 4\} \), see [19, 17, 28].
Comparing with the known [36] dimensions of Jacobi cusp forms \( J_{k,N}^{\text{cusp}} \), we see that \( S_k(K(N)) \)
for \( N \in \{1, 2\} \) and \( k \leq 14 \) does not have any nonlifts. Thus we need only consider \( N \in \{4, 8, 16\} \)
in this section. Our first task is to compute the dimension of each of these spaces, and this will
entail finding upper and lower bounds that are equal.

### 3.1 Paramodular forms and Fourier expansions

A paramodular form \( f \in S_k(K(N)) \) has a Fourier expansion
\[
\sum_t a(t; f)e(\langle \Omega, t \rangle)
\]
where the sum is over \( t \in \mathcal{X}_2(N) = \{ \left[ \begin{array}{rr} n & r/2 \\ r/2 & Nm \end{array} \right] > 0 : n, r, m \in \mathbb{Z} \} \), and where \( \langle \Omega, t \rangle = \text{tr}(\Omega t) \).
The similarity group \( \{ u \in \text{GL}(2, \mathbb{R}) : \left[ \begin{array}{rr} a & b \\ c & d \end{array} \right] \in K(N) \} \) equals \( \hat{\Gamma}^0(N) = \langle \Gamma^0(N), [1 \, 0 \, -1] \rangle \), where, as usual, \( \Gamma^0(N) = \{ \left[ \begin{array}{rr} a & b \\ c & d \end{array} \right] \in \text{SL}(2, \mathbb{Z}) : b \equiv 0 \mod N \} \), and hence the Fourier coefficients satisfy the
following relations amongst themselves, for \( t[u] = t'u'u \),
\[
a(t[u]; f) = \det(u)^k a(t; f), \quad \text{for all } u \in \hat{\Gamma}^0(N).
\]
(14)

Another set of important relations among the Fourier coefficients comes from the Fricke involution \( \mu_N \); we have \( a(t; f)|\mu_N| = a(T(t); f) \) for
\[
t = \left[ \begin{array}{rr} n & r/2 \\ r/2 & Nm \end{array} \right], \quad T(t) = \left[ \begin{array}{rr} m & -r/2 \\ -r/2 & Nn \end{array} \right],
\]
(15)
so that \( t \mapsto T(t) \) gives the action of \( \mu_N \) on the Fourier coefficients. Therefore Fricke eigen-
forms obey the additional conditions
\[
a(T(t); f) = \epsilon a(t; f), \quad \text{for } f \in S_k(K(N))'.
\]
(16)

Note that twinning stabilizes \( \mathcal{X}_2(N) \) and respects \( \hat{\Gamma}^0(N) \)-classes. These observations follow
from the equation \( T(t) = \hat{F}_N t^t F_N \), for \( F_N = \left[ \begin{array}{rr} 1 & N1 \\ N & -1 \end{array} \right] \), the elliptic Fricke involution
on \( \Gamma_0(N) \). We may view the Fourier expansion as a map \( FE : S_k(K(N)) \to \prod_{t \in \mathcal{X}_2(N)} \mathbb{C} \) that
sends \( f \mapsto (a(t; f))_{t \in \mathcal{X}_2(N)} \). Relations (14) and (16) above show that the image of \( S_k(K(N))' \)
under \( FE \) lies in a very special subspace.
For a ring \( R \subseteq \mathbb{C} \), we define \( S_k(K(N))(R) \subseteq S_k(K(N)) \) to be the \( R \)-module of paramodular cusp forms \( f \in S_k(K(N)) \) with \( a(t; f) \in R \) for all \( t \in \mathcal{X}_2(N) \). Fundamental results of Shimura [35] show that general spaces of modular forms have integral bases, i.e., a basis with integral Fourier coefficients.

The natural reduction map \( \mathbb{R}_p : \mathbb{Z} \to \mathbb{F}_p \) allows us to define modular forms over \( \mathbb{F}_p \), a concept useful for both theory and computations: \( S_k(K(N))(\mathbb{F}_p) = \mathbb{R}_p \circ \text{FE}(S_k(K(N))(\mathbb{Z})) \). Thus paramodular forms over \( \mathbb{F}_p \) are formal series with coefficients in \( \mathbb{F}_p \) and the Fourier expansion map \( \text{FE} : S_k(K(N))(\mathbb{F}_p) \to \prod_{t \in \mathcal{X}_2(N)} \mathbb{F}_p \) is really the identity map. From the existence of an integral basis, it follows from the structure theorem for finitely generated \( \mathbb{Z} \)-modules that

\[
\dim_{\mathbb{C}} S_k(K(N))^e = \text{rank}_\mathbb{Z} S_k(K(N))^e(\mathbb{Z}) = \dim_{\mathbb{F}_p} S_k(K(N))^e(\mathbb{F}_p).
\]

For odd primes \( p \), we have the direct sum \( S_k(K(N))(\mathbb{F}_p) = S_k(K(N))^+(\mathbb{F}_p) \oplus S_k(K(N))^-(\mathbb{F}_p) \).

### 3.2 Good Hecke operators and their action on Fourier coefficients

A Hecke operator is called **good** when its similitude is prime to the level. For each prime \( q \) not dividing \( N \), we use the good Hecke operator \( T(q) : S_k(K(N)) \to S_k(K(N)) \) defined as follows. Decompose \( K(N) \text{ diag}(1, 1, q, q) K(N) = \bigcup_j K(N) \gamma_j \) into a union of distinct cosets. For \( f \in S_k(K(N)) \), set \( f|T(q) = \sum_j f|\gamma_j \), which is again in \( S_k(K(N)) \). Since \( T(q) \) commutes with the Fricke involution \( \mu_N \), \( T(q) \) also stabilizes \( S_k(K(N))^e \). The action of \( T(q) \) on the Fourier expansion of \( f \) is given by

\[
(a(t; f)T(q)) = a(qt; f) + q^{k-2} a \left( \frac{1}{q} t \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix}; f \right) + q^{k-2} \sum_{j \mod q} a \left( \frac{1}{q} t \begin{bmatrix} 1 & 0 \\ j & q \end{bmatrix}; f \right) + q^{2k-3} a \left( \frac{1}{q} t; f \right).
\]

For \( k \geq 2 \), this equation shows that \( T(q) \) stabilizes \( S_k(K(N))^e(R) \) and is \( R \)-linear for subrings \( R \) of \( \mathbb{C} \). On \( S_k(K(N))^e(\mathbb{F}_p) \), the reduction of \( T(q) \), \( T(q)_p \), is defined by \( (\mathbb{R}_p \circ \text{FE}(f)|T(q)) = \mathbb{R}_p \circ \text{FE}(f|T(q)) \) and also obeys equation (17).

A possible source of confusion is that equation (17) is valid for the **classical** normalization of the slash, setting \( \sigma = \left[ \begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right] \in \text{GSp}(4, \mathbb{R})^+ \) with similitude \( \mu = \mu(\sigma) = \det(\sigma)^{1/2} \),

\[
(f|\sigma)(\Omega) = \mu^{2k-3} \det(C\Omega + D)^{-k} f ((A\Omega + B)(C\Omega + D)^{-1}).
\]

In contrast, representation theory employs the **scalar invariant** slash where the power of the similitude is \( \mu^k \) instead of \( \mu^{2k-3} \). The tension between these normalizations is real because local Euler factors depend only upon the local representation for the scalar invariant action of the Hecke algebra, whereas \( T(q) \) is uniformly defined over \( \mathbb{Z} \) for weights \( k \geq 2 \) only for the classical action. Our concession to this tension is to write the scalar invariant action of the left and the classical action on the right, so that \( f|T(q) = q^{k-3} T(q) f \).
3.3 Fourier-Jacobi expansions, Jacobi forms, and Jacobi Hecke operators

The Fourier expansion of a paramodular cusp form $f \in S_k(K(N))$ may be rearranged to give the Fourier-Jacobi expansion, setting $\Omega = \left[ \frac{\tau}{z}, \frac{\omega}{z} \right] \in \mathcal{H}_2$, and $q = e(\tau)$, $\zeta = e(z)$,

$$f(\Omega) = \sum_{j=1}^{\infty} \phi_j(\tau, z) e(Nj\omega), \quad (18)$$

$$\phi_j(\tau, z) = \sum_{n,r \in \mathbb{Z}: 4nNj > r^2} a\left(\left[ \frac{n}{r/2}, \frac{r/2}{Nj} \right]; f\right) q^n \zeta^r. \quad (19)$$

When we want to indicate the dependence of the $\phi_j$ on $f$ we will write $\phi_j(\tau, z; f)$ instead of $\phi_j(\tau, z)$, or $\phi_j(f)$ instead of $\phi_j$. We recall the definition of a Jacobi form and the following subgroups, for rings $R \subseteq \mathbb{C}$,

$$P_{2,1}(R) = \begin{bmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \cap \text{Sp}(4, R); \quad GP_{2,1}(R) = \begin{bmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \cap \text{GSp}(4, R).$$

A **Jacobi form** $\phi \in J_{k,m}$ of weight $k \in \mathbb{Z}$ and index $m \in \mathbb{Z}_{\geq 0}$ is a holomorphic function $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ such that the associated function $E_m \phi : \mathcal{H}_2 \to \mathbb{C}$ given by $(E_m \phi)(\Omega) = \phi(\tau, z) e(m\omega)$ is invariant under $P_{2,1}(\mathbb{Z})$, and is bounded on domains of the type $\{ \Omega \in \mathcal{H}_2 : \text{Im} \Omega > Y_0 \}$. The boundedness condition is essential and, given the other assumptions, is equivalent to a Fourier expansion for $\phi$ of the form $\phi(\tau, z) = \sum_{n,r \in \mathbb{Z}: n \leq 0, 4nm \geq r^2} c(n, r; \phi) q^n \zeta^r$. For **Jacobi cusp forms** $\phi \in J_{k,m}^{\text{cusp}}$, we require $4mn > r^2$. For a **weakly holomorphic** $\psi \in J_{k,m}^{\text{wh}}$ we drop the boundedness condition and require $n \gg -\infty$. Indices with $4mn \leq r^2$ are called **singular**. Spaces of Jacobi forms have integral bases by [7] and so we may define $J_{k,m}^{\text{cusp}}(R)$ for $R$ a subring of $\mathbb{C}$ or for $\mathbb{F}_p$ as in the case of paramodular forms.

The subgroup $K_{\infty}(N) = P_{2,1}(\mathbb{Q}) \cap K(N)$ stabilizes the Fourier-Jacobi expansion (18) term by term, so that each $\phi_j \in J_{k,Nj}^{\text{cusp}}$ is a Jacobi form and the Fourier coefficients of the $\phi_j$ are

$$c(n, r; \phi_j) = a\left(\left[ \frac{n}{r/2}, \frac{r/2}{Nj} \right]; f\right). \quad (20)$$

The Fourier-Jacobi expansion defines a map, letting $\xi = e(\omega)$,

$$\text{FJ} : S_k(K(N)) \to \bigoplus_{j=1}^{\infty} J_{k,Nj}^{\text{cusp}}, \quad \text{via } f \mapsto \sum_{j=1}^{\infty} \phi_j \xi^{Nj}, \quad (21)$$

where we have identified the sum on the right with the vector $(\phi_j)$.

