Siegel modular forms of half integral weight and a lifting conjecture

By

Shuichi Hayashida* and Tomoyoshi Ibukiyama**

Abstract

A conjecture on lifting to Siegel cusp forms of half-integral weight \( k - 1/2 \) of degree two from each pair of cusp forms of \( SL_2(\mathbb{Z}) \) of weight \( 2k - 2 \) and \( 2k - 4 \) is given with a conjectural relation of the \( L \) functions and numerical evidences. We also describe the space of Siegel modular forms of half-integral weight, its “plus subspace” and Jacobi forms of degree two by explicitly given theta functions.

This paper has two aims.

1. We describe Siegel modular forms of half integral weight of \( \Gamma_0(4) \) of degree two explicitly.

2. We give a conjecture on lifting preserving \( L \) function from a pair of elliptic modular forms to Siegel modular forms of half integral weight of degree two with numerical evidences on coincidence of the Euler factors.

As for (1), we also describe the so-called “plus subspace” consisting of a kind of new forms which is isomorphic to the space of Jacobi forms of some sort. We state our results in section §1 (cf. Theorems 1.3, 1.5, 1.8, 1.9) and give the proof in §2. In the remaining sections we treat (2) (cf. Conjecture 3.1).

Now we explain more precise content of this paper. First of all, rough content of our conjecture mentioned above is as follows. We denote by \( M_{k-1/2}(\Gamma_0(4)) \) the space of Siegel modular forms of \( \Gamma_0(4) \) of degree two of weight \( k - 1/2 \) and by \( S_{k-1/2}(\Gamma_0(4)) \) the subspace of cusp forms. We denote by \( S^+_{k-1/2}(\Gamma_0(4)) \) the plus subspace of degree two. (This plus space was first introduced by W. Kohnen in case of one variable and later generalized by the present authors for general degree. As for the definition, see §2). Now our conjecture claims that for each pair of common eigen cusp forms \( f \) of weight

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2k − 2 and g of weight 2k − 4 belonging to $SL_2(\mathbb{Z})$, there should exist a common eigen Siegel cusp form $F \in S^+_k(\Gamma_0(4))$ of weight $k - 1/2$ of $\Gamma_0(4)$ such that $L(s, F) = L(s, f)L(s - 1, g)$ (cf. §3, Conjecture 3.1). Here $L(s, f)$ and $L(s, g)$ are the usual Hecke $L$ functions and $L(s, F)$ is a $L$ function defined by Zhuravlev [22] (cf. also [6]). His Hecke theory on Siegel modular forms of half integral weight and the precise definition of $L$ function will be reviewed in §3.

This conjecture is based on our numerical calculation of examples of $L$ functions of explicitly given Siegel cusp forms of half integral weight. So we explain our explicit results on Siegel modular forms. Denote by $M_{k-1/2}(\Gamma_0(4), \chi)$ the space of Siegel modular forms of weight $k - 1/2$ of $\Gamma_0(4)$ of degree two with character $\chi$. Then the direct sum $\bigoplus_{k=1}^\infty M_{k-1/2}(\Gamma_0(4), \chi)$ is not a ring. But we can regard it as a module over a certain ring of Siegel modular forms of integral weight, and we can give explicit generators of modules of Siegel modular forms of half integral weight (with or without character) by theta constants (cf. Theorem 1.1, 1.2, 1.3). By this, we can give a dimension formula of Siegel modular forms of half integral weight of $\Gamma_0(4)$ as a corollary (cf. Corollary 1.2, 1.5) which was first obtained by Tsushima [17] by using holomorphic Lefschetz Theorem.

Then we need a description of the plus subspace. In degree one case, this space is isomorphic to holomorphic or skew holomorphic Jacobi forms of index one. (Eichler-Zagier [2], Skoruppa [15]). We can generalize the notion of the plus space for general degree so that the plus space of weight $k - 1/2$ of degree $n$ is isomorphic to the space of holomorphic or skew holomorphic Jacobi forms of index one of weight $k$ of degree $n$ of $Sp(n, \mathbb{Z})$, depending on parity of $k$ or on character. (This is mostly known in Ibukiyama [9] and Hayashida [3]. The remaining case can be done in a similar way.) Now we have Tsushima’s dimension formula for Jacobi forms of degree two. By using his result, we can also give each basis of the space of holomorphic or skew holomorphic Jacobi forms, or of the plus subspace explicitly (cf. Theorem 1.8, 1.9, 1.10). The result is very simple. Each space is a free module over the ring isomorphic to Siegel modular forms of even weight belonging to $Sp(2, \mathbb{Z})$. Extracting modular forms with small weights, the Euler factors at small primes in the plus space can be given by computer calculations, and we see that these examples support our conjecture (cf. §3).

The authors would like to express their thanks to Professor Tsushima for showing us his new results on dimensions.

1. Modules of Siegel modular forms

1.1. Graded rings of modular forms of integral weights

Let $n$ or $N$ be any natural number. For any commutative ring $R$, we denote by $Sp(n, R)$ the symplectic group of size $2n$ with components in $R$.

$$Sp(n, R) = \{ g \in M_{2n}(R); gJ^tg = J \}$$
where $J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ and $1_n$ is the unit matrix of size $n$. We put

$$\Gamma_0^{(n)}(N) = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{Z}); C \equiv 0 \mod N \right\}.$$

Sometimes a conjugate of $\Gamma_0^{(n)}(4)$ is easier to treat, so we put $\rho_2 = \begin{pmatrix} 1_n & 0 \\ 0 & 2 \cdot 1_n \end{pmatrix}$, and put $\Gamma^{(n)} = \rho_2^{-1} \Gamma_0^{(n)}(4) \rho_2$. Then, we get

$$\Gamma^{(n)} = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{Z}); B \equiv C \equiv 0 \mod 2 \right\}.$$

If we define

$$\psi(g) = \left( \frac{-1}{\det(D)} \right)$$

for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(n)}(4) \cup \Gamma^{(n)}$, then this gives a character of the group $\Gamma_0^{(n)}(4)$ or $\Gamma^{(n)}$. For any integer $k$, any discrete subgroup $\Gamma'$ of $Sp(n, \mathbb{R})$ with $\text{vol}(\Gamma' \backslash Sp(n, \mathbb{R})) < \infty$ and a character $\chi$ of $\Gamma'$, and any function $F(\tau)$ on the Siegel upper half space

$$H_n = \{ \tau = X + iY = \tau \in M_n(\mathbb{C}); X, Y \in M_n(\mathbb{R}), Y > 0 \ (\text{positive definite}) \},$$

we write

$$(F|_{k, \chi}) \gamma) = \chi(\gamma)^{-1} \det(C\gamma + D)^{-k} F(\gamma \tau).$$

We say that a holomorphic function $F$ on $H_n$ is a modular form of weight $k$ with character $\chi$ belonging to $\Gamma'$ if it satisfies

$$F|_{k, \chi} \gamma = F$$

for all $\gamma \in \Gamma'$ and is bounded at each cusps of $\Gamma'$. The space of these modular forms is denoted by $M_k(\Gamma', \chi)$ and cusp forms by $S_k(\Gamma', \chi)$. When $\chi$ is the trivial character, we may sometimes omit $\chi$ in the above notation. For simplicity, we write

$$M(\Gamma', \chi) = \bigoplus_{k=0}^{\infty} M_k(\Gamma', \chi^k).$$

Then, this is obviously a graded ring.

In this paper, we mainly treat the case $n = 2$. So we write $\Gamma_0(4) = \Gamma_0^{(2)}(4)$ and $\Gamma = \Gamma^{(2)}$. The following formula for $n = 2$ was calculated by Tsushima, using [10] and [16].
Proposition 1.1 (Tsushima [17]).

\[
\sum_{k=0}^{\infty} \dim M_k(\Gamma_0(4), \psi^k) t^k = \frac{1}{(1-t)(1-t^2)(1-t^3)},
\]

\[
\sum_{k=0}^{\infty} \dim M_k(\Gamma_0(4)) t^k = \frac{(1 + t^4)(1 + t^{11})}{(1 - t^2)^3(1 - t^9)},
\]

\[
\sum_{k=0}^{\infty} \dim M_{2k}(\Gamma_0(4), \psi) t^{2k} = \frac{t^{12} + t^{14}}{(1 - t^2)^3(1 - t^9)}.
\]

First, we shall obtain the graded ring \( \oplus_{k=0}^{\infty} M_k(\Gamma_0(4), \psi^k) \). Instead of \( \Gamma_0(4) \), we consider \( \Gamma \), partly because \( \Gamma \subset \Gamma_0(2) \) and \( M(\Gamma_0(2)) \) has been known in Ibukiyama [7]. Indeed the ring \( \bigoplus_{k=0}^{\infty} M_{\text{even}}(\Gamma_0(2)) \) of modular forms of even weights is generated by four algebraically independent modular forms \( X, Y, Z, K \) of degree two defined by

\[
X = ((\theta_{0000})^4 + (\theta_{0001})^4 + (\theta_{0010})^4 + (\theta_{0011})^4)/4,
\]

\[
Y = (\theta_{0000}\theta_{0001}\theta_{0010}\theta_{0011})^2,
\]

\[
Z = ((\theta_{0100})^4 - (\theta_{0110})^4)^2/16384,
\]

\[
K = (\theta_{0100}\theta_{0110}\theta_{1000}\theta_{1001}\theta_{1100}\theta_{1110})^2/4096,
\]

(see [7]), where \( \theta_m \) is the theta constant on \( H_2 \) defined by

\[
\theta_m(\tau) = \sum_{p \in \mathbb{Z}^2} e \left( \frac{1}{2} t \left( p + \frac{m'}{2} \right) \tau \left( p + \frac{m'}{2} \right) + t \left( p + \frac{m''}{2} \right) \right),
\]

for \( m = (m', m'') \in \mathbb{Z}^4, m', m'' \in \mathbb{Z}^2, \tau \in H_2 \) and \( e(x) = e^{2\pi ix} \). (cf. Igusa [11]). Now, we put

\[
f_1 = (\theta_{0000})^2,
\]

\[
f_2 = f_1^2,
\]

\[
g_2 = (\theta_{0000})^4 + (\theta_{0100})^4 + (\theta_{1000})^4 + (\theta_{1100})^4
\]

\[
f_3 = (\theta_{0001}\theta_{0010}\theta_{0011})^2,
\]

\[
\chi_5 = \theta_{0000}\theta_{0001}\theta_{0010}\theta_{0011}\theta_{0100}\theta_{0110}\theta_{1000}\theta_{1001}\theta_{1100}\theta_{1110},
\]

\[
f_6 = (\theta_{0001} - \theta_{0010})(\theta_{0001} - \theta_{0011})(\theta_{0010} - \theta_{0011}),
\]

\[
f_{11} = f_6\chi_5,
\]

\[
f_{21/2} = f_{11}/\theta_{0000}.
\]

By definition, we have \( Y = f_1f_3 \) and it is not difficult to show that \( Z = (g_2 + 2X - 3f_2)^2/36864 \). (Since all the relations between \( \theta_m \) are known by Igusa [10], it is a routine calculation to show this anyway. We omit the details here.) Here the form \( f_{21/2} \) is obviously holomorphic. The notation \( f_6 \) and \( \chi_5 \) are introduced to make notation simpler.
Theorem 1.1. We have
\[ M_\Gamma = \bigoplus_{k=0}^\infty M_k(\Gamma, \psi^k) \] and hence \( X, f_2, g_2, f_3 \) are also algebraically independent.

We denote by \( B \) the weighted polynomial ring generated by \( X, f_2, g_2, K \).
\[
B = \mathbb{C}[X, f_2, g_2, K].
\]

Theorem 1.1. The ring \( M(\Gamma, \psi) = \bigoplus_{k=0}^\infty M_k(\Gamma, \psi^k) \) is given by a weighted polynomial ring
\[
M(\Gamma, \psi) = \mathbb{C}[f_1, g_2, X, f_3].
\]
Also, the ring \( M(\Gamma) = \bigoplus_{k} M_k(\Gamma) \) is given by
\[
M(\Gamma) = B \oplus YB \oplus f_{11}(B \oplus YB).
\]
The formula for \( f_{11} \) is easily obtained but the result is complicated and not so interesting, so we omit it here.

Theorem 1.2. The module of Siegel modular forms of even weight of \( \Gamma \) with character \( \psi \) is given by
\[
\bigoplus_{k=0}^\infty M_{2k}(\Gamma, \psi) = f_{11}f_1B \oplus f_{11}f_3B.
\]
We note that the result for \( M_k(\Gamma, \psi) \) for odd \( k \) is already contained in Theorem 1.1.

We can rewrite the above results for \( \Gamma \) to those for \( \Gamma_0(4) \) very easily, since
\[
M_k(\Gamma_0(4), \psi) = \{ F(2\tau); F \in M_k(\Gamma, \psi) \} \text{ and } M_k(\Gamma_0(4)) = \{ F(2\tau); F \in M_k(\Gamma) \}.
\]
The latter spaces are also described by usual theta constants by using Riemann’s theta relations (cf. Igusa [11, p. 233]). For example, we get
\[
f_1(2\tau) = (\theta_{0000}(\tau)^2 + \theta_{0001}(\tau)^2 + \theta_{0010}(\tau)^2 + \theta_{0011}(\tau)^2)/4,
\]
\[
X(2\tau) = (2X(\tau) + 12(\theta_{0000}(\tau)^2\theta_{0001}(\tau)^2\theta_{0010}(\tau)^2\theta_{0011}(\tau)^2)
+ 3(\theta_{0000}(\tau)^2\theta_{0001}(\tau)^2 + \theta_{0000}(\tau)^2\theta_{0010}(\tau)^2 + \theta_{0000}(\tau)^2\theta_{0011}(\tau)^2
+ \theta_{0001}(\tau)^2\theta_{0010}(\tau)^2 + \theta_{0001}(\tau)^2\theta_{0011}(\tau)^2 + \theta_{0010}(\tau)^2\theta_{0011}(\tau)^2))/32,
\]
g_2(2\tau) = X(\tau).
1.2. Modular forms of half integral weights

We put $\theta(\tau) = \sum_{p \in \Z^n} \exp(2\pi i \langle \tau, p \rangle)$. Let $F$ be a holomorphic function on $H_n$. For any integer $k \geq 1$, we say that $F$ is a Siegel modular form of weight $k - 1/2$ belonging to $\Gamma_0(4)$ with character $\chi$, if $F$ satisfies the following condition:

$$F(\gamma \tau) = \chi(\gamma) \left( \frac{\theta(\gamma \tau)}{\theta(\tau)} \right)^{2k-1} F(\tau) \quad \text{for every } \gamma \in \Gamma_0(n) \ .$$

We denote the space of above forms by $M_{k-1/2}(\Gamma_0(n), \chi)$. When $\chi$ is $\psi$ or the trivial character, we also put $M_{k-1/2}(\Gamma_0(n), \chi) = \{ f(\tau/2); \ f \in M_{k-1/2}(\Gamma_0(n), \chi) \}$.