The infinite direct sum $\bigoplus_{j=1}^{\infty} J_{k,Nj}^{\text{cusp}}$ is an inverse limit with respect to the projection maps

$$\text{proj}_d : \bigoplus_{j=1}^{\infty} J_{k,Nj}^{\text{cusp}} \to \bigoplus_{j=1}^{d} J_{k,Nj}^{\text{cusp}}, \quad \text{for } d \leq u.$$  

The projection onto the first $u$ Fourier-Jacobi coefficients

$$\text{proj}_u \circ \text{FJ} : S_k(K(N))^{\xi} \to \bigoplus_{j=1}^{u} J_{k,Nj}^{\text{cusp}} \quad (22)$$
Table 3: A sufficient number $u_0$ to make projection from $S_k(K(N))^\epsilon$ onto the first $u_0$ Jacobi coefficients injective. An improved number $u_1^+ \cup u_1^-$ is given in the second set.

<table>
<thead>
<tr>
<th>$u_0$</th>
<th>$u_1^+ \cup u_1^-$</th>
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<tbody>
<tr>
<td>$k$</td>
<td>$K(4)$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
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<td>3</td>
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<tr>
<td>14</td>
<td>6</td>
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injects for sufficiently large $u$ and algorithms to find $u_0$ such that the map (22) injects for $u \geq u_0$ may be found in [3]. When $N$ is a prime power for example, $u_0$ is roughly $Nk/5$ and Table 3 displays $u_0$ for $1 \leq k \leq 14$ and $N \in \{4, 8, 16\}$. We write $S_k(K(N))^\epsilon[S_k(K(N))]\epsilon$ for the projection of $S_k(K(N))^\epsilon$ onto its first $u$ Fourier-Jacobi coefficients, i.e.,

$$S_k(K(N))^\epsilon[S_k(K(N))]\epsilon = \text{proj}_u^\infty \circ \text{FJ}(S_k(K(N))^\epsilon).$$

One cannot take an arbitrary sequence of Jacobi forms $\phi_j$ and obtain the Fourier-Jacobi expansion $\sum_{j=1}^\infty \phi_j \xi^{N_j}$ of some paramodular form. Indeed, the Fourier-Jacobi coefficients of a paramodular Fricke eigenform satisfy the following symmetries. Let $f \in S_k(K(N))^\epsilon$ have the Fourier-Jacobi expansion $\sum_{j=1}^\infty \phi_j \xi^{N_j}$. Then

for all $t_1 = \begin{bmatrix} n_1 & r_{1/2}^2 \\ r_{1/2} \sqrt{m_1} \end{bmatrix}$, $t_2 = \begin{bmatrix} n_2 & r_{2/2}^2 \\ r_{2/2} \sqrt{m_2} \end{bmatrix} \in \mathcal{X}_2(N)$, and $u \in \mathcal{F}^0(N)$,

if $t_1[u] = t_2$, then $c(n_1, r_1; \phi_{m_1}) = \det(u)^k c(n_2, r_2; \phi_{m_2})$, (23)

and

for all $t = \begin{bmatrix} n & r^2 \\ r \sqrt{m} \end{bmatrix} \in \mathcal{X}_2(N)$, $c(n, r; \phi_m) = (-1)^k \epsilon c(m, r; \phi_n)$. (24)
Equations (23) and (24) are consequences of equations (14) and (16). We refer to equation (24) as the **involution conditions**. Formal series of Jacobi forms which satisfy equations (23) and (24), and converge in an appropriate sense, are in fact Fourier-Jacobi expansions of paramodular forms, see [18].

Following Gritsenko [12], we present the action of $T(q)$ on the Fourier-Jacobi coefficients of a paramodular cusp form in terms of the Jacobi raising and lowering operators, $V_q$ and $W_q$. The raising operator $V_q : J_{k,m} \to J_{k,mq}$ is defined, for primes $q$, by

$$
(\phi|V_q)(\tau, z) = q^{k-1}\phi(q\tau, qz) + \frac{1}{q} \sum_{\lambda \mod q} \phi\left(\frac{\tau + \lambda}{q}, z\right),
$$

or equivalently by

$$
c(n, r; \phi|V_q) = q^{k-1}c\left(\frac{n}{q}, \frac{r}{q}; \phi\right) + c(qn, r; \phi),
$$

as in [7]. The lowering operators $W_q : J_{k,m} \to J_{k,\frac{m}{q}}$ were introduced in a special case in [21]. Their image is zero when the prime $q$ does not divide $m$. When $q$ divides $m$, we have

$$
(\phi|W_q)(\tau, z) = q^{k-2} \sum_{\lambda \mod q} \phi(q\tau, z + \lambda\tau)e\left(\frac{m}{q}(2\lambda z + \lambda^2\tau)\right) + q^{-2} \sum_{\lambda, \mu \mod q} \phi\left(\frac{\tau + \lambda}{q}, \frac{z + \mu}{q}\right),
$$

or equivalently

$$
c(n, r; \phi|W_q) = c(qn, qr; \phi) + q^{k-2} \sum_{\lambda \mod q} c\left(\frac{n + \lambda r + \frac{m}{q}\lambda^2}{q}, -\frac{r + \frac{2m}{q}\lambda}{q}; \phi\right). \quad (26)
$$

The invariance properties of the raising and lowering operators, i.e., that they send Jacobi forms to Jacobi forms, can be obtained by considering them as the Hecke operators $V_q = K_\infty(N) \text{diag}(q, q, 1, 1) K_\infty(N)$ and $W_q = K_\infty(N) \text{diag}(1, 1, q, q) K_\infty(N)$ for the noncommutative Jacobi Hecke algebra for $K_\infty(N)$ inside $G\text{P}_{2,1}(\mathbb{Q})$, see [12]. The action of $T(q)$ on the Fourier-Jacobi expansion of an $f \in S_k(K(N))$ is given by

$$
\text{FJ}(f) = \sum_{j=1}^{\infty} \phi_{j\xi}^N j; \quad \text{FJ}(f|T(q)) = \sum_{j=1}^{\infty} \left(\phi_{jq}|W_q + q^{k-2}\phi_{jq}|V_q\right) \xi^{Nj},
$$

as can be directly verified by comparing equations (25) and (26) with (17) using (20).

### 3.4 Jacobi restriction and upper bounds

In this section we define the Jacobi restriction spaces $J_\epsilon^u(R)$ for $R$ being $\mathbb{F}_p$ or a subring of $\mathbb{C}$. Jacobi restriction is described in [18, 3] but we cover it here in further detail because the extension of $T(q)$ to $J_\epsilon^u(\mathbb{F}_p)$ in subsection 3.7 is subtle.

By collectively ordering the index sets of the Fourier expansions of $J_{k,Nj}^{\text{cusp}}$ for all $j \in \mathbb{N}$ in some way, we view $\bigoplus_{j=1}^{\infty} J_{k,Nj}^{\text{cusp}}(R) \subseteq R^\infty$. 
3.4.1 Definition. Let \( N, u, D_0 \in \mathbb{N} \), \( k \in \mathbb{Z} \), and \( \epsilon \in \{-1, 1\} \). Let \( R \) be \( \mathbb{F}_p \) or a subring of \( \mathbb{C} \). The \( R \)-module

\[
\mathcal{J}_u^\epsilon(R) \subseteq \bigoplus_{j=1}^u J_{k,N_j}^{\text{cusp}}(R) \subseteq R^\infty
\]

consists of the \( f = \sum_{j=1}^u \mathbf{f}_j \xi^{NJ} \in \bigoplus_{j=1}^u J_{k,N_j}^{\text{cusp}}(R) \) that satisfy the following conditions,

for all \( t_1 = \left[ \begin{array}{c} n_1 \\ r_1/2 N m_1 \end{array} \right] \), \( t_2 = \left[ \begin{array}{c} n_2 \\ r_2/2 N m_2 \end{array} \right] \in \mathcal{X}_2(N) \), and \( U \in \mathcal{F}^0(N) \),

if \( t_1[U] = t_2 \), and \( \det(2t_1), \det(2t_2) \leq D_0 \) and \( m_1, m_2 \leq u \),

then \( c(n_1, r_1; f_{m_1}) = \det(U)^k c(n_2, r_2; f_{m_2}) \), \hspace{1cm} (28)

and

for all \( t = \left[ \begin{array}{c} n \\ r/2 N m \end{array} \right] \in \mathcal{X}_2(N) \), if \( \det(2t) \leq D_0 \) and \( n, m \leq u \),

then \( c(n, r; f_m) = (-1)^k \epsilon c(m, r; f_n) \). \hspace{1cm} (29)

This important construction calls for a number of comments. The defining equations in Definition 3.4.1 are truly elementary, one coordinate in \( R^\infty \) equals \( \pm 1 \) times another, so that \( \mathcal{J}_u^\epsilon(R) \) is defined over the various commutative rings \( R \). The \( R \)-module \( \mathcal{J}_u^\epsilon(R) \) also depends on \( N, k, \) and \( D_0 \) so that \( \mathcal{J}_u^\epsilon(R,N,k,D_0) \) would be more proper, but we suppress \( N, k, \) and \( D_0 \) to lighten the notation somewhat. When no ring is indicated the field of complex numbers is meant, so \( \mathcal{J}_u^\epsilon = \mathcal{J}_u^\epsilon(\mathbb{C}) \). We have written a program, which we call Jacobi restriction, for the cases \( R = \mathbb{Z} \) and \( R = \mathbb{F}_p \). This program accepts input \( (N, k, \epsilon, D_0, u, R) \) and returns initial expansions, out to \( (n, r) \) satisfying \( 4nNj - r^2 \leq D_0 \), of an \( R \)-basis of \( \mathcal{J}_u^\epsilon(R) \). We always choose \( D_0 \) large enough so that elements of \( J_{k,N_j}^{\text{cusp}}(R) \) for \( j \leq u \) are determined by their initial expansions out to \( 4nNj - r^2 \leq D_0 \); thus, the output characterizes a basis of \( \mathcal{J}_u^\epsilon(R) \), and \( \mathcal{J}_u^\epsilon(R) \) is an \( R \)-module of finite rank very amenable to computation. In particular, \( \text{rank}_R \mathcal{J}_u^\epsilon(R) \) is always known. Finally, because the spaces \( J_{k,m}^{\text{cusp}} \) have integral bases, the output for \( R = \mathbb{Z} \) also works for any subring \( R \subseteq \mathbb{C} \).

The next lemma shows that \( \mathcal{J}_u^\epsilon \) is an upper approximation of the space \( S_k(K(N))^\epsilon[u] \).

3.4.2 Lemma. Let \( N, u \in \mathbb{N} \), \( k \in \mathbb{Z} \), and \( \epsilon \in \{-1, 1\} \). We have

\[
\text{proj}_u^\infty \circ \text{FJ} : S_k(K(N))^\epsilon \to S_k(K(N))^\epsilon[u] \subseteq \mathcal{J}_u^\epsilon.
\]

Proof. By equations (23) and (24), the Fourier-Jacobi expansion of an \( f \in S_k(K(N))^\epsilon \) satisfies the conditions in Definition 3.4.1 for all choices of indices. The conditions defining \( \mathcal{J}_u^\epsilon \) are thus a subset of the conditions satisfied by \( (\text{proj}_u^\infty \circ \text{FJ})(f) \).

3.4.3 Corollary. Let \( u \in \mathbb{N} \) be such that \( \text{proj}_u^\infty \circ \text{FJ} : S_k(K(N))^\epsilon \to S_k(K(N))^\epsilon[u] \) injects. Then \( \dim S_k(K(N))^\epsilon \leq \dim \mathcal{J}_u^\epsilon \).
3.5 Jacobi restriction modulo $p$

Jacobi restriction can also be run modulo a prime $p$. As in the appendix of [1], for a subset $H \subseteq \mathbb{C}^\infty$, let $H_p = R_p (H \cap \mathbb{Z}^\infty) \subseteq \mathbb{F}_p^\infty$ denote the reduction of $H \cap \mathbb{Z}^\infty \mod p$. If $H_1, H_2 \subseteq \mathbb{C}^\infty$ are subspaces with integral bases and $L : H_1 \to H_2$ is a linear map whose matrix in these bases is integral, then $L$ also has a reduction, $L_p : H_{1p} \to H_{2p}$, with the defining property that $(L(h))_p = L_p(h_p)$ for $h \in H_1$. To give some examples, for paramodular forms we have $(\text{FE}(S_k(K(N))))_p \subseteq S_k(K(N))(\mathbb{F}_p)$ and for Jacobi forms $(\text{FE}(J_{k,m}^{\text{cusp}}))_p = J_{k,m}^{\text{cusp}}(\mathbb{F}_p)$. The good Hecke operator $T(q) : S_k(K(N))^\epsilon(\mathbb{Z}) \to S_k(K(N))^\epsilon(\mathbb{Z})$ has, for $k \geq 2$, an integral matrix by (17), and so induces a map $T(q)_p : S_k(K(N))^\epsilon(\mathbb{F}_p) \to S_k(K(N))^\epsilon(\mathbb{F}_p)$ given by: $f \mapsto T(q)_p f = g$ means there exists an $f \in S_k(K(N))^\epsilon(\mathbb{Z})$ such that $R_p(\text{FE}(f)) = f$ and $R_p(\text{FE}(f(T(q)))) = g$.

Because spaces of modular forms have integral bases, important information survives the reduction mod $p$. For example, $\dim \mathbb{C} S_k(K(N))^\epsilon[u] = \dim \mathbb{F}_p S_k(K(N))^\epsilon[u]_p \leq \dim J_u^\epsilon$. Hence if $u \geq u_0$, for some basic $u_0$ making $\text{proj}^{u_0} \circ \text{FJ}$ injective, we have $\dim \mathbb{C} S_k(K(N))^\epsilon \leq \dim J_u^\epsilon$ as well. We easily have $J_u^\epsilon \subseteq J_u^\epsilon(\mathbb{F}_p)$ and examples show that the containment can be proper. Noting Lemma 3.4.2, the hope when we run Jacobi restriction is that all the following spaces have the same dimension,

$$S_k(K(N))^\epsilon \xrightarrow{\text{proj}^{u_0} \circ \text{FJ}} S_k(K(N))^\epsilon[u] \xrightarrow{\text{mod } p} S_k(K(N))^\epsilon[u]_p \subseteq J_{u,p}^\epsilon \subseteq J_u^\epsilon(\mathbb{F}_p). \quad (30)$$

When these spaces do have the same dimension we can, in retrospect, regard the computations as having been performed in any one of them; however it is the space $J_u^\epsilon(\mathbb{F}_p)$ that is most amenable to computation, being a finite dimensional $\mathbb{F}_p$-vector space with a known basis. Especially, we can row reduce and compute the smallest $u_1^\epsilon$ for which the projection

$$\text{proj}_u^{u_1^\epsilon} : J_u^\epsilon(\mathbb{F}_p) \to \bigoplus_{j=1}^{u_1^\epsilon} J_{k,Nj}^{\text{cusp}}(\mathbb{F}_p)$$

is injective. For $u = u_0$, Table 3 also gives particular values of $u_1^\epsilon$ with this property for $1 \leq k \leq 14$, $N \in \{4,8,16\}$, $p = 12347$, and various $D_0$. The choice of $D_0$ was 400 for $K(4)$, 800 for $K(8)$ when $k \leq 8$ and 1000 for larger $k$, and 1600 for $K(16)$ when $k \leq 3$ and 2000 for larger $k$. The caption of Table 3, however, instead reports that the projection from $S_k(K(N))^\epsilon[u_1^\epsilon]$ to $S_k(K(N))^\epsilon$ is injective. The injectivity in these cases follows from the proof in section 3.10 that $\dim S_k(K(N))^\epsilon = \dim J_{u_1^\epsilon}^\epsilon(\mathbb{F}_p)$, and so $p$ and $D_0$ are not reported in Table 3.

3.6 Extending $T(q)$ to $J_u^\epsilon(\mathbb{C})$

Our goal in this section is to lift the map $T(q) : S_k(K(N))^\epsilon \to S_k(K(N))^\epsilon$ to another map $\hat{T}(q) : J_u^\epsilon \to J_u^\epsilon$ such that the following diagram commutes

$$\begin{array}{ccc}
J_u^\epsilon & \xrightarrow{\hat{T}(q)} & J_u^\epsilon \\
\downarrow{\text{proj}^{\infty} \circ \text{FJ}} & & \downarrow{\text{proj}^{\infty} \circ \text{FJ}} \\
S_k(K(N))^\epsilon & \xrightarrow{T(q)} & S_k(K(N))^\epsilon
\end{array} \quad (31)$$

Admittedly, this diagram will only be useful for $u$ large enough to make the vertical map injective. We proceed in two steps and need to make certain assumptions about the space $J_u^\epsilon$. Because
we can compute with \( J^\epsilon_u \) it is reasonable to impose needed conditions on \( J^\epsilon_u \) as long as they can be checked in practice. First, define a map

\[
\tilde{T}(q) : \bigoplus_{j=1}^u J_{k,Nj}^{\text{cusp}} \rightarrow \bigoplus_{j=1}^u J_{k,Nj}^{\text{cusp}}
\]

\[
\sum_{j=1}^u \phi_j \xi^{Nj} \mapsto \sum_{j=1}^u \left( q^{k-2} \phi_{j/q} |V_q + \phi_{qj} |W_q \right) \xi^{Nj}.
\]

This definition reflects the computational fact that the operator \( T(q) \) returns shorter Fourier-Jacobi expansions than it receives. Since the above action agrees with equation (27) we have

\[
\text{proj}_{u/q}^u \text{proj}^\infty_{u/q} \text{FJ}(f|T(q)) = (\text{proj}_{u/q}^u \text{FJ}(f)) \mid \tilde{T}(q).
\]

We introduce the notion of one map being relatively stable with respect to another. Let \( \pi : A \rightarrow \pi A \) and \( T : A \rightarrow \pi A \) be maps and \( B \subseteq A \). We say \( T \) is relatively stable on \( B \) with respect to \( \pi \) when \( T(B) \subseteq \pi(B) \). This is equivalent to saying that \( T : A \rightarrow \pi A \) extends to a relative map \( T : (A,B) \rightarrow (\pi(A,B),\pi(B)) \). We will require that \( \tilde{T}(q) \) be relatively stable on \( J^\epsilon_u \) with respect to \( \text{proj}_{u/q}^u \). When \( J^\epsilon_u \) has successfully been computed, we will need to check whether or not \( \tilde{T}(q) : J^\epsilon_u \rightarrow \text{proj}_{u/q}^u J^\epsilon_u \subseteq \bigoplus_{j=1}^u J_{k,Nj}^{\text{cusp}} \). We will also require that \([u/q] \geq u'_1 \), so that \( \text{proj}_{u/q}^u \) injects on \( J^\epsilon_u \).

### 3.6.1 Proposition. Let \( N, u \in \mathbb{N}, k \in \mathbb{Z}, \text{ and } \epsilon \in \{-1,1\} \). Let \( q \) be a prime with \( q \nmid N \). Assume that

i) \( \tilde{T}(q) \) is relatively stable on \( J^\epsilon_u \) with respect to \( \text{proj}_{u/q}^u \).

ii) The restriction of \( \text{proj}_{u/q}^u \) to \( J^\epsilon_u \) is injective.

Then \( \tilde{T}(q) : J^\epsilon_u \rightarrow J^\epsilon_u \) is well-defined by: \( \hat{f}(\tilde{T}(q) = g \) means \( \hat{f}(\tilde{T}(q) = \text{proj}_{u/q}^u g \). Under these hypotheses, diagram (31) commutes.

**Proof.** Assume that \( f \in J^\epsilon_u \). Because \( \tilde{T}(q) \) is relatively stable there exists a \( g \in J^\epsilon_u \) such that \( \hat{f}(\tilde{T}(q) = \text{proj}_{u/q}^u g \). Because \( \text{proj}_{u/q}^u g \) is injective, this \( g \) is unique, and thus \( \tilde{T}(q) \) is well-defined. The linearity of \( \tilde{T}(q) \) follows from the equation \( \hat{f}(\tilde{T}(q) = \text{proj}_{u/q}^u g \) and the uniqueness of \( g \).