The following Theorem 1.3 for $n = 2$ was first observed by Tsushima [17] by showing the dimension formulas in the Corollary 1.2 by Riemann Roch theorem and by comparing the dimensions of both sides in Theorem 1.3. We use a different argument, that is, without using the dimension formula, we first prove the following theorem directly by using ring theoretic argument, and next gives a dimension formula of modular forms of half integral weights as a corollary of this theorem.

**Theorem 1.3.** We get

$$\bigoplus_{k=0}^{\infty} M_{k+1/2}(\Gamma_0(4)) = \theta_{0000}(2\tau) \bigoplus_{k=0}^{\infty} M_k(\Gamma_0(4), \psi^k).$$

We put

$$M^{(1/2)}(\Gamma) = \bigoplus_{k=0}^{\infty} M_k(\Gamma, \psi^k) + \bigoplus_{k=0}^{\infty} M_{k+1/2}(\Gamma).$$

Since

$$\frac{\theta_{0000}(\gamma \tau)}{\theta_{0000}(\tau)} = \psi(\gamma) \det(c\tau + d)$$

for $\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma$, the module $M^{(1/2)}(\Gamma)$ is a graded ring corresponding to the automorphic factors $\theta_{0000}(\gamma \tau)/\theta_{0000}(\tau)^k$ ($k = 0, 1, \ldots, \in \Gamma$). By Theorem 1.1 and 1.3, we get

**Corollary 1.1.**

$$M^{(1/2)}(\Gamma) = \mathbb{C}[\theta_{0000}, g_2, X, f_3].$$

**Corollary 1.2.**

$$\sum_{k=0}^{\infty} \dim M_{k+1/2}(\Gamma_0(4)) t^k = \sum_{k=0}^{\infty} \dim M_{k+1/2}(\Gamma) t^k = \frac{1}{(1-t)(1-t^2)(1-t^3)}.$$

We denote by $S(\Gamma)$ the space of cusp forms in $\bigoplus_{k=0}^{\infty} M_{k+1/2}(\Gamma)$. The description of cusp forms is given as follows.
Theorem 1.4. The space $S(\Gamma)$ is generated as a $\bigoplus_{k=0}^{\infty} M_{2k}(\Gamma)$ module by four cusp forms

$$\theta_{0000}(f_3(3f_2 - 2X - g_2)), \quad \theta_{0000}(g_2 - 4X)(-3f_3 + f_1(g_2 + 8X - 6f_2)), \quad \theta_{0000}(8X + g_2 - 6f_2)(g_2 - 4X)(3f_2 - 2X - g_2)$$

of weight $11/2, 11/2, 13/2, 13/2$.

This module is not a free module. We can describe the module structure precisely (cf. the proof in §2) and get the following dimension formula of cusp forms, which was first obtained by Tsushima by Riemann Roch Theorem. Here we shall give a simple alternative proof based on the above theorem.

Corollary 1.3.

$$\sum_{k=0}^{\infty} \dim S_{k + 1/2}(\Gamma) t^k = \frac{2t^5 + t^7 + t^9 - 2t^{11} + 4t^6 - t^8 + t^{10} - 3t^{12} + t^{14}}{(1 - t^2)^3(1 - t^5)}.$$ 

We also give Siegel modular forms of half integral weight with character $\psi$. The following Corollary 1.5 was also obtained by Tsushima first (cf. [17]). Our proof is independent of his argument.

Theorem 1.5. We have

$$\bigoplus_{k=0}^{\infty} M_{k + 1/2}(\Gamma, \psi) = f_{21/2} \bigoplus_{k=0}^{\infty} M_k(\Gamma, \psi^k).$$

We denote by $S_{k + 1/2}(\Gamma, \psi)$ the subspace of cusp forms of $M_{k + 1/2}(\Gamma, \psi)$. Then we have

Corollary 1.4.

$$\bigoplus_{k=0}^{\infty} S_{k + 1/2}(\Gamma, \psi) = \bigoplus_{k=0}^{\infty} M_{k + 1/2}(\Gamma, \psi).$$

Corollary 1.5.

$$\sum_{k=0}^{\infty} \dim M_{k + 1/2}(\Gamma_0(4), \psi) t^k = \sum_{k=0}^{\infty} \dim S_{k + 1/2}(\Gamma_0(4), \psi) t^k = \frac{t^{10}}{(1 - t)(1 - t^2)^2(1 - t^3)}.$$ 

1.3. The plus subspace of Siegel modular forms of half integral weight

In order to explain the relation between the plus space and Jacobi forms shortly, first we introduce holomorphic Jacobi forms of general degree following Ziegler [23]. Let $k$ be a natural number and let $F(\tau, z)$ be a holomorphic function on $\tau, z \in H_n \times \mathbb{C}^n$. If $F$ satisfies the next three conditions (1), (2), (3), we say that $F$ is a holomorphic Jacobi form of weight $k$ of index 1 of degree $n$. 

(1) $F(M(\tau, z)) = e^{(t^i z(C\tau + D)^{-1} Cz)} \det(C\tau + D)^k F(\tau, z)$ for any $M \in \text{Sp}(n, \mathbb{Z})$, where $M(\tau, z) = (M\tau, t^i(C\tau + D)^{-1} z) \in H_n \times \mathbb{C}^n$.

(2) $F(\tau, z + \tau \lambda + \mu) = e(-i\lambda \tau \lambda - 2^t \lambda z) F(\tau, z)$ for any $\lambda, \mu \in \mathbb{Z}^n$.

(3) $F(\tau, z)$ has the Fourier expansion of the following form,

$$F(\tau, z) = \sum_{N, r} A(N, r) e(\text{tr}(N\tau + 4^t rz)).$$

where we denote by $L_n^*$ the set of all half integral symmetric matrices, and $N$ runs over all positive semi-definite elements in $L_n^*$, and $r$ runs over all elements in $\mathbb{Z}^n$ satisfying $4N - r^t r \geq 0$ (i.e. positive semi-definite).

Moreover, if the Fourier coefficients $A(N, r)$ are zero unless $4N - r^t r > 0$ (i.e. positive definite), then we say that $F$ is a holomorphic Jacobi cusp form.

Next we introduce skew holomorphic Jacobi forms following Skoruppa [15] and Arakawa [1]. Let $k$ be a natural number. Let $F(\tau, z)$ be a function on $(\tau, z) \in H_n \times \mathbb{C}^n$ which is real analytic in the real and the imaginary part of $\tau$ and holomorphic in $z$. If $F$ satisfies the next three conditions (1), (2) and (3), we say that $F$ is a skew holomorphic Jacobi form of weight $k$ of index 1 of degree $n$.

(1) $F(M(\tau, z)) = e^{(t^i z(C\tau + D)^{-1} Cz)} \det(C\tau + D)^k \det(C\tau + D)|F(\tau, z)$ for any $M \in \text{Sp}(n, \mathbb{Z})$.

(2) $F(t^i z + \tau \lambda + \mu) = e(-i\lambda \tau \lambda - 2^t \lambda z) F(\tau, z)$ for any $\lambda, \mu \in \mathbb{Z}^n$.

(3) $F(\tau, z)$ has the Fourier expansion of the following form,

$$F(\tau, z) = \sum_{N, r} A(N, r) e \left( \text{tr} \left( N\tau - \frac{1}{2} i(4N - r^t r) Y \right) \right) 4^t rz.$$
for some column vector $\lambda \in (\mathbb{Z}/2\mathbb{Z})^n$. We have a theorem for general degree $n$.

**Theorem 1.6.** We have the following isomorphisms.

$$J_{k,1}^{(n)} \cong M_{k-1/2}^+(\Gamma_0(4), \psi^k).$$

$$J_{k,1}^{(n),sk} \cong M_{k-1/2}^+(\Gamma_0(4), \psi^{k+1}).$$

When $k$ is even, then $\psi^k = id$ and the above first isomorphism is the claim in Ibukiyama [9], and the second isomorphism for any $k$ is the claim in Hayashida [3]. The remaining case is easy to prove and we omit the proof in this paper.

Now from now on, we consider the case $n = 2$ exclusively until the end of this paper. We put $J_{k,1} = J_{k,1}^{(2)}, J_{k,1}^{cusp} = J_{k,1}^{(2),cusp}$ and so on.

For any modular form $f(Z) \in M_k(\text{Sp}(2,\mathbb{Z}))$, if we take $g(Z) = f(4Z)$, then the Fourier coefficients of $g(Z)$ is non zero only at $T$ with $T \in 4\mathbb{L}_2$. Besides, we have $g(Z) \in M_k(\Gamma_0(4))$. Hence, if we put $A' = \{f(4Z); f \in \oplus_{n=0}^{\infty} M_{2k}(\text{Sp}(n,\mathbb{Z}))\}$, then $M_{k-1/2}^+(\Gamma_0(4), \psi^k)$ is $A'$-module. To make our calculation easier a little, in §1 and §2 we sometimes use the group $\Gamma = \rho_2^{-1}\Gamma_0(4)\rho_2$ instead of $\Gamma_0(4)$. So, for $l = 0$ or 1, we put

$$M_{k-1/2}^+(\Gamma, \psi^l) = \{f(\tau/2); f \in M_{k-1/2}^+ (\Gamma_0(4), \psi^l)\},$$

$$A = \{f(\tau/2); f \in A'\}.$$ Of course every result on $\Gamma$ can be easily interpreted to the one for $\Gamma_0(4)$ by taking the image of $f(\tau) \rightarrow f(2\tau)$. Also we put

$$M^+(\Gamma) = \oplus_{k=1}^{\infty} M_{k-1/2}^+(\Gamma),$$

$$M^+(\Gamma, \psi) = \oplus_{k=1}^{\infty} M_{k-1/2}^+(\Gamma, \psi).$$

Then $M^+(\Gamma)$ and $M^+(\Gamma, \psi)$ are $A$-modules. The following dimension formulae by Tsushima are very helpful to determine the $A$-module structures, and we can show they are free $A$-modules as the formulae may suggest.

**Theorem 1.7** (Tsushima [18]).

$$\sum_{k=0}^{\infty} \dim(J_{k,1})t^k = \frac{t^4 + t^6 + t^{10} + t^{12} + t^{21} + t^{27} + t^{29} + t^{35}}{(1 - t^4)(1 - t^6)(1 - t^{10})(1 - t^{12})}.$$ 

$$\sum_{k=0}^{\infty} \dim(J_{k,1}^{sk})t^k = \frac{t + t^7 + t^9 + t^{15} + t^{24} + t^{26} + t^{30} + t^{32}}{(1 - t^4)(1 - t^6)(1 - t^{10})(1 - t^{12})}.$$
\[
\sum_{k=0}^{\infty} \dim(J_{k,1}^{cusp}) t^k = \frac{t^4 + t^6 + t^8 - 2t^{26} + t^{28} + t^{29} + t^{35}}{(1 - t^4)(1 - t^6)(1 - t^{10})(1 - t^{12})}
\]

Then we have the following structure theorem.

To make our expression slightly shorter, we replace the generators \(g_2\) or \(f_3\) by
\[
R_2 = 6f_1^2 - 2g_2 - 4X,
\]
\[
V_3 = 2(f_1^2 - 2f_1X + f_3).
\]

and put
\[
P_{7/2} = \theta_{0000} (-48f_1^3 + 21V_3 + 112f_1X)/64,
\]
\[
P_{11/2} = \theta_{0000} (-1152f_1^5 - 11f_1R_2^2 + 792f_1^2V_3 + 792V_3X + 4224f_1X^2)/3072,
\]
\[
P_{19/2} = \theta_{0000} (f_1R_2^4 - 162V_3^3 + 36R_2^2V_3X)/135895496,
\]
\[
P_{23/2} = \theta_{0000} (16f_1^3R_2^3 + 3R_2^2V_3 + 4R_2^4f_1X + 18f_1R_2^2V_3^2 - 864f_1^4R_2^2V_3
+ 1728f_1^2R_2^2V_3X + 3888f_1^2V_3^3 - 6480V_3^3X
- 288R_2^2V_3X^2)/21743271936,
\]
\[
P_{1/2} = \theta_{0000},
\]
\[
P_{3/2} = \theta_{0000} (-192f_1^6 - 26f_1^2R_2^2 + 4992f_1^4f_3 + 7488f_3^2
+ 312f_1^3V_3 - 2808f_3V_3 + 117V_3^2)/12288,
\]
\[
P_{17/2} = \theta_{0000} (768f_1^6 + 13056f_1^4f_3 - 544f_1R_2^2f_3 + 104448f_1^2f_3^2 - 1632f_1^3V_3
+ 17f_1R_2^2V_3 - 1632f_1^2f_3V_3 + 408f_1^3V_3^2 + 78336f_3^2X - 29376f_3V_3X
+ 1224V_3^2X)/196608,
\]
\[
P_{29/2} = \theta_{0000} (144f_1^4R_2^2 + 5f_1^2R_2^2 + 3744f_1^3R_2^2f_3 + 720R_2^4f_3^2 - 684f_1^3R_2V_3
+ 180R_2^2f_3V_3 - 45R_2^2V_3^2 - 41472f_1R_2^2f_3^2V_3^2 + 9396f_1R_2^2V_3^3
- 69984f_2^2V_3^4 + 116640V_3^4X)/100192997081088.
\]

Then we have the following structure theorem.

**Theorem 1.8.** The vector space \(M^+ (\Gamma)\) is a free \(A\) module of rank 8, and we have
\[
M^+ (\Gamma) = AP_{7/2} \oplus AP_{11/2} \oplus AP_{19/2} \oplus AP_{23/2} \oplus AP_{1/2} \oplus AP_{17/2} \oplus AP_{29/2}.
\]
We put $S^+(\Gamma) = S(\Gamma) \cap M^+(\Gamma)$ and we denote by $A^{cusp}$ the space of cusp forms in $A$. We denote by $E_k(\tau)$ the Eisenstein series of $Sp(2, \mathbb{Z})$ of weight $k$ such that the constant term of the Fourier expansion is one. We also put $E'_k(\tau) = E_k(2\tau)$ and $B' = \mathbb{C}[E'_2, E'_6]$. Then $S^+(\Gamma)$ is given as follows.