In order to show the commutativity of the diagram we must check

\[
(\text{proj}_{u/q}^u (\text{FJ}(f))) \mid \tilde{T}(q) = (\text{proj}_{u/q}^u \circ \text{FJ}) (f|T(q)),
\]

or, by definition of \( \tilde{T}(q) \), we must show that

\[
(\text{proj}_{u/q}^u (\text{FJ}(f))) \mid \tilde{T}(q) = \text{proj}_{u/q}^u ((\text{proj}_{u/q}^u \circ \text{FJ}) (f|T(q))).
\]

Thus we must check \( \sum_{j=1}^u \phi_j \xi^{Nj} \mid \tilde{T}(q) = \text{proj}_{u/q}^\infty (f|T(q)) \). By equation (27) the right hand side is \( \sum_{j=1}^u \left( q^{k-2} \phi_{j/q} |V_q + \phi_{qj} |W_q \right) \xi^{Nj} \), which is the definition of the left hand side.
3.7 Extending $T(q)_p$ to $J'_a(F_p)$

Our goal in this section is to lift the map $T(q)_p : S_k(K(N))^\epsilon(F_p) \to S_k(K(N))^\epsilon(F_p)$ to a map $\mathcal{T}(q) : J'_a(F_p) \to J'_a(F_p)$ such that the following diagram commutes

$$
\begin{array}{ccc}
J'_a(F_p) & \xrightarrow{T(q)} & J'_a(F_p) \\
\downarrow{\text{proj}_{[u]}^c \circ FJ_p} & & \downarrow{\text{proj}_{[u]}^c \circ FJ_p} \\
S_k(K(N))^\epsilon(F_p) & \xrightarrow{T(q)_p} & S_k(K(N))^\epsilon(F_p). \\
\end{array}
$$

(33)

Recall the definition (32) of the map $\tilde{T}(q)$. By equations (25) and (26), the action of $V_q$ and $W_q$ is integral for $k \geq 2$; so we may consider the reduction of the map $\tilde{T}(q)$ mod $p$:

$$
\tilde{T}(q)_p : \bigoplus_{j=1}^u J^\text{cusp}_{k,N_j}^c(F_p) \to \bigoplus_{j=1}^{|u/q|} J^\text{cusp}_{k,N_j}^c(F_p),
$$

and restrict $\tilde{T}(q)_p$ to $J'_a(F_p) \subseteq \bigoplus_{j=1}^u J^\text{cusp}_{k,N_j}^c(F_p)$ to obtain $\tilde{T}(q)_p : J'_a(F_p) \to \bigoplus_{j=1}^{|u/q|} J^\text{cusp}_{k,N_j}^c(F_p)$. As in the previous section, it is reasonable to impose needed conditions on $J'_a(F_p)$ that are easy to check. We will require that $\tilde{T}(q)_p$ be relatively stable on $J'_a(F_p)$ with respect to $\text{proj}_{[u/q]}^u$. This condition is achieved whenever $S_k(K(N))^\epsilon[u]_p$ actually equals $J'_a(F_p)$, which is what the whole set-up aims to prove, so there is no harm in requiring relative stability. If relative stability fails, we should increase $u$ and try again. We will also require that $\text{proj}_{[u]}^u$ be injective on $J'_a(F_p)$. This second condition is achieved when $|u/q| \geq u_1$, which may be costly.

3.7.1 Proposition. Let $N, u \in \mathbb{N}$, $k \in \mathbb{Z}$, and $\epsilon \in \{-1, 1\}$. Let $p$ and $q$ be primes with $q \nmid N$. Assume

i) $\tilde{T}(q)_p$ is relatively stable on $J'_a(F_p)$ with respect to $\text{proj}_{[u]}^u$.

ii) The restriction of $\text{proj}_{[u]}^u$ to $J'_a(F_p)$ is injective.

Then $T(q) : J'_a(F_p) \to J'_a(F_p)$ is well-defined by: $f|T(q) = g$ means $f|\tilde{T}(q)_p = \text{proj}_{[u]}^u g$. Under these hypotheses, diagram (33) commutes.

Proof. We show that $T(q)$ is well-defined and $F_p$-linear. Take $f \in J'_a(F_p)$. Since $\tilde{T}(q)_p$ is relatively stable on $J'_a(F_p)$ with respect to $\text{proj}_{[u/q]}^u$, there exists a $g \in J'_a(F_p)$ such that $f|\tilde{T}(q)_p = \text{proj}_{[u]}^u g$. If there were another such $g'$, then $g' = g$ because $\text{proj}_{[u]}^u$ is injective on $J'_a(F_p)$. This shows that $T(q)$ is well-defined. Linearity follows from $f|\tilde{T}(q)_p = \text{proj}_{[u]}^u g$ and the uniqueness of $g$.

In order to show the commutativity of the diagram, take $f \in S_k(K(N))^\epsilon(F_p)$. We must show

$$(\text{proj}_{[u]}^u \circ FJ_p(f)) |T(q) = (\text{proj}_{[u]}^u \circ FJ_p) (f|T(q)_p),$$

which, by definition of $T(q)$, means

$$(\text{proj}_{[u]}^u \circ FJ_p(f)) |\tilde{T}(q)_p = \text{proj}_{[u]}^u (\text{proj}_{[u]}^u \circ FJ_p) (f|T(q)_p),$$

and this is true.
or equivalently, 
\[(\text{proj}_{u,p}^\infty(\text{FJ}_p(f))) | \tilde{T}(q)_p = \text{proj}_{[u/q],p}^u \text{FJ}_p(f|T(q)_p). \tag{34}\]

There is an \(f \in S_k(K(N))^i(\mathbb{Z})\) such that \(\tilde{f} = \text{FJ}(f)_p\), so that (34) would follow by reduction from 
\[(\text{proj}_{u}^\infty(\text{FJ}(f)))) | \tilde{T}(q) = \text{proj}_{[u/q]}^u \text{FJ}(f|T(q)). \tag{35}\]

Writing \(\text{FJ}(f) = \sum_{j=1}^\infty \phi_j \xi^{N_j}\), we verify (35) from the definition of \(\tilde{T}(q)\) and equation (27),
\[\left(\sum_{j=1}^u \phi_j \xi^{N_j}\right) | \tilde{T}(q) = \sum_{j=1}^{[u/q]} \left(q^{k-2}\phi_{j/q}|V_q + \phi_{qj}|W_q\right) \xi^{N_j} = \sum_{j=1}^{[u/q]} \phi_j(f|T(q)) \xi^{N_j}. \]

### 3.8 Bootstrapping and lower bounds

We now explain the technique of **bootstrapping**, a combination of Jacobi restriction and Hecke spreading, which computes lower bounds for \(\dim S_k(K(N))^i\). As motivation, we first discuss Borcherds products. The theory of Borcherds products and the theory of Hecke operators bear little relation. A Borcherds product, for example, seems to only be a Hecke eigenform when forced to be by dimensional reasons. In general, if a Borcherds product is written as a linear combination of Hecke eigenforms it seems that the Borcherds product is often supported on every eigenspace with the same Atkin-Lehner signs as the Borcherds product. Thus repeated applications of \(T(q)\) on a Borcherds product are likely to span the entire Atkin-Lehner space of paramodular forms that the Borcherds product belongs to. Over \(\mathbb{Q}\), many iterations of \(T(q)\) on a Borcherds product are much too expensive, but over \(\mathbb{F}_p\) many iterations of \(T(q)\) on \(J^u_u(\mathbb{F}_p)\) are feasible. Let \(S \subseteq S_k(K(N))^i(\mathbb{F}_p)\). Define
\[B_p(S; T(q)) = \text{Span}_{\mathbb{F}_p} \{\tilde{f}|T(q)^i \in J^u_u(\mathbb{F}_p) : i \in \mathbb{Z}_{\geq 0}, \tilde{f} \in S\}.

#### 3.8.1 Lemma. Let \(u\) be large enough so that \(\text{proj}_{u,p}^\infty \circ \text{FJ}_p\) injects on \(S_k(K(N))^i(\mathbb{F}_p)\). Assume the hypotheses of Proposition 3.7.1. Then
\[\dim B_p(S; T(q)) \leq \dim S_k(K(N))^i(\mathbb{F}_p).\]

**Proof.** By the commutative diagram (33), the subspace \(B_p(S; T(q)) \subseteq J^u_u(\mathbb{F}_p)\) is the injective image under \(\text{proj}_{u,p}^\infty \circ \text{FJ}_p\) of the span of \(f|T(q)^i_p \in S_k(K(N))^i(\mathbb{F}_p)\) for \(i \in \mathbb{Z}_{\geq 0}\), and \(f \in S\). \[\]

### 3.9 Specific upper bounds: Jacobi restriction

We use the technique of Jacobi restriction to compute upper bounds for \(\dim S_k(K(N))^i\). Jacobi restriction over \(\mathbb{Q}\) requires a lot of memory. It is better, when sufficient, to run Jacobi restriction modulo \(p\). Table 3 gives \(u_0\) large enough to make projection onto the first \(u_0\) Jacobi coefficients injective. Using the containments in (30), Table 4 reports the resulting upper
Table 4: Dimensions of cusp forms of weight $k$. The signs $+$ and $-$ refer to the paramodular Atkin-Lehner sign, which is the same as the Fricke sign in these cases.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$K(1)$ lifts nonlifts</th>
<th>$K(2)$ lifts nonlifts</th>
<th>$K(4)$ lifts nonlifts</th>
<th>$K(8)$ lifts nonlifts</th>
<th>$K(16)$ lifts nonlifts</th>
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<td>+ -</td>
<td>+ -</td>
<td>+ -</td>
<td>+ -</td>
</tr>
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<td>7 9 3</td>
<td>14 46 14</td>
</tr>
</tbody>
</table>

bound $\dim_S k(K(N))^\ell = \dim_S k(K(N))^\ell[u_0] \leq \dim_J^0(F_p)$ given as output by the Jacobi restriction program, using the same determinant bounds $D_0$ and prime $p$ as in section 3.5. In Table 4 we have further refined these upper bounds to apply to the spaces of nonlifts, which is a direct adjustment because the dimensions of the lift spaces are known by [7]. Because $\dim_S k(K(4))$ is known and the upper bounds for the three subspaces of $S_k(K(4))$ add up to the known total dimension, the dimensions of the subspaces of $S_k(K(4))$ listed in Table 4 are the actual dimensions without further argument. We will prove that the upper bounds of the dimensions of the nonlift subspaces of $S_k(K(8))$ and $S_k(K(16))$ as listed in Table 4 are in fact the true dimensions in all cases. This illustrates the power of Jacobi restriction. The proof involves constructing enough paramodular forms to show these numbers are also lower bounds.

3.10 Specific lower bounds: Borcherds products and bootstrapping

In the previous section we computed the upper bounds for $\dim_S k(K(N))$ given in Table 4. This section will compute matching lower bounds, mainly by constructing Gritsenko lifts and Borcherds products, but also via Hecke operators, and oldform theory. The theory of Borcherds products [2, 13] creates meromorphic paramodular forms, transforming by a character $\chi$ of
products that turn out to be holomorphic and cuspidal with trivial character. There is an
index $N$.

3 PARAMODULAR CUSP FORMS OF WEIGHT $k \leq 14$ AND LEVEL $N = 16$

$K(N)$, in $M_k^{\text{mero}}(K(N))^\epsilon(\chi)$ from weakly holomorphic Jacobi forms $\psi \in J_{0,N}^{\text{wh}}$ of weight zero and index $N$ whose Fourier coefficients are integral on singular indices. We will only use Borcherds products that turn out to be holomorphic and cuspidal with trivial character. There is an algorithm [27] to find all Borcherds products in a given space $S_k(K(N))$, so we simply post the constructions of the Borcherds products that we use here on the website [40]. Given an appropriate $\psi \in J_{0,N}^{\text{wh}}$, we write $\text{Borch}(\psi) \in M_k^{\text{mero}}(K(N))^\epsilon(\chi)$ for the associated Borcherds product. If we write the Fourier expansion of $\psi$ as $\psi(\tau, z) = \sum_{n,r \in \mathbb{Z}} c(n, r) q^n z^r$, then $\text{Borch}(\psi)$ is defined by analytic continuation of the following infinite product for $\Omega = \left[ \frac{\tau}{z} \right] \in \mathcal{H}_2$,

$$\text{Borch}(\psi)(\Omega) = q^A \zeta^B \zeta^C \prod_{(m,n,r) \geq 0} (1 - q^n \zeta^r \zeta^{mN})^{c(mn,r)}.$$

The product is taken over $m, n, r \in \mathbb{Z}$ such that $m \geq 0$, and if $m = 0$ then $n \geq 0$, and if $m = n = 0$ then $r < 0$. Set $N = \{1, 2, 3, \ldots\}$. The exponents $A, B, C$ are given by $2A = \sum_{r < 0} c(0, r)$, $2B = \sum_{r < 0} rc(0, r)$, and $2C = \sum_{r < 0} r^2 c(0, r)$. Borcherds products always come with a Fricke sign. The sign $\epsilon$ is given by $\epsilon = (-1)^{d_\phi}$ where $d_\phi = \sum_{n \in \mathbb{N}} \sigma_0(n) c(-n, 0)$, and $\sigma_0(n)$ is the number of positive divisors of $n$.

Here are our methods for obtaining lower bounds on $\dim S_k(K(N))^\epsilon$. Fix $k, N$, and $\epsilon = \pm 1$. We search for Borcherds products in $S_k(K(N))^\epsilon$. If we find enough to span a space whose dimension equals that of the upper bound, then we are done. If not, we employ the method of bootstrapping from subsection 3.8. We check the hypotheses of Proposition 3.7.1: that $T(3)_p$ is relatively stable on $\mathcal{J}_{u_0}^\epsilon(\mathbb{F}_p)$ with respect to $\text{proj}_{[u_0/3]}^{u_0}$, and that $u_0 \geq 3u_1^\epsilon$ so that $\text{proj}_{[u_0/3]}^{u_0}$ is injective on $\mathcal{J}_{u_0}^\epsilon(\mathbb{F}_p)$. There are three places in Table 3 where $u_0 < 3u_1^\epsilon$, but these occur for $K(4)$ and weight $k \in \{7, 11, 12\}$ where the dimension is already known. Still using the $u_0$ from Table 3, we compute a matrix representation for $T(3)$ on a fixed basis for $\mathcal{J}_{u_0}^\epsilon(\mathbb{F}_p)$. We find a set $S \subseteq S_k(K(N))^\epsilon$ of Borcherds products and take $f \in S$, see [40] for the Borcherds products found. It is feasible to expand a Borcherds product $f$ out far enough to determine $(\text{proj}_{[u_0/3]}^{u_0} \text{FJ}(f))_p$ in this basis. Once we get the coordinates of $(\text{proj}_{[u_0/3]}^{u_0} \text{FJ}(f))_p$ in this basis, it is linear algebra to compute the bootstrapped subspace on $S_p$. Then $u_0 \geq 3u_1^\epsilon$ and Lemma 3.8.1 imply that $\dim B_p(S; T(3)) \leq \dim S_k(K(N))^\epsilon(\mathbb{F}_p)$. It turns out that the dimension of each bootstrapped subspace $B_p(S_p; T(3))$ gives the same lower bound as the upper bound $\dim \mathcal{J}_{u_0}^\epsilon(\mathbb{F}_p)$ in every case in Table 4 except in the single case $S_{14}(K(8))^-$. Thus we know $\dim \mathcal{J}_{u_0}^\epsilon S_k(K(N))^\epsilon[u_0] = \dim_{\mathbb{F}_p} S_k(K(N))^\epsilon[u_0] = \dim_{\mathbb{F}_p} \mathcal{J}_{u_0}^\epsilon(\mathbb{F}_p)$ in all cases in Table 4 except $S_{14}(K(8))^-$. There are no Borcherds products in $S_{14}(K(8))^-$. We now explain the additional argument needed for this exceptional case.

We know that $\dim S_{14}(K(8))^- \leq 3$. We found all the eigenforms in each of $S_{14}(K(N))^\pm$ for $N \in \{1, 2, 4, 8, 16\}$ except $S_{14}(K(8))^-$. We show there is an eigenform in $S_{14}(K(16))^-$ of $T(3)$-eigenvalue $3^{11} \lambda_3 = -1580472$ which is not a $T_{1,0}$-eigenform. The eigenspace of $S_{14}(K(16))^-$ with this $T(3)$-eigenvalue is one dimensional. Lemma 3.10.1 implies that there exists a newform $f_{\text{new}} \in S_{14}(K(2))^j$ for some $j \in \{0, 1, 2, 3\}$ with the same $T(3)$-eigenvalue. Looking at $T(3)$-eigenvalues for the lifts, we see that $f_{\text{new}}$ must be a nonlift. There are no nonlifts in $S_{14}(K(N))$ for $N \in \{1, 2\}$ and there are two nonlift eigenforms in $S_{14}(K(4))$. But the $T(3)$-eigenvalue $-1580472$ does not show as an eigenvalue in $S_{14}(K(8))^+$ or in $S_{14}(K(4))$. We conclude that $f_{\text{new}}$ must be in $S_{14}(K(8))^-$. Together with the two oldforms in $S_{14}(K(8))^-$ coming from the two newforms in $S_{14}(K(4))$, we conclude that $\dim S_{14}(K(8))^- \geq 2 + 1 = 3$. 
3.10.1 Lemma. Let $N$ be a positive integer and $p$ be a prime dividing $N$. Let $W \subset S_k(K(N))$ be a non-zero eigenspace for a Hecke operator $T$ at some good place $q \mid N$. Assume that the operators $T_{0,1}(p)$, $T_{1,0}(p)$ and the Atkin-Lehner $\alpha_p$ are not simultaneously diagonalizable on $W$. Then there exists a new-eigenform $f_{\text{new}} \in S_k(K(M))$ for some $M \mid N$ with $v_p(M) < v_p(N)$ and with the same $T$-eigenvalue as the elements of $W$.

Proof. Since Hecke operators at good places commute, we can find a basis $f_1, \ldots, f_n$ of $W$ consisting of eigenforms for almost all good Hecke operators, including the place $q$.

By Theorem 2.6 i) of [34], the adelization $\Phi_i$ of $f_i$ generates an irreducible, cuspidal, automorphic representation $\pi_i \cong \otimes_s \pi_{i,s}$ of $\mathrm{PGSp}(4, \mathbb{A}_Q)$, for each $i$. The automorphic form $\Phi_i$ corresponds to a sum of pure tensors $\sum_j (\otimes_s w_{i,s,j})$, where $w_{i,s,j}$ is in the space of $\pi_{i,s}$. After averaging, we may assume that $w_{i,s,j}$ is a paramodular vector of level $v_s(N)$, for each prime number $s$. In particular, each $w_{i,q,j}$ is a spherical vector in $\pi_{i,q}$, and hence an eigenvector for the local operator $T_q$ corresponding to $T$, with the same eigenvalue as $T$ on $W$.

We claim that there exists an $i \in \{1, \ldots, n\}$ such that the conductor exponent $a(\pi_{i,p})$ is less than $v_p(N)$. Clearly, we must have $a(\pi_{i,p}) \leq v_p(N)$ for each $i$, since $a(\pi_{i,p})$ is the smallest possible level of any paramodular vector in $\pi_{i,p}$ by Corollary 7.5.5 of [31]. Assume that we would have $a(\pi_{i,p}) = v_p(N)$ for all $i$. Then each $w_{i,p,j}$ would be a local newform in $\pi_{i,p}$, which is unique up to scalars by Theorem 7.5.4 of [31]. In particular, $T_{0,1}(p)$, $T_{1,0}(p)$ and $\alpha_p$ would be simultaneously diagonalizable on $W$, contradicting our hypothesis. This proves our claim that there exists an $i_0 \in \{1, \ldots, n\}$ such that $a(\pi_{i_0,p}) < v_p(N)$.

Let $\Phi_{\text{new}}$ be the automorphic form corresponding to the global holomorphic, paramodular newform in $\pi_{i_0}$. De-adelizing $\Phi_{\text{new}}$, we obtain a Siegel modular form $f_{\text{new}}$ with the desired properties.

We have now proven that Table 4 gives true dimensions and not just upper bounds. Once we know that the dimension of $S_k(K(N))^{\epsilon}$ agrees with our upper bound, we have $J_{u_0}^\epsilon(\mathbb{F}_p) = S_k(K(N))^{\epsilon}[u_0]_p$, and can use the improved $u_1^\epsilon$ in Table 3 for which the projection $\text{proj}^{u_1^\epsilon}_{u_0^\epsilon} : J_{u_0^\epsilon}(\mathbb{F}_p) \to J_{u_1^\epsilon}(\mathbb{F}_p)$ injects. It follows that $\text{proj}^{u_1^\epsilon} : S_k(K(N))^{\epsilon} \to S_k(K(N))^{\epsilon}[u_1^\epsilon]$, which injects. With these improved $u_1^\epsilon$, we run Jacobi restriction over $\mathbb{Q}$ to $u = 3u_1^\epsilon$ Jacobi coefficients and break $S_k(K(N))^{\epsilon}$ into $T(3)$-eigenspaces by verifying the hypotheses of Proposition 3.6.1 and using $\hat{T}(3)$. We stress that we postpone running Jacobi restriction over $\mathbb{Q}$ until we have the improved $u_1^\epsilon$ from Table 3 available for $S_k(K(N))^{\epsilon}$. We are eventually forced to run Jacobi restriction over $\mathbb{Q}$, however, in order to compute Hecke eigenspaces. Once we have $S_k(K(N))^{\epsilon}$ broken into one dimensional eigenspaces, we can revert, if we wish, to using $T(q)$ to compute further good rational eigenvalues inside $J_{u_1^\epsilon}(\mathbb{F}_p)$. The point here is that, for $T(q)f = \lambda_q f$, good eigenvalues have simple archimedean bounds $|\lambda_q| \leq (1 + q)(1 + q^2)$, see [9], page 269, Hilfsatz 4.8, and $q^{k-3}\lambda_q$ is integral for $k \geq 2$. In the next section, however, we are more interested in computing eigenvalues at the bad primes, as a step toward identifying the local representations.

3.11 Nonlift newforms

From Table 4, we can count how many of each dimension of nonlifts are oldforms from lower levels using the global theory of newforms in [30]. Table 5 breaks $S_k(K(16))^{\epsilon}$ into the dimension of newforms and oldforms.
By computing the eigenvalue $\lambda_3$ for all the nonlift eigenforms, we are able to distinguish the newforms from the oldforms. See Table 6 for the eigenvalues of nonlift newforms for $S_k(K(4))$ and $S_k(K(8))$ for $k \leq 14$. Note that there are no nonlifts for $S_k(K(N))$ for $N \in \{1, 2\}$ and $k \leq 14$. The eigenvalues of the nonlift newforms for $S_k(K(16))$ with $k \leq 14$ are in Table 7 along with other eigenvalues. We were able to easily distinguish the newforms because it turns out that these newforms have different $\lambda_3$ eigenvalues than the oldforms of the same level.