**Theorem 1.9.**

\[ S^+(\Gamma) = A^{cusp} P_{1/2} \oplus A^{cusp} P_{11/2} \oplus AP_{9/2} \oplus AP_{23/2} \]

\[ \oplus A^{cusp} P_{1/2} \oplus A^{cusp} P_{13/2} \oplus A^{cusp} P_{17/2} \oplus B' P_{25/2} \oplus AP_{29/2} , \]

where

\[ P_{25/2} = (5E'_2)^3 P_{1/2} - 5(E'_6)^2 P_{1/2} + 6E'_6 P_{13/2} - 6E'_4 P_{17/2} )/17280. \]

Let $S^+(\Gamma, \psi)$ be the space of cusp forms in $M^+(\Gamma, \psi)$. In order to describe the explicit structure $M^+(\Gamma, \psi)$ and $S^+(\Gamma, \psi)$, we put

\[ P_{41/2} = f_{21/2} R_2 (2322432 f_3^3 V_3 X + 1008 f_1 R_2 V_3 + R_2^4 - 497664 V_3 f_1 X^2 \]

\[ + 9216 f_1 R_2^2 V_3 X - 1824768 f_1^5 V_3 + 217728 f_1^3 V_3^2 - 10368 V_3^2 X \]

\[ - 7962624 f_1^3 X + 7962624 f_1^4 X^2 \]

\[ - 2654208 f_1^2 V_3^3 + 2654208 f_1^8 )/521838526464, \]

\[ P_{53/2} = f_{21/2} R_2 (-4008 f_1^8 R_2^4 + 2 f_1^2 R_2^6 + 1296 f_1^3 R_2^4 V_3 + 4608 f_1^4 R_2^4 X \]

\[ + 9 R_2^4 V_3^2 + 144 R_2^4 V_4 f_1 + 41472 f_1^4 R_2^2 V_3^2 \]

\[ + 3888 f_1 R_2^2 V_3^3 + 41472 f_1^2 R_2^2 V_3^2 X - 279936 f_1^3 V_3^3 \]

\[ - 93312 V_3^4 X ) / 307792887033102336, \]

\[ P_{57/2} = f_{21/2} R_2 (-16 f_1^2 R_2^6 X - 72 R_2^4 V_3^2 X - 3456 f_1^5 R_2^4 V_3 \]

\[ + 27648 f_1^3 R_2^4 V_3 X + 1658880 f_1^4 R_2^2 V_3^2 X + 62208 V_3^3 R_2^2 f_1 X \]

\[ + 165888 V_3^2 R_2^2 f_1^2 X^2 - 1152 R_2^4 V_3 X^3 f_1 - 48 f_1^3 R_2^6 \]

\[ + 2239488 f_1^3 V_3^4 + 1119744 f_1 V_3^6 + 746496 V_3^3 X^2 \]

\[ - 8957952 f_1^3 V_3^3 X + 73728 f_1^2 R_2^4 X^2 - 73728 f_1^6 R_2^4 X \]

\[ + 248832 f_1^3 R_2^3 V_3^3 + 2376 f_1^2 R_2^3 V_3^2 - 497664 f_1^3 R_2^2 V_3^2 \]

\[ + 3 f_1 R_2^6 V_3 ) / 4924686192529637376, \]

\[ P_{69/2} = f_{21/2} R_2 (135 f_1 R_2^6 V_3^3 - 870912 f_1^5 R_2^4 V_3^3 + 71663616 f_1^9 R_2^2 V_3^3 \]

\[ + 5474304 f_1^8 R_2^4 V_3^2 - 35831808 f_1^6 R_2^2 V_3^4 + 9072 f_1^4 R_2^6 V_3^2 \]

\[ - 6718464 V_3^6 X^2 - 10077696 f_1 V_7^7 + 181398528 f_1^4 V_6^6 \]

\[ - 32248627 f_1 V_5^5 + 110592 f_1^{10} R_2^6 + 26542080 f_1^9 R_2^2 V_3 X \]

\[ + 64512 f_1^5 R_2^6 V_3 X - 11114496 f_1^8 R_2^4 V_3 X^2 - 31850496 f_1^7 R_2^4 V_3 X^2 \]

\[ + 2448 f_1^2 R_2^6 V_3^2 X + 752467968 f_1^5 V_3^5 X - 648 R_2^4 V_3^4 X \]

\[ + 68040 f_1^2 R_2^4 V_3^4 + 18 f_1^{10} R_2^8 V - 76032 f_1^7 R_2^6 V \]

\[ - 7962624 f_1^{11} R_2^6 V - 36864 f_1^4 R_2^6 V_3 X^3 + 107495424 V_3^5 X^3 f_1 \]

\[ + 128 f_1^4 R_2^8 X - 537477120 f_1^3 V_3^5 X^3 - 134369280 f_1^2 V_3^6 X \]
Then we have the following structure theorem.

**Theorem 1.10.** The vector space $M^+ (\Gamma, \psi)$ is a free $A$ module of rank 8, and we have

$$M^+ (\Gamma, \psi) = S^+ (\Gamma, \psi)$$

$$= AP_{41/2} \oplus AP_{53/2} \oplus AP_{57/2} \oplus AP_{69/2} \oplus AP_{47/2} \oplus AP_{51/2} \oplus AP_{63/2}.$$
2. Proofs on explicit structures of modular forms

2.1. Generators

We quote the theta transformation formula from Igusa [11, p. 227]. For even theta characteristics \( m = t^t (m', m'') \) with \( m', m'' \in \mathbb{Z}^2 \) (column vectors) and \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2, \mathbb{Z}) \), we write

\[
M \cdot m = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} m + \begin{pmatrix} (c^t d)_0 \\ (a^t b)_0 \end{pmatrix},
\]

where for any symmetric matrix \( x \), we denote \( x_0 \) the vector whose components consist of diagonal elements of \( x \). Then we get

\[
\theta_{M,m}(M\tau) = \kappa(M)e(\phi_m(M)) \det(c \tau + d)^{\frac{1}{2}} \theta_m(\tau),
\]

where \( \kappa(M) \) is a certain eighth root of unity, \( e(x) = e^{2\pi i x} \) and

\[
\phi_m(M) = -\frac{1}{8} (t m' t b m' + t m' t a c m'' - 2 t m' t b c m'' - 2 (a^t b)_0 (d m' - c m'')).
\]

For any natural number \( N \), we denote by \( \Gamma(N) \) the principal congruence subgroup of \( Sp(2, \mathbb{Z}) \) of level \( N \). Then \( \Gamma \supset \Gamma(2) \), and any coset in \( \Gamma / \Gamma(2) \) is represented by some element of the form \( M = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \).

\textit{Proof of Proposition 1.2.} The assertion for \( X, Y, K \) are in [7]. We shall show the rest. For the above \( M \) with \( b = c = 0 \), we get \( \phi_m(M) = 0 \) for all \( m \), and

\[
M \cdot m = \begin{pmatrix} d m' \\ a m'' \end{pmatrix}.
\]

So, we get \( f_2 | 2 M = f_2 \), and the action \( f \to f|2 M \) gives a permutation on \( \{ (\theta_{0001})^4, (\theta_{0010})^4, (\theta_{0100})^4 \} \), or \( \{ (\theta_{0000})^4, (\theta_{0100})^4, (\theta_{1100})^4 \} \). Hence we get \( f_2, y_2 \in A_2(\Gamma) \). The assertion for modular forms of odd weight can be obtained similarly. Now, we show the relation of modular forms given in the proposition. Using the notation of Igusa [10], we put

\[
y_0 = (\theta_{0110})^4, \quad y_1 = (\theta_{0100})^4, \quad y_2 = (\theta_{0000})^4,
\]

\[
y_3 = -(\theta_{1000})^4 - (\theta_{0110})^4, \quad y_4 = - (\theta_{1100})^4 - (\theta_{0110})^4.
\]

It is known that these forms generate the graded ring \( A_{even}(\Gamma(2)) \) of even weights modular forms with the fundamental relation

\[
(y_1 y_1 + y_0 y_2 + y_1 y_2 - y_3 y_4)^2 - 4y_0y_1y_2(y_0 + y_1 + y_2 + y_3 + y_4) = 0.
\]

Using Riemann’s theta relation (cf. [10, Lemma 1]), and using the reduction process in Igusa [10, p. 393], we get
\[ f_2 = y_2, \]
\[ g_2 = -2y_0 + y_1 + y_2 - y_3 - y_4, \]
\[ X = (y_0 + y_1 + 4y_2 + 2y_3 + 2y_4)/4, \]
\[ Y = (-y_0y_1 + y_0y_2 + y_1y_2 + y_3y_4 + 2y_0^2 + 2y_2y_3 + 2y_2y_4)/2, \]
\[ K = (-y_0^2y_1 + y_0^2y_2 - y_0y_1^2 - 2y_0y_1y_2 - 2y_0y_1y_3 - y_0y_1y_4 - 2y_0y_1y_4 + y_1^2y_2 - y_1y_3y_4)/8192, \]
\[ f_2^2 = (y_2 + y_3)(y_2 + y_4)(y_0 + y_1 + y_2 + y_3 + y_4). \]

By direct calculation, we get
\[ f_3^2 = -4096K + f_2(4g_2X - 6f_2g_2 + 24f_2X + g_2^2 - 8X^2)/9 + (4X - 2f_2)Y. \]

Hence we prove (1) and (3). Now we show that \( X, f_2, g_2, K \) are algebraically independent. We define the Witt operator \( W \) on any function \( F(Z) \) on \( H_2 \) by
\[ (WF)(\tau_1, \tau_2) = F \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}. \]

For \( i = 1, 2 \), we put \( x_i = \theta_0(\tau_i), y_i = \theta_{10}(\tau_i), z_i = \theta_{00}(\tau_i) \). It is well known that \( z_i^4 = x_i^4 + y_i^4 \). We get
\[ W(X) = (x_1^4 + z_1^4)(x_2^4 + z_2^4)/4, \]
\[ W(f_2) = (z_1z_2)^4, \]
\[ W(g_2) = \prod_{i=1}^2(2z_i^4 - x_i^4), \]
\[ W(K) = 0. \]

Since the four forms \( x_1, x_2, z_1, z_2 \) are algebraically independent, three forms \( W(X), W(f_2), W(g_2) \) are also algebraically independent. Now, assume that \( P(X, f_2, g_2, K) = 0 \) for a polynomial \( P(X_1, X_2, X_3, X_4) \) of four variables. Writing \( P = P_1(X_1, X_2, X_3) + X_4P_2(X_1, X_2, X_3, X_4) \) and applying \( W \), we get
\[ P_1(W(X), W(f_2), W(g_2)) = 0. \]

Hence we get \( P_1 = 0 \) as a polynomial. Hence \( P_2(X, f_2, g_2, K) = 0 \). Since the degree of \( P_2 \) is smaller than the one of \( P \), we get \( P = 0 \) by induction. By using the relation between \( f_3^2 \) and \( K \), we also get that \( f_2, g_2, X \) and \( K \) are algebraically independent. \( \square \)

**Lemma 2.1.** If \( F + YG = 0 \) for any \( F, G \in B \), then \( F = G = 0 \).

**Proof.** Let \( P_i \) (\( i = 1, 2 \)) be polynomials of four variables and assume that
\[ (2.1) \quad F + YG = 0 \]
for \( F = P_1(X, f_2, g_2, K) \) and \( G = P_2(X, f_2, g_2, K) \). For each \( i = 1, 2 \), we take polynomials \( Q_{1i} \) of three variables and \( Q_{2i} \) of four variables such that
\[ P_i(X_1, X_2, X_3, X_4) = Q_{1i}(X_1, X_2, X_3) + X_4Q_{2i}(X_1, X_2, X_3, X_4). \]
Taking the image of the Witt operator $W$ of both sides of (2.1), we get

$$Q_{11}(W(f_2), W(g_2), W(X)) + W(Y)Q_{21}(W(f_2), W(g_2), W(X)) = 0.$$ 

Since we have $W(Y) = (z_1 z_2 x_1 x_2)^4 = W(f_2(g_2 - 6 f_2 + 8 X))/3$ and three forms $W(f_2), W(g_2), W(X)$ are algebraically independent, we get

$$Q_{11}(X_1, X_2, X_3) = -\frac{1}{3}X_1(-6X_1 + X_2 + 8X_3)Q_{21}(X_1, X_2, X_3).$$

So, if we put $f = Y - f_2(g_2 - 6 f_2 + 8X)/3$, then

$$fQ_{21}(f_2, g_2, X) + K(Q_{12}(f_2, g_2, X, K) + YQ_{22}(f_2, g_2, X, K)) = 0.$$ 

Now, we shall show $Q_{21} = 0$. Put

$$\gamma = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

Then, we get

$$(\theta_{0000})^4|_{2\gamma} = (\theta_{0000})^4,$$

$$(\theta_{1000})^4|_{2\gamma} = (\theta_{1000})^4,$$

$$(\theta_{0100})^4|_{2\gamma} = (\theta_{0100})^4,$$

$$(\theta_{1100})^4|_{2\gamma} = (\theta_{1111})^4,$$

and $X|_{2\gamma} = X, Y|_{4\gamma} = Y, K|_{6\gamma} = K$. Hence $W(f|_{4\gamma}) = z_1^4 z_2^4 (x_1^4 - z_1^4) (x_2^4 - z_2^4) \neq 0$. We get $W(f_2|_{2\gamma}) = (z_1 z_2)^4, W(g_2|_{2\gamma}) = (z_1^4 + x_1^4) (x_2^4 + z_2^4) - 3x_1^4 x_2^4$, $W(X|_{2\gamma}) = (x_1^4 + z_1^4) (x_2^4 + z_2^4)/4$, so these three forms are also algebraically independent, and since $W(K|_{6\gamma}) = 0$, we get $Q_{21} = 0$ as a polynomial. Hence, we get

$$P_{12}(f_2, g_2, X, K) + Y P_{22}(f_2, g_2, X, K) = 0.$$ 

Since the degree of $P_{12}$, or $P_{22}$ is less than the degree of $P_4$, or $P_2$, respectively, we get $P_{12} = P_{22} = 0$ by induction. Hence $F = G = 0$. q.e.d.

It is also obvious that $f_1 B + f_3 B$ is a direct sum. By comparing the dimensions, we get Theorem 1.1 and 1.2.

Finally, we shall prove Theorem 1.3. We shall show

**Proposition 2.1.** If $f \in M_{k+1/2}(\Gamma, \psi)$, then $f/\theta_{0000}$ is holomorphic on $H_2$.

Theorem 1.3 and Corollary are easily obtained from this. For the proof of this Proposition, we use the explicit structure of $M(\Gamma, \psi)$. We need several Lemmas.
Lemma 2.2. For any \( F \in C = \mathbb{C}[g_2, X, K] \), assume that \( F/f_1 \) is holomorphic. Then \( F = 0 \).

Proof. By Theorem 1.1, it is easy to see that \( \sum_{k=1}^{\infty} M_{2k-1}(\Gamma, \psi) = f_1B \oplus f_1YB \oplus f_0C \). Since \( F/f_1 \in M_{2k-1}(\Gamma, \psi) \) for some integer \( k \), we write \( F = f_1(f_1\alpha_1 + f_1\alpha_2Y + f_3\alpha_3) \) for some \( \alpha_1, \alpha_2 \in B \) and \( \alpha_3 \in C \). Hence \( F - f_2\alpha_1 = f_2Y\alpha_2 + Y\alpha_3 \in B \cap YB = \{0\} \) and we get \( F = f_2\alpha_1 \). Since \( F \in C \) and \( f_2, g_2, X, K \) are algebraically independent, we get \( F = 0 \).