### 3.12 Computing $T_{0,1}$ and $T_{1,0}$

The global Hecke operators at the bad primes have their origin in the local theory [30]. The global operators $T_{0,1}(p)$ and $T_{1,0}(p)$ at a bad prime $p$ were defined and studied in [26], where eigenvalues were computed that required information from Fourier expansions at multiple zero dimensional cusps. From Proposition 5.2 of [26], the two bad Hecke operators $T_{0,1}(2)$ and $T_{1,0}(2)$ may be written on $S_k(K(16))$ as

\[
T_{0,1}F = \sum_{x,y,z \in \{0,1\}} F \begin{bmatrix} 1 & 0 & x & y \\ 0 & 1 & y/16 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} + \sum_{x,y \in \{0,1\}} F \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & z/16 \\ 0 & 0 & 1 & -2 \end{bmatrix} + \sum_{x,y \in \{0,1\}} F \begin{bmatrix} 1 & -16y & x & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}
\]

\[
T_{1,0}F = \sum_{x,y \in \{0,1\}} F \begin{bmatrix} 2 & 0 & 0 & 2y \\ 0 & 1 & y - xy + z/16 & 0 \\ 0 & 2 & -2x & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} + \sum_{x,y \in \{0,1\}} F \begin{bmatrix} 1 & -16y & 0 & 0 \\ 0 & 0 & 1 & -3/8 \\ -x/2 & 1 + 8xy & y/2 & 1/32 \\ 0 & 0 & 0 & 16y \end{bmatrix}
\]
Table 6: Eigenvalues $3^{k-3}\lambda_3$ of nonlift newforms. Here $\alpha_{13,8}$ represents the four roots of $151059326544225331200 = 28599118413428736x - 271045699200x^2 + 463392x^3 + x^4$ and $\alpha_{14,8}$ represents the three roots of $70155550286581248 = -1194997748544x + 186408x^2 + x^3$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$K(4)$</th>
<th>$K(8)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>-18360</td>
</tr>
<tr>
<td>11</td>
<td>-13464</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>-88488</td>
<td>-14760</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>-154440</td>
<td>-685224</td>
</tr>
<tr>
<td></td>
<td></td>
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</tr>
<tr>
<td>14</td>
<td>-1422360</td>
<td>-1176984</td>
</tr>
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</tbody>
</table>
The zero dimensional cusps of $K(16)$ are given by the disjoint union, see Theorem 1.3 of [28], $\text{GSp}(4, \mathbb{Q})^+ = K(16) \text{GP}_{2,0}(\mathbb{Q}) \cup K(16) C_0(2) \text{GP}_{2,0}(\mathbb{Q}) \cup K(16) C_0(4) \text{GP}_{2,0}(\mathbb{Q})$, where

$$C_0(m) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad \text{GP}_{2,0}(R) = \begin{bmatrix} * & 0 & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix} \cap \text{GSp}(4, R).$$

The difficulty in computing $T_{0,1}F$ and $T_{1,0}F$ is that although most of the coset representatives defining $T_{0,1}$ and $T_{1,0}$ lie in the first cusp, a few lie in the second. As in [26], we overcome this difficulty by using the technique of restriction to a modular curve to compute the restrictions $F(s\tau + s')$ and $(T_{0,1}F)(s\tau + s')$ for some serviceable choice of $s, s'$. The point is that it is straightforward to compute $(F|u)(s\tau + s')$ when $u \in \text{GP}_{2,0}(\mathbb{Q})$, but a trick is required to compute $(F|C_0(2)u)(s\tau + s')$ for the last coset representative in $T_{0,1}$. The strategy of Section 4.2 in [26] is to access the cusp $K(N)C_0(m)\text{GP}_{2,0}(\mathbb{Q})$ by finding $\sigma = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \text{SL}(2, \mathbb{Z})$ and a positive definite $s_0 \in \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$ such that $\begin{bmatrix} aI & \beta s_0 \\ \gamma s_0 & \delta I \end{bmatrix} \in K(N)C_0(m)W_0$ for some $W_0 \in \text{GP}_{2,0}(\mathbb{Q})$. Setting $W_1 = \begin{bmatrix} A_1 & B_1 \\ 0 & D_1 \end{bmatrix} = u^{-1}W_0$ and $s\tau + s' = W_1(s_0\tau) = (A_1s_0\tau + B_1)D_1^{-1}$, it formally follows that

$$(F|C_0(m)u)(s\tau + s') = \text{det}(A_1D_1)^{-k/2} \text{det}(D_1)^{k/2} (g|_{2k}\sigma)(\tau),$$

for $g(\tau) = F(s_0\tau)$. For $\ell$ with $\ell s_0^{-1} \in \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$, we have $g \in S_{2k}(\Gamma(\ell))$, and we have reduced the problem of specializing $F$ at the $C_0(m)$-cusp to transforming an elliptic modular form.

By choosing $\ell = 16$ and $\sigma, s_0, W_0, s, s'$ as follows,

$$\sigma = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix}, \quad s_0 = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}, \quad W_0 = \begin{bmatrix} -8 & 6 \\ -2 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad s = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad s' = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix},$$

we get that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 & 16 & 7 \\ -3 & -8 & 1 \\ -15/16 & 1/8 & 2 \end{bmatrix} (s\tau + s') = (1/4)^{-k/2}(1)^k (g|_{2k}\sigma)(\tau),$$

where $g(\tau) = F(s_0\tau) \in S_{2k}(\Gamma(16))$. We therefore need to be able to work with cusp forms in $S_{2k}(\Gamma(16))$, namely we need to compute a basis of $S_{2k}(\Gamma(16))$ and the action of $\sigma$ on this basis. We show how to do this in Lemma 3.12.1.

To be able to compute the restrictions $F(s\tau + s')$ and $(T_{0,1}F)(s\tau + s')$, for $F \in S_k(K(16))$ and some choice of $s, s'$, we follow the instructions of Section 4.4 in [26]. For $T_{1,0}$, the delicate issue is simultaneously computing $(F|C_0(2)u)(s\tau + s')$ for the last two coset representatives in $T_{1,0}$. By choosing $\ell = 16$ and $\sigma, s_0, s, s', \tau_0, W_0$ as follows,

$$\sigma = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix}, \quad s_0 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \quad \tau_0 = 1/2,$$

$$s = \begin{bmatrix} 9441370 & -2347216 \\ -2347216 & 4668925/8 \end{bmatrix}, \quad s' = \begin{bmatrix} 3152523 & -313499/4 \\ -313499/4 & 12470225/64 \end{bmatrix}, \quad W_0 = \begin{bmatrix} -24 & 8 & -65 \\ -1055/2 & 176 & -1739 & -14897/16 \\ 0 & 0 & -44 & -1055/8 \end{bmatrix},$$

$$C_0(m) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad \text{GP}_{2,0}(R) = \begin{bmatrix} * & 0 & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix} \cap \text{GSp}(4, R).$$
we get the following, for \( g(\tau) = F(s_0 \tau) \in S_{2k}(\Gamma_0(16)) \),

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
2 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
8 & 32 & 14 & -3 \\
-3 & -16 & 2 & 5/16 \\
0 & 0 & 2 & -3/8 \\
0 & 0 & 4 & -1
\end{pmatrix}
\begin{pmatrix}
s \tau + s' = (\frac{1}{4})^{-k/2}(1)^k(g|_{2k}\sigma)(\tau), \\
s \tau + s' = (\frac{1}{4})^{-k/2}(1)^k(g|_{2k}\sigma)(\tau + \tau_0).
\end{pmatrix}
\]

The last thing we need before using this choice to compute \( T_{0,1}F \) is a knowledge of how forms in \( M_k(\Gamma_0(16)) \) transform by \( \sigma = \lt \frac{3}{8} \frac{1}{4} \gt \). We discuss the ring generators of \( M(\Gamma_0(16)) = \oplus_{k=0}^\infty M_k(\Gamma_0(16)) \). Let

\[
E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^n = 1 - 24q - 72q^2 - 96q^3 - 168q^4 - 144q^5 - \cdots
\]

be the nearly modular weight two Eisenstein series transforming, for all \( \lt a \ b \ c \ d \gt \in \text{SL}(2, \mathbb{Z}) \), by

\[
(E_2|\vartheta \lt a \ b \ c \ d \gt)(\tau) = E_2(\tau) - \frac{3}{\pi^2} \left( \frac{2\pi i c}{c\tau + d} \right).
\]

For \( d > 1 \), we define \( E_{2,d}^- \in M_2(\Gamma_0(d)) \) by \( E_{2,d}^-(\tau) = \frac{1}{\tau^d} (E_2(\tau) - dE_2(d\tau)) \). We define five elements in \( M_2(\Gamma_0(16)) \) by

\[
\begin{align*}
 a(\tau) &= \frac{1}{2}E_{2,2}^-(\tau) - 3E_{2,4}^-\tau + \frac{7}{2}E_{2,8}^-\tau = 1 - 24q^2 + 24q^4 - 96q^6 + 24q^8 - 144q^{10} + \cdots \\
 b(\tau) &= -\frac{1}{48}E_{2,2}^-\tau + \frac{7}{48}E_{2,8}^-\tau - \frac{5}{8}E_{2,16}^-\tau + \frac{1}{2}\vartheta\lt \begin{pmatrix} 2 & 0 & 0 \ 0 & 2 & 0 \ 0 & 0 & 8 \end{pmatrix} \gt (\tau) = q - 4q^3 + 6q^5 - 8q^7 + 13q^9 \cdots \\
 c(\tau) &= -\frac{1}{6}E_{2,2}^-\tau + \frac{7}{6}E_{2,8}^-\tau = 1 + 8q^2 + 24q^4 + 32q^6 + 24q^8 + 48q^{10} + \cdots \\
 d(\tau) &= \frac{1}{16}E_{2,2}^-\tau - \frac{1}{16}E_{2,4}^-\tau = q + 4q^3 + 6q^5 + 8q^7 + 13q^9 + \cdots \\
 e(\tau) &= \frac{1}{4}E_{2,4}^-\tau - \frac{7}{4}E_{2,8}^-\tau + \frac{5}{2}E_{2,16}^-\tau = 1 - 8q^4 + 24q^8 - 32q^{12} + \cdots.
\end{align*}
\]

The theta series \( \vartheta[Q] \) of an even \( m \)-by-\( m \) quadratic form, used above to define basis element \( b \), is defined by \( \vartheta[Q](\tau) = \sum_{n \in \mathbb{Z}^m} e\left(\frac{1}{2}Q[n] \tau\right) \). If \( \ell \) is even then \( \vartheta[Q] \in M_0(\Gamma_0(\ell), \chi) \) for some character \( \chi \). The character is trivial when \( \det(Q) \) is a square and \( 4 | m \), see [9], page 203. Using Satz 0.3 of [9], we also have, for even \( m \),

\[
\vartheta[Q]F_\ell = \epsilon^{m/4}\det(Q)^{-1/2}(-i)^{m/2}\vartheta[\ell Q^{-1}], \quad \text{for } F_\ell = \frac{1}{\sqrt{\ell}}\lt \begin{pmatrix} 0 & -1 \\
1 & 0 \end{pmatrix} \gt.
\]

A \( D_4 \)-subgroup of the normalizer of \( \Gamma_0(16) \) in \( \text{SL}(2, \mathbb{Q}) \), modulo \( \lt \pm I, \Gamma_0(16) \gt \), acts on \( M_k(\Gamma_0(16)) \). This representation of \( D_4 \) on \( M_2(\Gamma_0(16)) \) is 5-dimensional and decomposes into a 2-dimensional irreducible representation and three 1-dimensional representations. The basis of \( M_2(\Gamma_0(16)) \) defined above was selected to decompose this representation into its irreducible components.
3 PARAMODULAR CUSP FORMS OF WEIGHT $k \leq 14$ AND LEVEL $N = 16$ 32

3.12.1 Lemma. The graded ring $M(\Gamma_0(16))$ consists of homogeneous polynomials in the five elements $a, b, c, d, e \in M_2(\Gamma_0(16))$, subject to the six relations:

$$2e^2 = c^2 + ac; 32d^2 = c^2 - ac; c^2 = a^2 + 64b^2; \quad cd = 2be - ad; \quad ce = ae + 32bd; \quad de = bc.$$

Every element in $M_k(\Gamma_0(16))$ can be uniquely written as $P_k(a, b) + C_{k-2}(a, b)c + D_{k-2}(a, b)d + E_{k-2}(a, b)e$, where $P_k$ is a homogeneous polynomial of degree $k/2$ and the $C_{k-2}, D_{k-2}, E_{k-2}$ are homogeneous of degree $(k-2)/2$. $A = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$ normalize $\Gamma_0(16)$ and generate a subgroup isomorphic to the dihedral group $D_4$, with $T = AF = \begin{bmatrix} 2 & -1/4 \\ 4 & 0 \end{bmatrix}$ of order four, and $\sigma = T^3F = \begin{bmatrix} 3 & 1/2 \\ 8 & 3 \end{bmatrix}$ of order two. For the representation $\rho : D_4 \to GL(5, \mathbb{C})$ defined by $(a, b, c, d, e)\rho(g)$, we have

$$\rho(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}; \rho(F) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1/4 \end{bmatrix}; \rho(T) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -1/4 & 0 \end{bmatrix}; \rho(\sigma) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Proof. The transformation under $A$ is obvious and the transformation under $F$ may be worked out using (36) and (37). A helpful intermediate step is $(E_{2, d}^{-1})F(\tau) = -16/\pi E_{2, d}^{-1}(16 \pi \tau)$. The normalizer in SL(2, $\mathbb{Q}$) of $\Gamma_0(16)$, modulo $(\pm I, \Gamma_0(16))$, contains a dihedral group of order 8: $\langle A, F \rangle = \langle T, \sigma \rangle$. The index of $\Gamma_0(16)$ in SL(2, $\mathbb{Z}$) is 24, so, by the Valence Inequality, to prove equality in dim $M_k(\Gamma_0(16))$ it suffices to check the equality of the first $2k + 1$ Fourier coefficients. In this way we verify the six given relations and the images of $\rho$.

Every modular form in $M_k(\Gamma_0(16))$ that can be written as a polynomial in $a, b, c, d, e$, may be written in the form $P_k(a, b) + C_{k-2}(a, b)c + D_{k-2}(a, b)d + E_{k-2}(a, b)e$, by applying the given relations in the order given. We will show that no nontrivial relation of the given form can be zero. First, by applying $T^2$, we would have both $P_k(a, b) + C_{k-2}(a, b)c = 0$ and $D_{k-2}(a, b)d + E_{k-2}(a, b)e = 0$. Second, applying $T$ to the first we obtain $P_k(a, b) - C_{k-2}(a, b)c = 0$ and hence $P_k(a, b) = C_{k-2}(a, b) = 0$. The modular forms $a$ and $b$ have the same weight, and so are algebraically independent because $b/a$ is nonconstant. Hence the polynomials $P_k$ and $C_{k-2}$ are also trivial. Third, applying $T$ to the second we obtain $D_{k-2}(a, b)(4c) - E_{k-2}(a, b)(d/4) = 0$ as well. Over the field of meromorphic functions, we thus have $E_{k-2}(a, b) = \pm 4D_{k-2}(a, b)$ and this is also an equality among holomorphic functions. From $0 = D_{k-2}(a, b)d + E_{k-2}(a, b)e = D_{k-2}(a, b)(d \pm 4e)$, we conclude that $D_{k-2}$ and $E_{k-2}$ are zero as polynomials. The dimension of $\mathbb{C}[a, b, c, d, e] \cap M_k(\Gamma_0(16))$ is then $(k^2 + 1) + 3(k^2 - 1) = 2k + 1$. By the Riemann-Roch Theorem, dim $M_k(\Gamma_0(16)) = 2k + 1$ for even $k \geq 0$, and thus $M(\Gamma_0(16)) = \mathbb{C}[a, b, c, d, e]$ as graded rings.

We have all the ingredients to apply the techniques of Section 4.2 and 4.4 of [26] to compute the eigenvalues $\lambda_{0,1}$ and $\lambda_{1,0}$. We successfully computed the eigenvalues $\lambda_{0,1}$ and $\lambda_{1,0}$ of the nonlift newforms in $S_k(K(16))^{\pm}$ for $k \leq 14$. The results are in Tables 7 and 8. By applying the knowledge of these eigenvalues to Table A.14 of [31], we also identify the possibilities for the corresponding local representations at $p = 2$ of the underlying automorphic representations. Further information on the entries of these tables may be found at [40].

3.13 Supercuspidal forms found

From Tables 7 and 8, we see that we found supercuspidal forms in weights 9, 11, 13, 14. The website [40] gives formulas for these supercuspidal forms. For the odd weights $k = 9, 11, 13$, the
Table 7: Eigenvalues $\lambda_3$, $\lambda_{0,1}$ and $\lambda_{1,0}$ of nonlift newforms in $S_k(K(16))^\pm$. The definition of the algebraic numbers $\alpha_*$ and any corresponding eigenvalue $t_*$ are given in Table 9 or at the website [40].

<table>
<thead>
<tr>
<th>$k$</th>
<th>AL</th>
<th>$3^{k-3} \lambda_3$</th>
<th>$\lambda_{0,1}$</th>
<th>$\lambda_{1,0}$</th>
<th>type</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>+</td>
<td>$-96$</td>
<td>$-5$</td>
<td>0</td>
<td>X</td>
</tr>
<tr>
<td>7</td>
<td>−</td>
<td>$-600$</td>
<td>$-2$</td>
<td>$-4$</td>
<td>XIa</td>
</tr>
<tr>
<td></td>
<td>−</td>
<td>$-144$</td>
<td>$-3$</td>
<td>0</td>
<td>I, IIa, or X</td>
</tr>
<tr>
<td>8</td>
<td>+</td>
<td>$-1992$</td>
<td>0</td>
<td>$-4$</td>
<td>VII, VIIIa or IXa</td>
</tr>
<tr>
<td></td>
<td>+</td>
<td>912</td>
<td>$-3$</td>
<td>0</td>
<td>X</td>
</tr>
<tr>
<td></td>
<td>+</td>
<td>$-168$</td>
<td>$-2$</td>
<td>$-4$</td>
<td>XIa</td>
</tr>
<tr>
<td></td>
<td>+</td>
<td>$-864 \pm 112\sqrt{33}$</td>
<td>$1/8(-7 \mp \sqrt{33})$</td>
<td>0</td>
<td>X</td>
</tr>
<tr>
<td>9</td>
<td>−</td>
<td>$-8136$</td>
<td>2</td>
<td>$-4$</td>
<td>XIa</td>
</tr>
<tr>
<td></td>
<td>−</td>
<td>5856</td>
<td>$-5$</td>
<td>0</td>
<td>I, IIa, or X</td>
</tr>
<tr>
<td></td>
<td>−</td>
<td>$-2280$</td>
<td>0</td>
<td>$-4$</td>
<td>sc(16)</td>
</tr>
<tr>
<td></td>
<td>−</td>
<td>$-1920$</td>
<td>$1/4$</td>
<td>0</td>
<td>I, IIa, or X</td>
</tr>
<tr>
<td></td>
<td>−</td>
<td>1464</td>
<td>$-2$</td>
<td>$-4$</td>
<td>XIa</td>
</tr>
<tr>
<td></td>
<td>−</td>
<td>$\pm 480\sqrt{33}$</td>
<td>$1/4(-3 \mp \sqrt{33})$</td>
<td>0</td>
<td>I, IIa, or X</td>
</tr>
<tr>
<td>10</td>
<td>+</td>
<td>$-12888$</td>
<td>2</td>
<td>$-4$</td>
<td>XIa</td>
</tr>
<tr>
<td></td>
<td>+</td>
<td>5928</td>
<td>$-2$</td>
<td>$-4$</td>
<td>XIa</td>
</tr>
<tr>
<td></td>
<td>+</td>
<td>$-3768$</td>
<td>0</td>
<td>$-4$</td>
<td>VII, VIIIa or IXa</td>
</tr>
<tr>
<td></td>
<td>+</td>
<td>$-1080$</td>
<td>0</td>
<td>$-4$</td>
<td>VII, VIIIa or IXa</td>
</tr>
<tr>
<td></td>
<td>+</td>
<td>$7248 \pm 240\sqrt{505}$</td>
<td>$1/8(-19 \mp \sqrt{505})$</td>
<td>0</td>
<td>X</td>
</tr>
<tr>
<td></td>
<td>+</td>
<td>$\alpha_{10,16}$ (degree 5)</td>
<td>$t_{10}$</td>
<td>0</td>
<td>X</td>
</tr>
<tr>
<td>11</td>
<td>+</td>
<td>$-66096$</td>
<td>$-29/8$</td>
<td>0</td>
<td>X</td>
</tr>
<tr>
<td></td>
<td>−</td>
<td>8040</td>
<td>0</td>
<td>$-4$</td>
<td>sc(16)</td>
</tr>
<tr>
<td></td>
<td>−</td>
<td>$24(-1245 \pm 32\sqrt{21})$</td>
<td>2</td>
<td>$-4$</td>
<td>XIa</td>
</tr>
<tr>
<td></td>
<td>−</td>
<td>$120(111 \pm 8\sqrt{69})$</td>
<td>$-2$</td>
<td>$-4$</td>
<td>XIa</td>
</tr>
<tr>
<td></td>
<td>−</td>
<td>$-73584$</td>
<td>$9/2$</td>
<td>0</td>
<td>I, IIa, or X</td>
</tr>
<tr>
<td></td>
<td>−</td>
<td>18768</td>
<td>1</td>
<td>0</td>
<td>I, IIa, or X</td>
</tr>
<tr>
<td></td>
<td>−</td>
<td>35568</td>
<td>$-3/4$</td>
<td>0</td>
<td>I, IIa, or X</td>
</tr>
<tr>
<td></td>
<td>−</td>
<td>$48(425 \pm 2\sqrt{3961})$</td>
<td>$1/32(-107 \mp \sqrt{3961})$</td>
<td>0</td>
<td>I, IIa, or X</td>
</tr>
<tr>
<td></td>
<td>−</td>
<td>$\alpha_{11,16}$ (degree 4)</td>
<td>$t_{11}$</td>
<td>0</td>
<td>I, IIa, or X</td>
</tr>
<tr>
<td>12</td>
<td>+</td>
<td>$-12456$</td>
<td>0</td>
<td>$-4$</td>
<td>VII, VIIIa or IXa</td>
</tr>
<tr>
<td></td>
<td>+</td>
<td>$72(819 \pm 64\sqrt{85})$</td>
<td>0</td>
<td>$-4$</td>
<td>VII, VIIIa or IXa</td>
</tr>
<tr>
<td></td>
<td>+</td>
<td>$72(-521 \pm 128\sqrt{5})$</td>
<td>2</td>
<td>$-4$</td>
<td>XIa</td>
</tr>
<tr>
<td></td>
<td>+</td>
<td>$72(831 \pm 8\sqrt{85})$</td>
<td>$-2$</td>
<td>$-4$</td>
<td>XIa</td>
</tr>
<tr>
<td></td>
<td>+</td>
<td>$\alpha_{12,16,a}$ (degree 5)</td>
<td>$t_{12,a}$</td>
<td>0</td>
<td>X</td>
</tr>
<tr>
<td></td>
<td>+</td>
<td>$\alpha_{12,16,b}$ (degree 8)</td>
<td>$t_{12,b}$</td>
<td>0</td>
<td>X</td>
</tr>
<tr>
<td></td>
<td>−</td>
<td>$-185616$</td>
<td>$-21/8$</td>
<td>0</td>
<td>I, IIa, or X</td>
</tr>
</tbody>
</table>
3 Paramodular Cusp Forms of Weight \( k \leq 14 \) and Level \( N = 16 \)