Lemma 2.3. For any \( F \in M(\Gamma, \psi) = \oplus_{k=0}^{\infty} M_k(\Gamma, \psi^k) \), assume that \( F/\theta_{0000} \) is holomorphic. Then, \( F/f_1 \) is also holomorphic.

Proof. First, we assume that \( F \) is of odd weight. Then \( F = f_1\alpha_1 + f_3\alpha_2 \) for \( \alpha_2 \in C \) and \( \alpha_1 \in B + YB \). Since \( F/\theta_{0000} \) is holomorphic, and \( f_1 = \theta_{0000}^2 \), we see \( (f_3\alpha_2/\theta_{0000}) \) is holomorphic and hence \((f_3\alpha_2/\theta_{0000})^2 \) also. Since \( f_2^2 = -4096K + f_1h \) for some holomorphic function \( h \), we see \( K\alpha_2^2/f_1 \) is also holomorphic. Since the numerator belongs to \( C \), we get \( \alpha_2 \) to be 0. Hence \( F/f_1 = \alpha_1 \) is holomorphic. Secondly, we assume \( F \) is of even weight. We write \( F = f_1 + f_2\alpha_2 + Y\alpha_3 \), where \( \alpha_1 \in C \), \( \alpha_2 \), \( \alpha_3 \in B \). Since \( f_2 = \theta_{0000}^2, Y = \theta_{0000}^2 f_3 \), and \( F/\theta_{0000} \) is holomorphic, we see \( \alpha_1/\theta_{0000} \) is holomorphic, hence \( \alpha_1^2/f_1 \) also. Since \( \alpha_1^2 \in C \), we get \( \alpha_1 \) to be 0 by the previous lemma. Hence \( F/f_1 = f_1\alpha_2 + f_3\alpha_3 \) is holomorphic.

Proof of Proposition 2.1. Since \( f \in M_{k+1/2}(\Gamma) \), we see that \( F := \theta_{0000} f \in M_{k+1}(\Gamma, \psi^{k+1}) \). Since \( f = F/\theta_{0000} \) is holomorphic, \( F/f_1 = f/\theta_{0000} \) is again holomorphic by Lemma 2.3.

2.2. Cusp forms

We define a maximal parabolic subgroup \( P_1(\mathbb{Q}) \) of \( \text{Sp}(2, \mathbb{Q}) \) corresponding to the one dimensional cusps by

\[
P_1(\mathbb{Q}) = \left\{ \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \right\} \in \text{Sp}(2, \mathbb{Q}) \right\}.
\]

We can take a complete set of representatives of \( \Gamma \setminus \text{Sp}(2, \mathbb{Q})/P_1(\mathbb{Q}) \) as follows.

\[
M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},
\]

\[
M_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.
\]
A modular form $F$ of $\Gamma$ is a cusp form, if and only if $\Phi(F|_k M_1) = 0$ for all $i$ with $1 \leq i \leq 4$, where $\Phi$ is the usual Siegel $\Phi$-operator. For the characteristic $m \in \mathbb{Z}^+$ whose second component is odd, we get $\Phi(\theta_m) = 0$. Otherwise, we get $\Phi(\theta_{m_1,0,m_3,m_4}) = \theta_{m_1,m_3}$. Now we put $x = \theta_{01}$ and $z = \theta_{00}$. Then $\theta_{10} = z^4 - x^4$. By the theta transformation formula, the forms $\Phi(F|M) = \Phi(F|_k M_1)$ is obtained for generators $F$ as follows.

1. $\Phi(K) = \Phi(f_3 + f_1 f_2 - 2X f_1) = \Phi(2X - 3f_2 + g_2) = 0$. $\Phi(f_1) = z^2$ and $\Phi(X) = (x^4 + z^4)/2$ are algebraically independent.
2. $\Phi(K|M_2) = \Phi(f_3 M_2) = \Phi((g_2 + 8X - 6f_2)|M_2) = 0$. $\Phi(f_1|M_2) = z^2$ and $\Phi(X|M_2) = (x^4 + z^4)/4$ are algebraically independent.
3. $\Phi(K|M_3) = \Phi(f_3 M_3) = \Phi(f_3 |M_3) = 0$. $\Phi(X|M_3) = (x^4 + z^4)/4$ and $\Phi(g_2| M_3) = 2z^4 - x^4$ are algebraically independent.
4. $\Phi(K|M_4) = \Phi(f_3 M_4) = \Phi((g_2 - 4X)|M_4) = 0$. $\Phi(f_1|M_4) = z^2$ and $\Phi(X|M_4) = (x^4 + z^4)/4$ are algebraically independent.

For any $f = \theta_{0000}F \in M_{k+1/2}(\Gamma)$ with odd $k$, we see that $f$ is a cusp form if and only if $F$ is a cusp form. Indeed, $\Phi(\theta_{0000}|M_1) = 0$ only for $i = 3$ in the above four cases. In this case, we also get $\Phi(f_1| M_3) = \Phi(f_3|M_3) = 0$, so $\Phi(F|M_3) = 0$ for any $F = f_1 G + f_2 H \in M_k(\Gamma, \psi)$. When $k$ is even, the above observation is false. For example, $F = (2X - 3f_2 + g_2)(g_2 + 8X - 6f_2)(g_2 - 4X)$ is not a cusp form, while $\theta_{0000} F$ is a cusp form. Anyway, in order to show that $f = \theta_{0000} F$ is a cusp form, we have to check only the conditions (1), (2), (4) for $F$ of integral weight. We first treat the case of odd weight for $\Gamma$. Since $K$ is a cusp form, we can assume that

$$F = f_1(P_1(X,f_2,g_2) + Y P_2(X,f_2,g_2)) + f_3 p_3(X,g_2),$$

up to the ideal generated by $K$. From (4) above, we get $P_1(X_1,X_2,4X_1) = 0$ and hence $P_1 = (X_3 - 4X_1)Q_1(X_1,X_2,X_3)$ for some polynomial $Q_1$. Also from (2), we get $P_1 = (X_3 + 8X_1 - 6X_2)(X_3 - 4X_1)Q_2(X_1,X_2,X_3)$ for some polynomial $Q_2$. Now, we put $X_2 = X_2 - (2X_1 + X_3)/3$. Then from the condition (1), we get

$$-9(4X_1 - X_3)^2 Q_2(X_1, X_2^* + (2X_1 + X_3)/3, X_3) + (2X_1 + X_3)(4X_1 - X_3) P_2(X_1, X_2^* + (2X_1 + X_3)/3, X_3) + 3(4X_1 - X_3) P_3(X_1, X_2^* + (2X_1 + X_3)/3, X_3) = 0.$$

This means that there are polynomials $R_i$ ($i \leq i \leq 4$) such that

$$P_1 = (4X_1 - X_3 - 6X_2^*)(X_3 - 4X_1)(R_1(X_1,X_3) + X_2^* R_2 (X_1,X_2^*,X_3),$$

$$P_2 = R_3(X_1,X_3) + X_2^* R_4 (X_1,X_2^*,X_3),$$

$$P_3 = 3(4X_1 - X_3) R_1(X_1,X_3) + \frac{1}{3}(2X_1 + X_3) R_3(X_1,X_3).$$

Hence, $\theta_{0000} F$ is a cusp form for odd weight $F$, if and only if
for some polynomials \( R_i \) (1 \( \leq \) i \( \leq \) 7), where we put \( f'_2 = f_2 - (2X + g_2)/3 \). By the structure Theorem 1.1, we can see that the above polynomials \( R_i \) depends only on \( F \). Hence, the generating function of the dimension of cusp forms is given by

\[
\frac{2t^5}{(1-t^2)^2} + \frac{2t^7}{(1-t^2)^3} + \frac{t^9(1+t^4)}{(1-t^2)^3(1-t^6)} + \frac{t^9}{(1-t^2)^2(1-t^6)}.
\]

Now, we assume that \( k \) is even. Then, we can assume \( F = P_1(X, f_2, g_2) + YP_2(X, f_2, g_2) \) as before. By the condition (2) and (4), we get \( P_1(X_1, X_2, X_3) = (X_3 - 4X_1)(X_3 + 8X_1 - 6X_2)Q_1(X_1, X_2, X_3) \) for some polynomial \( Q_1 \). Now we put \( X_3^* = X_3 + 2X_1 - 3X_2 \). We write

\[
Q_1(X_1, X_2, X_3) = R_1(X_1, X_2) + X_3^*R_2(X_1, X_2, X_3^*),
\]

\[
P_2(X_1, X_2, X_3) = R_3(X_1, X_2) + X_3^*R_4(X_1, X_2, X_3^*).
\]

Then by the condition (1), we get

\[
9(X_3 - 2X_1)R_1(X_1, X_2) + X_2R_3(X_1, X_2) = 0.
\]

So, we get \( R_1(X_1, X_2) = X_2R_5(X_1, X_2) \) and \( R_3(X_1, X_2) = 9(2X_1 - X_2)R_5(X_1, X_2) \) for some polynomial \( R_5 \). Hence, any modular form \( \theta_{0000} F \) with \( F \) with even weight is a cusp form if and only if

\[
F = Y(g_2 + 2X - f_2)R_4(X, f_2, g'_2)
\]

\[
+ Y(g_2 - 4X)(g_2 + 8X - 6f_2)(g_2 + 2X - 3f_2)R_2(X, f_2, g'_2)
\]

\[
(2f_2 - 4X)(g_2 + 8X - 6f_2) + 9Y(2X - f_2)R_5(X, f_2)
\]

\[
+ K(R_6(X, f_2, g_2, K) + YR_7(X, f_2, g_2, K))
\]

for some polynomials \( R_i \) (i = 2, 4, 5, 6, 7), where we put \( g'_2 = g_2 + 2X - 3f_2 \). We can show by the structure Theorem 1.1 that these polynomials \( R_i \) are uniquely determined by \( F \). Hence the generating function of the dimension of cusp forms is given by

\[
\frac{2t^6}{(1-t^2)^3} + \frac{t^6}{(1-t^2)^2} + \frac{t^6(1+t^4)}{(1-t^2)^3(1-t^6)}.
\]

Thus we complete the proof of Theorem 1.4 and its Corollary 1.3. \( \square \)
Let $f \in M_{k-1/2}(\Gamma, \psi)$, then $\theta_{0000} f \in M_k(\Gamma, \psi^{k+1})$. By Theorem 1.1, 1.2, we have

$$\oplus_{k=0}^{\infty} M_k(\Gamma, \psi) = f_{11}(B \oplus YB \oplus f_1 B \oplus f_3 B)$$

$$= f_{11} C[f_1, X, g_2, f_3]$$

$$= f_{11} M(\Gamma, \psi).$$

Moreover, $f_{11} = \theta_{0000} f_{21/2}$. Hence we have Theorem 1.5.

Let $M_i$ ($i=1,2,3,4$) be representatives of $\Gamma \backslash Sp(2, \mathbb{Q})/P_i(\mathbb{Q})$ which were defined before. For each $i$, we see $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in M_{i+1} \Gamma M_i$. So, for any $f \in M_{k-1/2}(\Gamma, \psi)$, we have $\Phi(f|M_i) = \Phi(f|M_i \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}) = -\Phi(f|M_i)$.

Hence $f$ belongs to $S_{k-1/2}(\Gamma, \psi)$. Thus we prove Corollary 1.4.

Corollary 1.5 is obvious by Theorem 1.5 and Corollary 1.4.

### 2.3. Plus space

**Proof of Theorem 1.6.** This is mostly known by [9] and [3]. The remaining case uncovered by these papers can be easily proved in the same way and the proof is omitted here.

**Proof of Theorem 1.8.** Although the plus space is originally defined for $\Gamma_0(4)$, we are taking its conjugate $\Gamma$ for some convenience of calculation. As we explained, the original plus space is obtained by taking $f(2\tau)$ for our $f(\tau)$ for $\Gamma$. Now if the basis of $M_{k-1/2}(\Gamma)$ is concretely given, the basis of the plus space $M_{k-1/2}^+(\Gamma)$ is obtained in principle as follows. The condition of the plus space is the linear conditions on Fourier coefficients. By demanding this condition for Fourier coefficients at several $T$, we get a linear subspace $M$ of $M_{k-1/2}(\Gamma)$ which contains $M_{k-1/2}^+(\Gamma)$. If we impose the condition at more $T$, the space $M$ becomes smaller or is unchanged. Since we know $\dim M_{k-1/2}^+(\Gamma)$ by Theorem 1.7, we can continue the process until we get $\dim M = \dim M_{k-1/2}^+(\Gamma)$ and we get the plus space.

For example, in the case of weight $7/2$, a basis of $M_{7/2}^+(\Gamma)$ is given by $\theta_{0000} f_3^1$, $\theta_{0000} f_1 g_2$, $\theta_{0000} f_1 X$, and $\theta_{0000} f_3$. For any $f \in M_{k-1/2}(\Gamma)$, we have the following Fourier expansion

$$F(\tau) = \sum_T c(T) e\left(\frac{1}{2} tr(T \tau)\right)$$

where $T$ runs over half integral symmetric matrices. We give Fourier coefficients of the above four forms in the following table, where $(a, c, b)$ means the Fourier
coefficient \( c(T) \) at \( T = \left( \begin{array}{c} a \\ b/2 \\ c \end{array} \right) \).

\[
\begin{array}{|c|c|c|c|}
\hline
& \theta_{0000} f_1^3 & \theta_{0000} f_1 g_2 & \theta_{0000} f_1 X & \theta_{0000} f_3 \\
\hline
(1, 0, 0) & 14 & 30 & 6 & -2 \\
(1, 1, 0) & 168 & 456 & 24 & 8 \\
(1, 1, 1) & 0 & 192 & 0 & 0 \\
\hline
\end{array}
\]

Writing down the condition that a linear combination vanishes at \((1, 0, 0), (1, 1, 0)\) and \((1, 1, 1)\), the following modular form is the unique candidate of the element of the plus space up to constant:

\[-3\theta_{0000} f_1^3 + 14\theta_{0000} f_1 X + 21\theta_{0000} f_3.\]

Since \( \dim M_{7/2}(\Gamma) = 1 \), this actually belongs to the plus space. By similar method, we can give basis of \( M_{k-1/2}(\Gamma) \) for \( k = 7/2, 11/2, 19/2, 23/2, 1/2, 13/2, 17/2, 29/2 \), using the fact that the dimension of the plus space of each of these weights is 1, 1, 3, 3, 1, 2, 2, 5, respectively. We also see that \( P_{k-1/2} \in M_{k-1/2}(\Gamma) \). Now, we would like to show that these elements are linearly independent over \( \mathbb{A} \). A set of generators of \( \oplus_{k=0}^{\infty} M_{2k}(\text{Sp}(2, \mathbb{Z})) \) is given by

\[
E_4(\tau) = 4X^2 - 3Y + 12288Z,
E_6(\tau) = -8X^3 + 9XY + 73728XZ - 27648K,
\chi_{10}(\tau) = YK,
\chi_{12}(\tau) = 3Y^2Z - 2XYK + 2072K^2,
\]

where \( E_4(\tau) \) and \( E_6(\tau) \) are the Eisenstein series of weight 4 or 6 with constant term one as before and \( \chi_{10} \) or \( \chi_{12} \) is a cusp form of weight 10 or 12, respectively, which is normalized so that the coefficient at \( \left( \frac{1}{2}, \frac{1}{2} \right) \) is one. (cf. Igusa [10]). We write \( E_k^*(\tau) = E_k(2\tau) \) for \( k = 4, 6 \) and \( \chi_k^*(\tau) = \chi_k(2\tau) \) for \( k = 10, 12 \). Then we have \( A = \mathbb{C}[E_4^*, E_6^*, \chi_{10}^*, \chi_{12}^*] \). We also get

\[
E_4^* = (-768 Z + X^2 + 3Y) / 4,
E_6^* = (-3456 K - X^3 + 9XY + 1152XZ) / 8,
\chi_{10}^* = KZ,
\chi_{12}^* = (3K^2 - 2KXZ + 3YZ^2) / 4.
\]

Let \( W \) be the Witt operator defined before. For \( i = 1, 2 \), we put \( x_i = \)
I, G

It is clear that \( A_{0} \), since the weights of the forms are \( A_{0} \) for the latter. We shall show linear independence of four forms \( k_{P_{10}} \) we get

\[
W(\theta_{0000}P_{7/2}) = 2^{-6} \prod_{i=1}^{2} (x_{i}^{4} + y_{i}^{4}) (8 x_{i}^{4} + y_{i}^{4}),
\]

\[
W(\theta_{0000}P_{11/2}) = 2^{-10} \prod_{i=1}^{2} (x_{i}^{4} + y_{i}^{4}) (32 x_{i}^{8} + 20 x_{i}^{4} y_{i}^{4} - y_{i}^{8}),
\]

\[
W(\theta_{0000}P_{19/2}) = 2^{-23} \prod_{i=1}^{2} x_{i}^{4} y_{i}^{12} (x_{i}^{4} + y_{i}^{4}),
\]

\[
W(\theta_{0000}P_{23/2}) = 2^{-27} \prod_{i=1}^{2} x_{i}^{4} y_{i}^{12} (x_{i}^{4} + y_{i}^{4}) (4 x_{i}^{4} + 5 y_{i}^{4}).
\]

It is clear that the \( A \)-module spanned by \( P_{7/2}, P_{11/2}, P_{19/2}, P_{23/2} \) and the \( A \)-module spanned by \( P_{1/2}, P_{3/2}, P_{7/2}, P_{9/2}, P_{29/2} \) have no intersection except for \( 0 \), since the weights of the forms are \( k - 1/2 \) with even \( k \) for the former but odd \( k \) for the latter. We shall show linear independence of four forms \( P_{7/2}, P_{11/2}, P_{19/2}, P_{23/2} \) over \( A \). Linear independence of four forms \( P_{1/2}, P_{3/2}, P_{7/2}, P_{29/2} \) are shown almost in the same way and the proof will be omitted here.

We assume that there exist polynomials \( Q_{i}(X_{1}, X_{2}, X_{3}, X_{4})(i = 1, 2, 3, 4) \) which satisfy the following relation

\[
\sum_{i=1}^{4} Q_{i}(E_{4i}^{*}, E_{6i}^{*}, \chi_{10i}, \chi_{12i}) G_{i} = 0,
\]

where we put \( G_{1} = \theta_{0000}P_{7/2}, G_{2} = \theta_{0000}P_{11/2}, G_{3} = \theta_{0000}P_{19/2}, G_{4} = \theta_{0000}P_{23/2} \). If we define polynomials \( R_{i}, R_{i}' \) by

\[
Q_{i}(X_{1}, X_{2}, X_{3}, X_{4}) = R_{i}(X_{1}, X_{2}, X_{3}) + X_{3}R_{i}'(X_{1}, X_{2}, X_{3}, X_{4}) \quad (i = 1, 2, 3, 4)
\]

we get

\[
\sum_{i=1}^{4} R_{i}(E_{4i}^{*}, E_{6i}^{*}, \chi_{12i}) G_{i} + \chi_{10} \sum_{i=1}^{4} R_{i}'(E_{4i}^{*}, E_{6i}^{*}, \chi_{10i}, \chi_{12i}) G_{i} = 0.
\]
By taking the image of both sides under Witt operator, we get
\[
\sum_{i=1}^{4} R_i(W(E_i^0), W(E_i^0), W(\chi_{12}^i)) W(G_i) = 0.
\]

As we wrote before, the forms \(W(E_1^0), W(E_2^0), W(\chi_{12}^1), W(G_1), W(G_2), W(G_3), W(G_4)\) are polynomials of four algebraically independent variables \(x_1, x_2, y_1, y_2\). For each \(W(f)\) for \(E_i^0\) etc. above, we denote by \(W(f)\) the polynomial of \(x_1, x_2, y_2\) obtained by substituting \(y_1^4\) by \(-2x_1^4\). Then we get
\[
W(E_1^0) = -3 x_1 (16 x_2^8 + 16 x_2^4 y_2^4 + y_2^8) / 2^6,
W(E_2^0) = 0,
W(\chi_{12}^1) = -3 x_1^{24} y_2^8 (x_2^4 + y_2^4) / 2^{20},
W(G_1) = -3 x_1 (x_2^4 + y_2^4), (8 x_2^4 + y_2^4) / 2^8,
W(G_2) = 3 x_1^{12} (x_2^4 + y_2^4) (32 x_2^8 + 20 x_2^4 y_2^4 - y_2^8) / 2^8,
W(G_3) = x_1^{20} x_2^4 y_2^{12} (x_2^4 + y_2^4) / 2^{20},
W(G_4) = -3 x_1^{24} x_2^4 y_2^{12} (x_2^4 + y_2^4) (4 x_2^4 + 5 y_2^4) / 2^{23}.
\]

Now we write \(R_i\) as
\[
R_i(X_1, X_2, X_3) = R_{i,1}(X_1, X_3) + X_2 R_{i,2}(X_1, X_2, X_3).
\]

Dividing the relation into the part where the total degree is 0 mod 16 and 8 mod 16, we get
\[
R_{1,1}(W(E_1^0), W(\chi_{12}^1) W(G_1) + R_{4,1}(W(E_1^0), W(\chi_{12}^1) W(G_4) = 0.
\]

and
\[
R_{2,1}(W(E_1^0), W(\chi_{12}^1) W(G_2) + R_{3,1}(W(E_1^0), W(\chi_{12}^1) W(G_3) = 0.
\]

Since four forms \(x_1, x_2, y_1, y_2\) are algebraic independent, \(W(G_4)\) is divisible by \(y_2\), so \(R_{1,1}(W(E_1^0), W(\chi_{12}^1) W(G_1)\) must be divisible by \(y_2\). But \(W(G_1)\) is not divisible by \(y_2\), so the polynomial \(R_{1,1}(X_1, X_3)\) is also divisible by \(X_3\). In the same argument we see that \(R_{4,1}(X_1, X_3)\) is divisible by \(X_3\). Repeating the process, we see \(R_{1,1}(X_1, X_3) = R_{4,1}(X_1, X_3) = 0\). We get \(R_{2,1}(X_1, X_3) = R_{3,1}(X_1, X_3) = 0\) in the same way. So we get \(R_0(X_1, X_2, X_3) = 0\), and \(Q_1(X_1, X_2, X_3, X_4) = 0\). Thus we have proved Theorem 1.8.

The proof of Theorem 1.10 is almost same as the proof of Theorem 1.8. But the computation is more complicate. To determine \(P_{69/2}\), we need Fourier coefficients of basis of \(M_{69/2}(\Gamma, \psi)\) at \(\left(\frac{a}{4}, \frac{b}{2}, c\right)\) with \(0 \leq a \leq 20, 0 \leq c \leq 20,\) and \(0 \leq b \leq 40\). We omit details here.
Next we shall prove Theorem 1.9. By applying the Siegel $\Phi$ operator at each cusp, we can show that $P_{19/2}$, $P_{23/2}$, $P_{29/2}$ are all cusp forms. So, in order to prove the theorem, it is sufficient to determine cusp forms in $AP_{7/2} \oplus AP_{11/2}$ and in $AP_{1/2} \oplus AP_{13/2} \oplus AP_{17/2}$. We see

$$
\Phi(E_4^*) = 2^{-4} \left( 16x_1^8 + 16x_1^4y_1^4 + y_1^8 \right),
$$

$$
\Phi(E_6^*) = 2^{-6} \left( 2x_1^4 + y_1^4 \right) \left( 32x_1^8 + 32x_1^4y_1^4 - y_1^8 \right),
$$

$$
\Phi(\theta_{0000}P_{7/2}) = 2^{-3} \left( x_1^4 + y_1^4 \right) \left( 8x_1^4 + y_1^4 \right),
$$

$$
\Phi(\theta_{0000}P_{11/2}) = 2^{-5} \left( x_1^4 + y_1^4 \right) \left( 32x_1^8 + 20x_1^4y_1^4 - y_1^8 \right).
$$

We assume that $R_1(E_4^*, E_6^*)P_{7/2} + R_2(E_4^*, E_6^*)P_{11/2}$ is a cusp form for some polynomials $R_i(X_1, X_2)$. By definition we get

$$
R_1(\Phi(E_4^*), \Phi(E_6^*))\Phi(\theta_{0000}P_{7/2}) + R_2(\Phi(E_4^*), \Phi(E_6^*))\Phi(\theta_{0000}P_{11/2}) = 0.
$$

For $f = E_4^*$ etc., we denote by $(\Phi(f))_0$ the polynomial of $x_1$ obtained from $\Phi(f)$ by substituting $y_1^4$ by $-2x_1^4$. Then,

$$
(\Phi(E_4^*))_0 = -3x_1^8/4,
$$

$$
(\Phi(E_6^*))_0 = 0,
$$

$$
(\Phi(\theta_{0000}P_{7/2}))_0 = -3x_1^8/4,
$$

$$
(\Phi(\theta_{0000}P_{11/2}))_0 = 3x_1^{12}/8.
$$

Let $R_i(X_1, X_2) = R_{i,1}(X_1) + X_2 R_{i,2}(X_1, X_2)$. We get

$$
R_{1,1}(\Phi(E_4^*))_0 + R_{2,1}(\Phi(E_4^*))_0 + R_{1,1}(\Phi(E_6^*))_0 + R_{2,1}(\Phi(E_6^*))_0 = 0.
$$

Regarding this as an equality between polynomials of $x_1$, we get $R_i(X_1, X_2) = 0$. So, there are no cusp forms in $\mathbb{C}[E_4^*, E_6^*]P_{7/2} + \mathbb{C}[E_4^*, E_6^*]P_{11/2}$ except for $0$. By similar calculation, we can show that there are no cusp forms in $\mathbb{C}[E_4^*, E_6^*]P_{1/2} + \mathbb{C}[E_4^*, E_6^*]P_{13/2} + \mathbb{C}[E_4^*, E_6^*]P_{17/2}$ except for the ideal generated by $P_{25/2}$. Thus we complete the proof of Theorem 1.9. \hfill \Box

### 3. A lifting conjecture

#### 3.1. Statement of Conjecture

For Siegel cusp forms of half integral weight of degree two, we propose the following conjecture.

**Conjecture 3.1.** For any $f \in S_{2k-2}(SL(2, \mathbb{Z}))$ and $g \in S_{2k-4}(SL(2, \mathbb{Z}))$ which are common eigen forms of Hecke operators, there exists a common eigen form $F \in S_{2k-1}(\Gamma_0(4))$ of Hecke operators such that

$$
L(s, F) = L(s, f) L(s - 1, g).
$$
Here $L(s, f)$ and $L(s, g)$ are the usual Hecke $L$ function and $L(s, F)$ is the $L$ function of $F$ defined by Zhuravlev which will be reviewed in §3.2. This conjecture is based on numerical examples of Euler factors of cusp forms given in §3.5. Conceptually, it can also be regarded as half-integral analogue of vector valued version of Yoshida lifting in [19]. We have also some similar experimental results for Siegel cusp forms outside the plus space but our knowledge would be too vague to state any conjecture in that case.

As for the $L$ function of common eigen non-cusp forms of half integral weight, we can prove a theorem given below which is similar to the theorem by Zharkovskaya [20] for integral weight. The theorem below may be regarded as a non-cusp form version of the above conjecture, though this seems very different from usual liftings. Let $k$ be a positive integer and let $F$ be an element of $M_{k-1/2}^+(\Gamma_0(4))$. If $F$ is not a cusp form, then we get $\Phi(F) \neq 0$ and $\Phi(F)$ belongs to the plus space of modular forms of one variable.

**Theorem 3.1.** Let $F \in M_{k-1/2}^+(\Gamma_0(4))$ be a Hecke eigen form with $\Phi(F) \neq 0$. Then $\Phi(F)$ is an eigen form of $T_1(p^2)$ of weight $k - 1/2$, where $T_1(p^2)$ is the usual Hecke operator on modular forms of half integral weight of degree one at $p$. Besides we have

$$L(s, F) = L(s - 1, E_{2k-4}) L(s, \Phi(F)),$$

where $E_{2k-4}$ is the Eisenstein series in $M_{2k-4}(SL(2, \mathbb{Z}))$.

The proof of this theorem will be given in §3.4. Almost the same theorem was given in [4] for $M_{k-1/2}^+(\Gamma_0^{(n)}(4))$, but here we assumed that $f$ is in the plus space, hence our theorem includes the claim for Euler 2 factors too.

### 3.2. Hecke theory for Siegel modular forms of half integral weigh of degree 2 at odd prime

The Hecke theory at odd primes on Siegel modular forms of half integral weight is developed in Zhuravlev [21], [22]. We review his result in case of degree two. The definition of $L$ function is not very clearly written there in terms of Hecke operators, so we review some argument also. (See also the definition in [6]). As for modular forms of $\Gamma_0(4)$, two is a bad prime, but if we restrict ourselves to the plus space, we have a good theory also at two. We shall explain this in the next section.

Now we put

$$GSp^+(2, \mathbb{R}) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M(4, \mathbb{R}) ; \ M J'M = n(M) J , \ n(M) \in \mathbb{R}^+ \right\}.$$ 

We denote by $GSp^{\sim}(2, \mathbb{R})$ the covering group of $GSp^+(2, \mathbb{R})$ defined as follows. The underlying set of $GSp^{\sim}(2, \mathbb{R})$ consists of pairs $(M, \phi(\tau))$, where $M \in GSp^+(2, \mathbb{R})$ and $\phi(\tau)$ is any holomorphic function on $H_2$ such that $|\phi(\tau)| = |\det M|^{-1/4} |C \tau + D|^{1/2}$. The group operation on $GSp(2, \mathbb{R})$ is given
by \((M, \phi(\tau))(M', \phi'(\tau)) = (MM', \phi(M')\phi'(\tau))\). Then, we can embed \(\Gamma_0(4)\) into the group \(\widetilde{GSp}^+(2, \mathbb{R})\) by
\[
\Gamma_0(4) \ni M \mapsto (M, \theta(M\tau)\theta(\tau)^{-1}) \in \widetilde{GSp}^+(2, \mathbb{R}),
\]
where \(\theta(\tau) = \theta_{0000}(2\tau)\).