Table 8: Continuation of Table 7.

<table>
<thead>
<tr>
<th>13</th>
<th>-183168</th>
<th>-33/8</th>
<th>0</th>
<th>X</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>-144(3879 ± 41\sqrt{609})</td>
<td>(-53 ± \sqrt{609})/32</td>
<td>0</td>
<td>X</td>
</tr>
<tr>
<td>-</td>
<td>-220968</td>
<td>2</td>
<td>-4</td>
<td>XIa</td>
</tr>
<tr>
<td>-</td>
<td>72(-333 ± 80\sqrt{609})</td>
<td>2</td>
<td>-4</td>
<td>XIa</td>
</tr>
<tr>
<td>-</td>
<td>(\alpha_{13,16,a}) (degree 3)</td>
<td>-2</td>
<td>-4</td>
<td>XIa</td>
</tr>
<tr>
<td>-</td>
<td>(\alpha_{13,16,b}) (degree 3)</td>
<td>0</td>
<td>-4</td>
<td>XIa</td>
</tr>
<tr>
<td>-</td>
<td>0</td>
<td>3/2</td>
<td>0</td>
<td>I, IIa, or X</td>
</tr>
<tr>
<td>-</td>
<td>725184</td>
<td>-1</td>
<td>0</td>
<td>I, IIa, or X</td>
</tr>
<tr>
<td>-</td>
<td>(\alpha_{13,16,c}) (degree 6)</td>
<td>(t_{13,c})</td>
<td>0</td>
<td>I, IIa, or X</td>
</tr>
<tr>
<td>-</td>
<td>(\alpha_{13,16,d}) (degree 4)</td>
<td>(t_{13,d})</td>
<td>0</td>
<td>I, IIa, or X</td>
</tr>
<tr>
<td>-</td>
<td>(\alpha_{13,16,e}) (degree 4)</td>
<td>(t_{13,e})</td>
<td>0</td>
<td>I, IIa, or X</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>14</th>
<th>517320</th>
<th>2</th>
<th>-4</th>
<th>XIa</th>
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</thead>
<tbody>
<tr>
<td>+</td>
<td>527688</td>
<td>-2</td>
<td>-4</td>
<td>XIa</td>
</tr>
<tr>
<td>+</td>
<td>216(-597 ± 16\sqrt{51})</td>
<td>2</td>
<td>-4</td>
<td>XIa</td>
</tr>
<tr>
<td>+</td>
<td>24(40387 ± 320\sqrt{25561})</td>
<td>-2</td>
<td>-4</td>
<td>XIa</td>
</tr>
<tr>
<td>+</td>
<td>-499608</td>
<td>0</td>
<td>-4</td>
<td>VII, VIIIa or IXa</td>
</tr>
<tr>
<td>+</td>
<td>216(2927 ± 56\sqrt{3889})</td>
<td>0</td>
<td>-4</td>
<td>VII, VIIIa or IXa</td>
</tr>
<tr>
<td>+</td>
<td>24(20759 ± 88\sqrt{8689})</td>
<td>0</td>
<td>-4</td>
<td>VII, VIIIa or IXa</td>
</tr>
<tr>
<td>+</td>
<td>(\alpha_{14,16,a}) (degree 8)</td>
<td>(t_{11,a})</td>
<td>0</td>
<td>X</td>
</tr>
<tr>
<td>+</td>
<td>(\alpha_{14,16,b}) (degree 13)</td>
<td>(t_{11,b})</td>
<td>0</td>
<td>X</td>
</tr>
<tr>
<td>-</td>
<td>-2434968</td>
<td>0</td>
<td>-4</td>
<td>XIa</td>
</tr>
<tr>
<td>-</td>
<td>-927072</td>
<td>-17/8</td>
<td>0</td>
<td>I, IIa, or X</td>
</tr>
<tr>
<td>-</td>
<td>-432(1935 ± 23\sqrt{2377})</td>
<td>(-97 ± \sqrt{2377})/32</td>
<td>0</td>
<td>I, IIa, or X</td>
</tr>
</tbody>
</table>

Table 9: Definition of algebraic numbers in Tables 7 and 8 for weights 10 and 11. The definitions for other weights are at the website [40]. Each \(\alpha\) is a root of the adjacent minimal polynomial. The \(\alpha\) in a definition of any \(t\) refer to the immediately preceding \(\alpha\).

<table>
<thead>
<tr>
<th>(\alpha)</th>
<th>minimal polynomial of (\alpha)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha_{10,16})</td>
<td>(-392100597530099712 + 36717761396736000x - 1936322592768x^2 - 384208896x^3 + 12000x^4 + x^5)</td>
</tr>
<tr>
<td>(t_{10})</td>
<td>((200684470423235227287552 + 94255611784369274880\alpha + 2115778851231744\alpha^2 - 1410266234784\alpha^3 - 54792385\alpha^4)/410907531887271468859392)</td>
</tr>
<tr>
<td>(\alpha_{11,16})</td>
<td>(332724999250575360 - 1154234880\alpha^2 + x^4)</td>
</tr>
<tr>
<td>(t_{11})</td>
<td>((858199620022272 + 28477875456\alpha - 1490544\alpha^2 - 53\alpha^3)/21539386294272)</td>
</tr>
</tbody>
</table>
supercuspidal form is given as a linear combination of Gritsenko lifts and repeated $T(3)$ images of one or more Borcherds products. For the even weight $k = 14$, the supercuspidal form is given as a linear combination of the repeated $T(3)$ images of one Borcherds product. We also give the formula for the weight 14 supercuspidal form here to provide a bridge to the database [40] and to aid any future reproduction of our results. Let $\Delta$ be the cusp form in $S_{12}(\text{SL}_2(\mathbb{Z}))$ normalized to have leading term $q$. Theta blocks are the invention of Gritsenko, Skoruppa, and Zagier, and the special case we use here may be defined, for $d_j \in \mathbb{N}$, by

$$TB_k(d_1, d_2, \ldots, d_{\ell})(\tau, z) = \eta(\tau)^{2k-\ell} \prod_{j=1}^{\ell} \vartheta(\tau, d_j z),$$

where $\eta$ is the Dedekind eta function and $\vartheta(\tau, z) = \sum_{n \in \mathbb{Z}} (-1)^n q^{(n+1/2)^2/2} \zeta^{n+1/2}$ is the odd Jacobi theta function. A basis $B$ of $J_{cusp}^{12,16}$ is given in Table 10 in terms of $W_2$ and $W_3$ images of theta blocks.

Define the following weight zero weakly holomorphic form $\psi_{14} \in J_{wh}^{0,16}(\mathbb{Z})$ using the vector $b_{14}$ in Table 11,

$$\phi_{14} = TB_{14}(1, 1, 1, 1, 1, 2, 2, 3, 3); \quad \psi_{14} = \frac{\phi_{14} | V_2}{\phi_{14}} + \frac{b_{14} \cdot B}{\Delta}.$$

We have $\text{Borch}(\psi_{14}) \in S_{14}(K(16))^\sim$. It happens that

$$\{T(3)^j \text{Borch}(\psi_{14}) : j = 0, \ldots, 13\}$$

is a basis of the space $S_k(K(16))^\sim$. Table 12 gives the linear combination vector $c_{14}$ that defines

$$f_{14} = \sum_{j=0}^{13} (c_{14})_j T(3)^j \text{Borch}(\psi_{14}).$$

We stopped at $k = 14$ because we found a supercuspidal paramodular form in an even weight space of the lowest possible level. Also, weight $k = 14$ for $K(16)$ is on the edge of tractability for the method of Jacobi restriction.

References


Table 12: Definition of vector $b_{14} \in \mathbb{Q}^{12}$. The relevant definitions for other weights are at the website [40].

\[
\begin{align*}
\mathbf{b}_{14} &= (-1155865602082817198192, -10565981369327462562477, \\
&\quad -2926740930944006282896, 9167023003084404792024, \\
&\quad 9262973271435152666448, 5762211536895867593392, \\
&\quad 2926740930944006282896, -575926067281640631444, \\
&\quad 191850399359964690328, -130078644330386905144, \\
&\quad 158496997375774748880, 3516352762727205084768, \\
&\quad 279268001096663167080660)
\end{align*}
\]

Table 12: Definition of vector $c_{14}$. The relevant definitions for other weights are at the website [40].

\[
\begin{align*}
\mathbf{c}_{14} &= (34824415729731223242461994879166366309741352437451593600, \\
&\quad -1231050015269758711257977743259890444383052096717455360, \\
&\quad 185755810226739139339249977817168623131512255710101504, \\
&\quad 12549839981599670823967381930835695798087486464384, \\
&\quad -12747907866219085265737676925516146197487989292955136, \\
&\quad -487551611392210229659800521268238302688265271514538816, \\
&\quad -51410212284561459894870136498517909876263689224454144, \\
&\quad 637535128963431721866818319112205804800646176286703616, \\
&\quad 20452299868556686562499034505713458565710475824857088, \\
&\quad 112323978289166189088890862245818983032675662143488, \\
&\quad -129183614695895171865843448868621940960348703520, \\
&\quad -86511023193385793107563673002328290272960212707520, \\
&\quad -635577298399001689023522775734788662836903493392, \\
&\quad -1717925069106704436788203765885404424234035840667) \\
&\quad /37257382163580423563364831824348829583722116066312192000
\end{align*}
\]
REFERENCES


REFERENCES