For any odd prime \(p\), we put
\[
\begin{pmatrix}
1 & p \\
p^2 & p
\end{pmatrix},
\]
\[
\begin{pmatrix}
1 & p^2 \\
p & p^2
\end{pmatrix}.
\]
Then \((K_s, p^{2s}) \in GSp^+(2, \mathbb{R}) (s = 1, 2)\). We put \(K_s = (K_s, p^{2s})\).

The left \(\tilde{\Gamma}_0(4)\)-coset decomposition of \(\tilde{\Gamma}_0(4)K_0\tilde{\Gamma}_0(4)\) is explicitly given by Zha-
ravlev [21]. For the sake of simplicity, we put
\[
X_0(p) = \tilde{\Gamma}_0(4)(p14, 1)\Gamma_0(4),
X_1(p) = \tilde{\Gamma}_0(4)K_1\Gamma_0(4),
X_2(p) = \tilde{\Gamma}_0(4)K_2\Gamma_0(4).
\]
We take a left \(\tilde{\Gamma}_0(4)\)-coset decomposition
\[
X_s(p) = \bigcup_v \tilde{\Gamma}_0(4)\tilde{M}_v,
\]
where \(\tilde{M}_v = (M_v, \phi_v(\tau)) \in GSp^+(2, \mathbb{R})\). We define an action of \(\tilde{M}_v = (M_v, \phi_v(\tau)) \in GSp^+(2, \mathbb{R})\) on \(F \in M_{k-2}(\Gamma_0(4), \psi)\) by
\[
F|_{k-2, \psi} M_v = n(M_v)^k \frac{\psi(M_v)}{\psi(M_v^{-1})} (\phi_v(\tau))^{-2k+1} F(M_v^{-1} \psi),
\]
and an action of \(X_s(p)\) by \(F| X_s(p) = \sum_v F|_{k-2, \psi} M_v\). By abuse of language, we denote these operators also by \(X_s(p)\) which are double cosets originally.

Let \(L_p\) be a Hecke ring generated by operators \(X_0(p)^{\pm 1}\), \(X_1(p), X_2(p)\). According to [22], this \(L_p\) is isomorphic to a certain ring of \(W_2\)-invariant polynomials \(\mathbb{C}W_2[x_0^{\pm 1}, x_1^{\pm 1}, x_2^{\pm 1}]\), where \(W_2\) is the automorphism group generated by the following elements:
\[
\sigma : x_0 \rightarrow x_0, x_1 \rightarrow x_2, x_2 \rightarrow x_1,
\]
\[
\sigma_i : x_0 \rightarrow x_0 x_i, x_i \rightarrow x_i^{-1}, x_j \rightarrow x_j \quad (i = 1, 2, \; i \neq j),
\]
\[
\sigma' : x_0 \rightarrow -x_0, x_1 \rightarrow x_1, x_2 \rightarrow x_2.
\]
By using Proposition 4.1 and Lemma 3.2 in [22], we see there is an isomorphism \(\phi : L_p \rightarrow \mathbb{C}W_2[x_0^{\pm 1}, x_1^{\pm 1}, x_2^{\pm 1}]\) such that
\[
\phi(X_0(p)) = p^{-3} x_0^2 x_1 x_2,
\]
\[
\phi(X_1(p)) = p^{-1} x_0^2 (x_1 + x_2)(1 + x_1 x_2),
\]
\[
\phi(X_2(p)) = x_0^2 (1 + x_1^2 + x_2^2 + (1 - p^{-2})x_1 x_2 + x_1^2 x_2^2).
\]
Define a polynomial \( \gamma(z) \) of \( z \) by \( \gamma(z) = \prod_{i=1}^{2}(1 - x_i z)(1 - x_i^{-1} z) \), and write its expansion as \( \gamma(z) = \sum_{j=0}^{4} (-1)^j R_j z^j \). By using the above three relations, the inverse image \( \phi^{-1}(R_j) \) of \( R_j \) is given:

\[
\begin{align*}
\phi^{-1}(R_0) &= \phi^{-1}(R_4) = 1, \\
\phi^{-1}(R_1) &= \phi^{-1}(R_3) = p^{-2}X_0(p)^{-1}X_1(p), \\
\phi^{-1}(R_2) &= p^{-3}X_0(p)^{-1}X_2(p) + (1 + p^{-2}).
\end{align*}
\]

We say that \( F \in M_{k-1/2}(\Gamma_0(4), \psi^j) \) is a Hecke eigen form, if \( F \) is a common eigen function for the action of \( X_1(p), X_2(p) \) for all odd prime \( p \). For a Hecke eigen form \( F \in M_{k-1/2}(\Gamma_0(4), \psi^j) \), we denote by \( \beta_{j,p} \) (\( j = 0, \ldots, 4 \)) the Hecke eigen value of \( F \) of \( \phi^{-1}(R_j) \). Then, there exists \( \alpha_{i,p}^\pm, \alpha_{2,p}^\pm \) which satisfy

\[
(1 - \alpha_{i,p}^\pm)(1 - \alpha_{i,p}^{-1} z^j) = \prod_{i=1}^{2}(1 - \alpha_{i,p} z)(1 - \alpha_{i,p}^{-1} z).
\]

The \( L \)-function of \( F \) is defined in Zhuravlev [22] by

\[
L(s, F) = \prod_{p} \prod_{i=1}^{2} (1 - \psi(p)^i \alpha_{i,p} p^{-s+k-3/2})^{-1} (1 - \psi(p)^i \alpha_{i,p}^{-1} p^{-s+k-3/2})^{-1}.
\]

We rewrite this by eigen values. We denote by \( \lambda(p) \) or \( \omega(p) \) the Hecke eigen values of \( F \) of \( X_1(p) \) or \( X_2(p) \), respectively. Then, we have

\[
L(s, F) = \prod_{p} \left( 1 - \lambda^*(p)p^{-s} + (p \omega(p) + p^{2k-5}(1 + p^2))p^{-2s} \\
- \lambda^*(p)p^{2k-3-3s} + p^{4k-6-4s} \\
\right)^{-1},
\]

where we put \( \lambda^*(p) = \lambda(p) \left( \frac{1}{p} \right)^{k+7/2} \). In the above product, at moment we defined Euler \( p \) factors only for odd primes, but an Euler 2 factor will be defined for elements of plus subspace later.

Next, we explain how to get eigen values by using Fourier coefficients of Hecke eigen forms. First, we prepare some notations. For \( i, j \in \{0, 1, 2\}, i + j \leq 2 \), we put \( d_{i,j} = \begin{pmatrix} 1_{2-i-j} & p1_i \\ p^2 1_j \end{pmatrix} \in M_2(\mathbb{Z}). \) We denote by \( M_{i,m}(p^k) \)
a complete set of representatives of matrices of \( M_{i,m}(\mathbb{Z}) \) modulo \( p^\delta \), and put

\[
R_{s,i,j} = \begin{cases}
B = \begin{pmatrix}
0_{2-i-j} & 0 & 0 \\
0 & A & pb_1 \\
0 & 1 & 1 \end{pmatrix}; & A \in M_{i,i}(p), B_1 \in M_{i,j}(p), B_2 \in M_{j,j}(p^2), \\
& A = A, B_2 = B_2, \text{ and } \text{rank}_p(A) = i - 2 + s
\end{cases}
\]

For a matrix \( B = \begin{pmatrix}
0_{2-i-j} & 0 & 0 \\
0 & A & pb_1 \\
0 & 1 & 1 \end{pmatrix} \in R_{s,i,j} \) and for a fixed \( \gamma = i - 2 + s \),
we define a function \( \kappa(B) \) by \( \kappa(B) = 1 \) or \( \varepsilon_p \left( \frac{(-1)^\gamma \det A}{p} \right) \) for \( \gamma = 0 \) or \( \gamma > 0 \),
respectively, where \( \varepsilon_p = 1 \) or \( \sqrt{-1} \) if \( p \equiv 1 \) or \( 3 \mod 4 \), \( A_1 \) is any matrix of size \( \gamma \) such that \( A \equiv A_1 (\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) \mod p \) for some \( M \in M_{i,i}(p) \cap SL(i, \mathbb{Z}) \), and \( (\frac{2}{p}) \) means the Legendre symbol. We write the Fourier expansion of \( F(\tau) \in M_{k-1/2}(\Gamma_0(p), \psi) \) as \( F(\tau) = \sum_{N \geq 0} C(N)e(\text{tr}(N\tau)) \). We define \( \alpha_{s,i,j}(T) \) by

\[
\alpha_{s,i,j}(T) = p^{(s+2-i-2j)(k-\frac{j}{2})-3s} \psi(p^{i+2})^T \\
\times \sum_{U,B} C \left( \frac{1}{p^2} d_{i,j} U^T U d_{i,j} \right) e \left( \frac{1}{p^2} \text{tr}(U^T U d_{i,j} B) \right) \kappa(B)^{-2k+1}.
\]

where the matrix \( B \) runs over all elements of \( R_{s,i,j} \), \( U \) runs over a complete set of representatives of \( (d_{i,j}^T SL(2, \mathbb{Z}) d_{i,j} \cap SL(2, \mathbb{Z})) \backslash SL(2, \mathbb{Z}) \), and we regard \( C(M) = 0 \) if \( M \) is not a half integral matrix.

Now let \( F \) be a Hecke eigen form and we assume that \( C(T) \neq 0 \) for some
semi-positive definite \( T = \left( \begin{array}{cc} a & b/2 \\ b/2 & c \end{array} \right) \). By using [21, Proposition 7.1], we have

\[
\lambda^*(p) = C(T)^{-1} \sum_{1 \leq i \leq s} \alpha_{1,i,i}(T), \omega(p) = C(T)^{-1} \sum_{0 \leq i \leq s} \alpha_{2,i,i}(T).
\]

Now for any prime \( p \), we put

\[
R(p) = \{ (a, \frac{x}{2}) : a \mod p \}, \\
R(p^2) = \{ (a, \frac{x}{2}) : a \mod p^2, y \mod p \}.
\]

Then we can calculate each \( \alpha_{s,i,j}(T) \) for odd \( p \) explicitly as follows.
\[ \alpha_{1,0}(T) = \sum_{U \in R(p)} C \left( \left( \begin{array}{c} p^{-1} 1 \\ \psi \end{array} \right) UT^4 \left( \begin{array}{c} p^{-1} 1 \\ \psi \end{array} \right) \right), \]
\[ \alpha_{1,1}(T) = \sum_{U \in R(p)} C \left( \left( \begin{array}{c} 1 \\ \psi \end{array} \right) UT^4 \left( \begin{array}{c} 1 \\ \psi \end{array} \right) \right), \]
\[ \alpha_{1,2}(T) = \left\{ \begin{array}{ll}
(\frac{(-1)^{k+i-3}}{p}) p^{k-2} C(T) & \text{if } p \nmid a \text{ and } p \mid \det 2T, \\
0 & \text{otherwise},
\end{array} \right. \]
\[ \alpha_{2,0}(T) = \sum_{U \in R(p)} C \left( \left( \begin{array}{c} p^{-1} \\ \psi \end{array} \right) UT^4 \left( \begin{array}{c} p^{-1} \\ \psi \end{array} \right) \right), \]
\[ \alpha_{2,1}(T) = \sum_{U \in R(p)} \left( \frac{(-1)^{k+i-1}a_U}{p} \right) C \left( \left( \begin{array}{c} 1 \\ \psi \end{array} \right) UT^4 \left( \begin{array}{c} 1 \\ \psi \end{array} \right) \right), \]
\[ \alpha_{2,2}(T) = \left\{ \begin{array}{ll}
-p^{2k-6} C(T) & \text{if } p \nmid \det 2T, \\
(p-1)p^{2k-6} C(T) & \text{if } p \mid \det 2T.
\end{array} \right. \]

3.3. Hecke theory on plus space at two

Although 2 is a bad prime for \( \Gamma_0(4) \), it is a good prime for the plus subspace, since it is isomorphic to the space of Jacobi forms of “level” one. Namely we know that for odd primes the Hecke theory of Jacobi forms and Siegel modular forms of half integral weight corresponds well (cf. [9], [3], also see the correction at the end of this paper.), and for \( F \in M_{k-1/2}^+(\Gamma_0(4), \psi^!) \) and for every odd prime \( p \), we can interpret \( X_1^+(p) = \psi(p)^4 p^{-k+7/2} X_1(p) \) or \( X_2(p) \) as a pull back of a Hecke operator on Jacobi forms. Now we can define operators \( X_1^+(2) \) and \( X_2(2) \) on \( M_{k-1/2}^+(\Gamma_0(4), \psi^!) \) as the pull backs of the same Hecke
operators at two on Jacobi forms. Hence we say that $F$ is a common eigen form if the image of $F$ in Jacobi forms of index one is a common eigen form of all the Hecke operators on Jacobi forms, including those at two. For $F$ we can define $\lambda^*(2)$ and $\omega(2)$ in the same way by using $X_1^*(2)$ and $X_2^*(2)$, and also the Euler 2 factor is defined in the same formula as in the case of odd primes. Hence $L(s, F)$ is defined as the product of Euler factors at all primes $p$ by the formula in the previous section.

The formula for $\lambda^*(2)$ and $\omega(2)$ using the Fourier coefficients is almost the same as in the odd case. Here we explain the necessary modification of the formula in the previous section. Let $C(T)$ be the Fourier coefficient of $F$ as before. If $C(T) \neq 0$ for $T = \begin{pmatrix} a & b/2 \\ b & c \end{pmatrix}$, then since $F$ belongs to the plus space, we get

$$a = (-1)^{k+l-1} \lambda_1^2 + 4\alpha,$$

$$b = (-1)^{k+l-1} 2\lambda_1 \lambda_2 + 4\beta,$$

$$c = (-1)^{k+l-1} \lambda_2^2 + 4\gamma,$$

where $(\lambda_1, \lambda_2) = (1, 1), (1, 0), (0, 1)$ or $(0, 0)$ and $\alpha, \beta, \gamma$ are integers. So, we get

$$\det(T) = 4(4\alpha\gamma - \beta^2) + 4(-1)^{k+l-1}(\alpha \lambda_2^2 + \gamma \lambda_1^2 - \beta \lambda_1 \lambda_2),$$

and hence $\det(T) \equiv 0 \mod 4$. The condition that $p \mid \det(2T)$ or $p \nmid \det(2T)$ in the previous formula should be replaced by the condition that $8 \nmid \det(T)$ or $8 \nmid \det(T)$, respectively. The Legendre symbol $\left( \frac{p}{x} \right)$ for odd $p$ in the formula before is now interpreted as follows. First of all, we can easily show that each $x$ which appears in the Legendre symbol, namely each of $x = (-1)^{k+l-1}a,$ $\begin{cases} -1 \end{cases}$

$$(-1)^{k+l-1}c,$$

$$(-1)^{k+l-1}a_U$$

or $(-1)^{k+l-1}c_U$ for $p = 2$ satisfies the condition $x \equiv 0$ or $1 \mod 4$. So, we define

$$\left( \frac{x}{p} \right) = \begin{cases} 0 & \text{if } x \equiv 0 \mod 4, \\ 1 & \text{if } x \equiv 1 \mod 8, \\ -1 & \text{if } x \equiv 5 \mod 8. \end{cases}$$

3.4. Proof of Theorem 3.1

For any prime $p$ including $p = 2$ and any eigen form $F \in \text{M}^{+}_{k-1/2}(\Gamma_0(4))$, we take the operators $X_1(p), X_2(p)$ such that $F|X_1(p) = p^{k-7/2}\lambda^*(p) F$ and $F|X_2(p) = \omega(p) F$ as before. First, we calculate Fourier coefficients of $F|X_1(p))$ and $F|X_2(p))$.

If a Siegel modular form $F$ has the Fourier expansion

$$F(\tau) = \sum_{T \geq 0} C(T) e(\text{tr}(T\tau)),$$

then we have $\Phi(F)(z) = \sum_{m \geq 0} c(m) e(mz)$, where we put $c(m) = C\left( \begin{array}{cc} m & 0 \\ 0 & 0 \end{array} \right)$. For $s = 1, 2$, we write $\Phi(F|X_s(p)))(z) = \sum_{m \geq 0} a_s(m) e(mz)$. Then we get $a_s(m)$


\[
= \sum_{2-s \leq i \leq j \leq 2} \alpha_{s,i,j} \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix},
\]

where \( \alpha_{s,i,j}(T) \) is the same notation as in the previous section. By straightforward calculation, we have:

\[
a_1(m) = c(m/p^2)p^{2k-3} + c(p^2m) + c(m) \left( p^{2k-4} + p + \left( \frac{(-1)^{k-1}m}{p} \right) p^{k-2} \right),
\]

\[
a_2(m) = c(m/p^2)(p^{4k-8} + p^{2k-3}) + c(p^2m)(p^{2k-5} + 1) + c(m) \left( p^{3k-7} \left( \frac{-1}{k-1} \right) m + p^{k-2} \left( \frac{-1}{k-1} \right) \right) + p^{2k-6}(p^2 - 1),
\]

where the symbol \( \left( \frac{2}{p} \right) \) for \( p = 2 \) is defined as before. Let \( f \) be a modular form of weight \( k - 1/2 \) of degree 1 in the plus-subspace and take its Fourier expansion \( f(z) = \sum b(m)e(mz) \). Then for any prime \( p \) including \( p = 2 \), the Hecke operator \( T_1(p^2) \) is defined by

\[
(f|T_1(p^2))(z) = \sum_{m \geq 0} \left( b(p^2m) + p^{2k-3}b(m/p^2) \right)
\]

\[
+ \left( \frac{-1}{k-1} \right) m + p^{k-2}b(m)) e(mz).
\]

Therefore we have

\[
\Phi(F)|p^{-k+7/2}X_1(p)) = \Phi(F)|T_1(p^2) + (p + p^{2k-4})\Phi(F),
\]

\[
\Phi(F)|X_2(p)) = (p^{2k-5} + 1)(\Phi(F)|T_1(p^2) + (p^{2k-4} - p^{2k-6})\Phi(F).
\]

If \( \Phi(F) \neq 0 \), then by the above relation, it is obvious that \( \Phi(F) \) is also an eigenform of \( T_1(p^2) \). If we put

\[
\Phi(F)|T_1(p^2) = \mu(p^2)\Phi(F),
\]

then we get

\[
\lambda^*(p) = \mu(p^2) + p + p^{2k-4},
\]

\[
\omega(p) = (p^{2k-5} + 1)\mu(p^2) + p^{2k-4} - p^{2k-6}.
\]

so we have,

\[
L(s, F) = \prod_p \left( (1 - \mu(p^2)p^{-s} + p^{2k-3-2s})(1 - p^{1-s})(1 - p^{2k-4-s}) \right)^{-1}
\]

\[
= L(s, \Phi(F))\zeta(s-1)\zeta(s-2k+4)
\]

\[
= L(s, \Phi(F))L(s-1, E_{2k-4}).
\]

This completes the proof of Theorem 3.1.
3.5. Numerical examples of Euler factors

In this section, we give some examples of Euler factors of forms in the plus space. The dimensions of the plus space and the space of elliptic modular forms are given in the following table for small weights.

<table>
<thead>
<tr>
<th>$k$</th>
<th>0 ~ 6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>dim $S_{k-1/2}(\Gamma_0(4))$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>dim $S_{2k-2}(SL(2,\mathbb{Z}))$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>dim $S_{2k-4}(SL(2,\mathbb{Z}))$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

For any Hecke eigen form $F \in S_{k-1/2}^+(\Gamma_0(4), \psi^j)$, we define the Hecke polynomial $H_p(T, F)$ at $p$ by

$$H_p(T, F) = 1 - \lambda^* (p) T + (p \omega(p) + p^{2k-5}(1 + p^2)) T^2 - \lambda^*(p)p^{2k-3}T^3 + p^{4k-6}T^4.$$ 

The dimension of cusp forms of plus space of weight 19/2 is 1. So $P_{19/2}(2\tau)$ is a Hecke eigen form, since the plus space is closed under the action of Hecke operators. Some of Fourier coefficients of $P_{19/2}(2\tau)$ are given as follows:

<table>
<thead>
<tr>
<th>wt $19/2$</th>
<th>$(3,3,2)$</th>
<th>$(24,3,0)$</th>
<th>$(11,8,8)$</th>
<th>$(19,4,4)$</th>
<th>$(27,27,18)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{19/2}(2\tau)$</td>
<td>1</td>
<td>-5022</td>
<td>-861</td>
<td>-3423</td>
<td>23088645</td>
</tr>
</tbody>
</table>

where $(a, c, b)$ means the Fourier coefficient at $\left( \frac{a}{b/2} \frac{b}{c} \right)$.

Next we calculate eigen values $\lambda^*(3)$ and $\omega(3)$ of $P_{19/2}(2\tau)$. We put $C(a, c, b) = C \left[ \left( \frac{a}{b/2} \frac{b/2}{c} \right) \right]$, then we have:

$$\lambda^*(3) = C(24,3,0) \times 2 + C(11,8,8) + C(19,4,4)$$

$$= -14328,$$

$$\omega(3) = C(3,3,2) \times 2 \times 3^{15} + C(27,27,18)$$

$$+ (C(11,8,8) - C(19,4,4)) \times 3^7 + C(3,3,2) \times (-3^{14})$$

$$= 52606584.$$

Then the Euler factor of $P_{19/2}(2\tau)$ at $p = 3$ is given by $H_3(3^{-s}, P_{19/2}(2\tau))$, where

$$H_3(T, P_{19/2}(2\tau)) = 1 + 14328T + 301308822T^2 + 1850320255464T^3 + 3^{34}T^4$$

$$= (1 + 10044T + 3^{17}T^2)(1 + 4284T + 3^{17}T^2).$$
We denote by $\Delta_{16}$ or $\Delta_{18}$ the normalized Hecke eigen form belonging to $SL(2,\mathbb{Z})$ of weight 16 or 18, respectively. For any common eigen form $f \in S_k(SL(2,\mathbb{Z}))$, we denote by $\lambda(p, f)$ the eigen value of the Hecke operator $T(p)$ on $f$ at prime $p$. We denote by $L_p(s, f)$ the Euler $p$-factor of the $L$ function of $f$. We see

$$\lambda(3, \Delta_{16}) = -3348, \lambda(3, \Delta_{18}) = -4284.$$ 

So, we get

$$H_3(3^{-s}, P_{19/2}(2\tau)) = L_3(s, \Delta_{18})L_3(s - 1, \Delta_{16}).$$

By similar calculation, we get $\lambda^* (2) = -96$ and $\omega(2) = -64896$ for $P_{19/2}(2\tau)$.

On the other hand, we see $\lambda(2, \Delta_{16}) = 216$, $\lambda(2, \Delta_{18}) = -528$, so we get

$$H_2(2^{-s}, P_{19/2}(2\tau)) = L_2(s, \Delta_{18})L_2(s - 1, \Delta_{16}).$$

Let $\Delta_k$ ($k = 20, 22, 26$) be the normalized cusp forms of weight $k$ in the one dimensional space $S_k(SL(2,\mathbb{Z}))$, and $\Delta_{24}^+, \Delta_{24}^-$ be a Hecke eigen basis of $S_{24}(SL(2,\mathbb{Z}))$. The Euler factors of Hecke eigen forms in the plus space of weight 21/2, 23/2, 25/2, 27/2, 29/2, 31/2, 33/2, and 35/2 are given as follows.

**weight 21/2.**

We have $\dim S^{+}_{21/2}(\Gamma_0(4)) = 1$ and this space $S^{+}_{21/2}(\Gamma_0(4))$ is generated by $\chi_{10}(4\tau)P_{1/2}(2\tau)$, which is of course a Hecke eigen form. We have

$$H_2(T, \chi_{10}(4\tau)P_{1/2}(2\tau)) = (1 + 1056T + 2^{19}T^2)(1 - 456T + 2^{19}T^2),$$

$$H_3(T, \chi_{10}(4\tau)P_{1/2}(2\tau)) = (1 + 12852T + 3^{19}T^2)(1 - 50652T + 3^{19}T^2),$$

and we also have

$$\lambda(2, \Delta_{18}) = -528, \quad \lambda(2, \Delta_{20}) = 456,$$

$$\lambda(3, \Delta_{18}) = -4284, \quad \lambda(3, \Delta_{20}) = 50652.$$ 

So we get

$$H_2(2^{-s}, \chi_{10}(4\tau)P_{1/2}(2\tau)) = L_2(s - 1, \Delta_{18})L_2(s, \Delta_{20}),$$

$$H_3(3^{-s}, \chi_{10}(4\tau)P_{1/2}(2\tau)) = L_3(s - 1, \Delta_{18})L_3(s, \Delta_{20}).$$

**weight 23/2.**

We have $\dim S^{+}_{23/2}(\Gamma_0(4)) = 1$ and $P_{23/2}(2\tau) \in S^{+}_{23/2}(\Gamma_0(4))$ is a Hecke eigen form. We have

$$H_2(T, P_{23/2}(2\tau)) = (1 - 912T + 2^{21}T^2)(1 + 288T + 2^{21}T^2),$$

$$H_3(T, P_{23/2}(2\tau)) = (1 - 151956T + 3^{21}T^2)(1 + 128844T + 3^{21}T^2),$$

and we also have

$$\lambda(2, \Delta_{20}) = 456, \quad \lambda(2, \Delta_{22}) = -288,$$

$$\lambda(3, \Delta_{20}) = 50652, \quad \lambda(3, \Delta_{22}) = -128844.$$
So we get

\[ H_2(2^{-s}, P_{23/2}(2\tau)) = L_2(s - 1, \Delta_{20}) L_2(s, \Delta_{22}), \]
\[ H_3(3^{-s}, P_{23/2}(2\tau)) = L_3(s - 1, \Delta_{20}) L_3(s, \Delta_{22}). \]

**weight 25/2.**

We have \( \dim S^+_{25/2}(\Gamma_0(4)) = 2 \). We put

\[ \chi^+_{25/2} = \left( 119 - \sqrt{144169} \right) \chi_{12}(4\tau) P_{1/2}(2\tau) + P_{25/2}(2\tau), \]
\[ \chi^-_{25/2} = \left( 119 + \sqrt{144169} \right) \chi_{12}(4\tau) P_{1/2}(2\tau) + P_{25/2}(2\tau). \]

Then \( \chi^{(\pm)}_{25/2} \in S^+_{25/2}(\Gamma_0(4)) \) and these are Hecke eigen forms. The Euler 2-factor and 3-factor of these forms are given by

\[ H_2(T, \chi^{(\pm)}_{25/2}) = (1 + 576T + 2^{23}T^2) \]
\[ \times \left( 1 - \left( 540 \mp 12\sqrt{144169} \right) T + 2^{23}T^2 \right), \]
\[ H_3(T, \chi^{(\pm)}_{25/2}) = (1 + 386532T + 3^{23}T^2) \]
\[ \times \left( 1 - \left( 169740 \pm 576\sqrt{144169} \right) T + 3^{23}T^2 \right). \]

The eigen values of \( \Delta_{22} \) and \( \Delta^{(\pm)}_{24} \) at 2 and 3 are given by

\[ \lambda(2, \Delta_{22}) = -288, \quad \lambda(2, \Delta^{(\pm)}_{24}) = 540 \mp 12\sqrt{144169}, \]
\[ \lambda(3, \Delta_{22}) = -128844, \quad \lambda(3, \Delta^{(\pm)}_{24}) = 169740 \pm 576\sqrt{144169}. \]

So we get

\[ H_2(2^{-s}, \chi^{(\pm)}_{25/2}) = L_2(s - 1, \Delta_{22}) L_2(s, \Delta^{(\pm)}_{24}), \]
\[ H_3(3^{-s}, \chi^{(\pm)}_{25/2}) = L_3(s - 1, \Delta_{22}) L_3(s, \Delta^{(\pm)}_{24}). \]

**weight 27/2.**

We have \( \dim S^+_{27/2}(\Gamma_0(4)) = 2 \). We put

\[ \chi^{(\pm)}_{27/2} = \left( -427 \pm \sqrt{144169} \right) \chi_{10}(4\tau) P_{7/2}(2\tau) + 9 E_4(4\tau) P_{19/2}(2\tau). \]

Then \( \chi^{(\pm)}_{27/2} \in S^+_{27/2}(\Gamma_0(4)) \) and these are Hecke eigen forms. We have

\[ H_2(T, \chi^{(\pm)}_{27/2}) = \left( 1 - \left( 1080 \mp 24\sqrt{144169} \right) T + 2^{25}T^2 \right) \left( 1 + 48T + 2^{25}T^2 \right), \]
\[ H_3(T, \chi^{(\pm)}_{27/2}) = \left( 1 - \left( 509220 \pm 1728\sqrt{144169} \right) T + 3^{25}T^2 \right) \left( 1 + 195804T + 3^{25}T^2 \right). \]
We also have
\[
\begin{align*}
\lambda(2, \Delta_{24}^{\pm}) &= 540 \pm 12 \sqrt{144169}, \\
\lambda(3, \Delta_{24}^{\pm}) &= 169740 \pm 576 \sqrt{144169},
\end{align*}
\]
\[
\lambda(2, \Delta_{26}) = -48, \\
\lambda(3, \Delta_{26}) = -195804.
\]
So we get
\[
\begin{align*}
H_2(2^{-s}, \chi_{27/2}^{\pm}) &= L_2(s - 1, \Delta_{24}^{\pm})L_2(s, \Delta_{26}), \\
H_3(3^{-s}, \chi_{27/2}^{\pm}) &= L_3(s - 1, \Delta_{24}^{\pm})L_3(s, \Delta_{26}).
\end{align*}
\]

**weight 29/2.**

We have \(\dim S^{+}_{29/2}(\Gamma_0(4)) = 2\). We put
\[
\chi^{(\pm)}_{29/2} = \left( 47 \pm \sqrt{18409} \right) E_4(4\tau) \chi_{10}(4\tau) P_{1/2}(2\tau) + 81 \chi_{29/2}(2\tau).
\]

Then \(\chi^{(\pm)}_{29/2} \in S^{+}_{29/2}(\Gamma_0(4))\) and these are Hecke eigen forms. We have
\[
\begin{align*}
H_2(T, \chi^{(\pm)}_{29/2}) &= (1 + 96T + 2^{27} T^2)(1 + (4140 \pm 108 \sqrt{18209}) T + 2^{27} T^2), \\
H_3(T, \chi^{(\pm)}_{29/2}) &= (1 + 587412 T + 3^{27} T^2) \\
&\quad (1 + (643140 \pm 20736 \sqrt{18209}) T + 3^{27} T^2).
\end{align*}
\]

We also have
\[
\begin{align*}
\lambda(2, \Delta_{26}) = -48, \\
\lambda(3, \Delta_{26}) = -195804.
\end{align*}
\]

So we get
\[
\begin{align*}
H_2(2^{-s}, \chi^{(\pm)}_{29/2}) &= L_2(s - 1, \Delta_{26})L_2(s, \Delta_{26}^{\pm}), \\
H_3(3^{-s}, \chi^{(\pm)}_{29/2}) &= L_3(s - 1, \Delta_{26})L_3(s, \Delta_{26}^{\pm}).
\end{align*}
\]

**weight 31/2.**

We have \(\dim S^{+}_{31/2}(\Gamma_0(4)) = 4\). For \(\epsilon_1 = \pm 1\) and \(\epsilon_2 = \pm 1\), we put
\[
\begin{align*}
\chi^{\epsilon_1, \epsilon_2}_{31/2} &= \\
&2(1087273 + 19401 \epsilon_1 \sqrt{18209} - \epsilon_2 \sqrt{51349}(6889 + 33 \epsilon_1 \sqrt{18209})) \\
&\times \chi_{12}(4\tau)P_{7/2}(2\tau) + 10(-661583 - 3855 \epsilon_1 \sqrt{18209} \\
&+ \epsilon_2 \sqrt{51349}(2327 - 27 \epsilon_1 \sqrt{18209}))\chi_{10}(4\tau)P_{11/2}(2\tau) \\
&+ (-1179 + 517 \epsilon_2 \sqrt{51349} - \epsilon_1 \sqrt{18209}(283 + \epsilon_2 \sqrt{51349}))E_6(4\tau)P_{19/2}(2\tau) \\
&+ 100590E_4(4\tau)P_{23/2}(2\tau).
\end{align*}
\]
These are in $S_{31/2}^+ (\Gamma_0(4))$ and Hecke eigen forms. Then we have

\[
H_2(T, \chi_{31/2}^{\epsilon_1, \epsilon_2}) = (1 + (8280 - 216 \epsilon_1 \sqrt{18209}) T + 2^{29} T^2) \\
\times (1 - (4320 - 96 \epsilon_2 \sqrt{51349}) T + 2^{29} T^2),
\]

\[
H_3(T, \chi_{31/2}^{\epsilon_1, \epsilon_2}) = (1 + (1929420 - 62208 \epsilon_1 \sqrt{18209}) T + 3^{29} T^2) \\
\times (1 + (2483820 - 52992 \epsilon_2 \sqrt{51349}) T + 3^{29} T^2).
\]

We also have

\[
\lambda(2, \Delta_{28}^{\pm}) = -4140 \pm 108 \sqrt{18209},
\]

\[
\lambda(2, \Delta_{30}^{\pm}) = 4320 \mp 96 \sqrt{51349},
\]

\[
\lambda(3, \Delta_{28}^{\pm}) = -643140 \pm 20736 \sqrt{18209},
\]

\[
\lambda(3, \Delta_{30}^{\pm}) = -2483820 \pm 52992 \sqrt{51349}.\]

So we get

\[
H_2(2^{-s}, \chi_{31/2}^{\pm, \pm}) = L_2(s - 1, \Delta_{28}^{\pm}) L_2(s, \Delta_{30}^{\pm}),
\]

\[
H_3(3^{-s}, \chi_{31/2}^{\pm, \pm}) = L_3(s - 1, \Delta_{28}^{\pm}) L_3(s, \Delta_{30}^{\pm}).
\]

**weight 33/2.**

We have dim $S_{33/2}^+ (\Gamma_0(4)) = 4$. For $\epsilon_1 = \pm 1$ and $\epsilon_2 = \pm 1$, we put

\[
\chi_{33/2}^{\epsilon_1, \epsilon_2} =
\]

\[
(-198304 + 10027 \epsilon_1 \sqrt{51349} - 128 \epsilon_2 \sqrt{18295489} \\
- \epsilon_1 \epsilon_2 \sqrt{51349} \sqrt{18295489}) E_6(4 \tau) \chi_{10}(4 \tau) P_{1/2}(2 \tau) \\
+ 189(1131 + 4 \epsilon_1 \sqrt{51349} + \epsilon_2 \sqrt{18295489}) E_4(4 \tau) \chi_{12}(4 \tau) P_{1/2}(2 \tau) \\
- 13608 \left(8 + \epsilon_1 \sqrt{51349}\right) \chi_{10}(4 \tau) P_{13/2}(2 \tau) + 189 E_4(4 \tau) P_{25/2}(2 \tau).
\]

Then $\chi_{33/2}^{\pm, \pm} \in S_{33/2}^+ (\Gamma_0(4))$ and these are Hecke eigen forms. We have

\[
H_2(T, \chi_{33/2}^{\epsilon_1, \epsilon_2}) = \left(1 - \left(8640 + 192 \epsilon_1 \sqrt{51349}\right) T + 2^{31} T^2\right) \\
\times \left(1 - \left(19980 + 12 \epsilon_2 \sqrt{18295489}\right) T + 2^{31} T^2\right),
\]

\[
H_3(T, \chi_{33/2}^{\epsilon_1, \epsilon_2}) = \left(1 + 108(68995 + 1472 \epsilon_1 \sqrt{51349}) T + 3^{31} T^2\right) \\
\times \left(1 - 324(26795 + 16 \epsilon_2 \sqrt{18295489}) T + 3^{31} T^2\right).
\]

We also have

\[
\lambda(2, \Delta_{30}^{\pm}) = 4320 \pm 96 \sqrt{51349},
\]

\[
\lambda(2, \Delta_{30}^{\pm}) = 19980 \pm 12 \sqrt{18295489},
\]

\[
\lambda(3, \Delta_{30}^{\pm}) = -36(68995 \pm 1472 \sqrt{51349}),
\]

\[
\lambda(3, \Delta_{30}^{\pm}) = 324(26795 \pm 16 \sqrt{18295489}).
\]
So we get

\[ H_2(2^{-s}, \chi_{33/2}^{\pm, \pm}) = L_2(s - 1, \Delta_{30}^{\pm}) L_2(s, \Delta_{32}^{\mp}), \]

\[ H_3(3^{-s}, \chi_{33/2}^{\pm, \pm}) = L_3(s - 1, \Delta_{30}^{\pm}) L_3(s, \Delta_{32}^{\mp}). \]

weight 35/2.

We have \( \dim S_{35/2}^{+}(\Gamma_0(4)) = 4 \). For \( \epsilon_1 = \pm 1 \) and \( \epsilon_2 = \pm 1 \), we put

\[ \chi_{35/2}^{\epsilon_1, \epsilon_2} = \]

\[ 80(-447232006969 + 489062419 \epsilon_2 \sqrt{2356201} + \epsilon_1 \sqrt{18295489}(-34677047 + 89597 \epsilon_2 \sqrt{2356201})) E_{4}(4 \tau) \chi_{10}(4 \tau) P_{\tau/2}(2 \tau) \]

\[ + 800(34455469783 - 39825301 \epsilon_2 \sqrt{2356201} + \epsilon_1 \sqrt{18295489}(-588847 + 1453 \epsilon_2 \sqrt{2356201})) \chi_{12}(4 \tau) P_{11/2}(2 \tau) \]

\[ + 3(-121215233603 - 796064447 \epsilon_2 \sqrt{2356201} + 137801891 \epsilon_1 \sqrt{18295489} - 83441(\epsilon_1 \sqrt{18295489} \epsilon_2 \sqrt{2356201})) E_{4}(4 \tau)^2 P_{19/2}(2 \tau) \]

\[ + 9492701472 E_{0}(4 \tau) P_{23/2}(2 \tau). \]

Then \( \chi_{35/2}^{\epsilon_1, \epsilon_2} \in S_{35/2}^{+}(\Gamma_0(4)) \) and these are Hecke eigen forms. We have

\[ H_2(T, \chi_{35/2}^{\epsilon_1, \epsilon_2}) = \left( 1 - \left( 39960 + 24 \epsilon_1 \sqrt{18295489} \right) T + 2^{33} T^2 \right) \]

\[ \times \left( 1 + \left( 60840 + 72 \epsilon_2 \sqrt{2356201} \right) T + 2^{33} T^2 \right), \]

\[ H_3(T, \chi_{35/2}^{\epsilon_1, \epsilon_2}) = \left( 1 - \left( 26044740 + 15552 \epsilon_1 \sqrt{18295489} \right) T + 3^{33} T^2 \right) \]

\[ \times \left( 1 - \left( 18959940 + 22464 \epsilon_2 \sqrt{2356201} \right) T + 3^{33} T^2 \right). \]

We also have

\[ \lambda(2, \Delta_{32}^{\pm}) = 19980 \pm 12 \sqrt{18295489}, \]

\[ \lambda(2, \Delta_{32}^{\pm}) = -60840 \pm 72 \sqrt{2356201}, \]

\[ \lambda(3, \Delta_{32}^{\pm}) = 8681580 \pm 5184 \sqrt{18295489}, \]

\[ \lambda(3, \Delta_{32}^{\pm}) = 18959940 \pm 22464 \sqrt{2356201}. \]

So we get

\[ H_2(2^{-s}, \chi_{35/2}^{\pm, \pm}) = L_2(s - 1, \Delta_{32}^{\pm}) L_2(s, \Delta_{34}^{\mp}), \]

\[ H_3(3^{-s}, \chi_{35/2}^{\pm, \pm}) = L_3(s - 1, \Delta_{32}^{\pm}) L_3(s, \Delta_{34}^{\mp}). \]

Finally we give examples of Siegel modular forms of weight 41/2 and 47/2.
which cannot be obtained by this kind of lifting. We put
\[ \chi_{41/2} = (738592 E_4^2(4\tau) \chi_{12}(4\tau) + 545630 E_4(4\tau) E_6(4\tau) \chi_{10}(4\tau) \\
+ 6582097600 \chi_{10}(\tau)^2) P_{1/2}(2\tau) - 395994 E_4(4\tau) \chi_{10}(4\tau) P_{1/2}(2\tau) \\
- 838926 \chi_{12}(4\tau) P_{17/2}(2\tau) + 3 E_4(4\tau)^2 P_{25/2}(2\tau) \\
- 191004 E_6(4\tau) P_{29/2}(2\tau). \]

Then \( \chi_{41/2} \) is a Hecke eigen form in \( S_{41/2}^+(\Gamma_0(4)) \). The Hecke polynomial of \( \chi_{41/2} \) at two is given by
\[ H_2(T, \chi_{41/2}) = 1 - 105600 T - 723412582400 T^2 \\
- 58054213946572800 T^3 + 2^{27} T^4, \]
which is irreducible over \( \mathbb{Q} \). We also put
\[ \chi_{47/2} = (-946246 E_4(4\tau)^2 \chi_{12}(4\tau) + 2194570 E_4(4\tau) E_6(4\tau) \chi_{10}(4\tau) \\
- 1553434778880 \chi_{10}(4\tau)^2) P_{7/2}(2\tau) + (-1747462 E_4(4\tau)^2 \chi_{10}(4\tau) \\
- 580106 E_6(4\tau) \chi_{12}(4\tau)) P_{11/2}(2\tau) + (-27675 E_4(4\tau)^2) E_6(4\tau) \\
+ 323725375872 E_4(4\tau) \chi_{10}(4\tau)) P_{19/2}(2\tau) + (38788 E_4(4\tau)^3 - 8377 E_6(4\tau)^2 \\
+ 26731596672 \chi_{12}(4\tau)) P_{23/2}(2\tau). \]

Then \( \chi_{47/2} \) is a Hecke eigen form in \( S_{47/2}^+(\Gamma_0(4)) \). The Hecke polynomial of \( \chi_{47/2} \) at two is given by
\[ H_2(T, \chi_{47/2}) = 1 - 3048960 T - 21597142384640 T^2 \\
- 107275743123965214720 T^3 + 2^{30} T^4, \]
which is irreducible over \( \mathbb{Q} \).

4. Appendix

In this appendix, we give data of Fourier coefficients of forms in \( S_{k-1/2}^+(\Gamma_0(4)) \) used in the numerical examples of liftings in the previous section. In the table below, \((a, c, b)\) means the Fourier coefficient at \( \left( \frac{a}{b}, \frac{c}{b} \right) \).

<table>
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<tr>
<th>wt 19/2, 23/2</th>
<th>( P_{19/2}(2\tau) )</th>
<th>( P_{23/2}(2\tau) )</th>
<th>wt 21/2</th>
<th>( \chi_{10}(4\tau) P_{1/2}(2\tau) )</th>
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<th>$x_{1\omega}(4\tau)P_{2\omega}/(2\tau)$</th>
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Siegel modular forms of half integral weight and a lifting conjecture

### Table 1

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<th>$E_4(4r)x_{10}(4r)x_{12}(4r)x_{14}(4r)x_{16}(4r)$</th>
<th>$x_{11}(4r)P_{12}(4r)$</th>
<th>$E_4(4r)P_{12}(4r)$</th>
<th>$E_4(4r)x_{12}(4r)P_{12}(4r)$</th>
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Correction

[9] p. 114 Theorem 2 line 5, \( p^{-2kn-j/2} \) in LHS should read \( p^{-(2k+1)n-j/2} \).
[9] p. 123 line 1, \((\mathbb{Z}/p\mathbb{Z})^n\) should read \((\mathbb{Z}/p^2\mathbb{Z})^n\).
[9] p. 123 line 10, the LHS should be multiplied by \( p^{-m} \).
[3] p. 208 Theorem 2 line 3, \( p^{-2kn-s/2} \) in LHS should read \( p^{-(2k+1)n-s/2} \).
[3] p. 216 Lemma 7 line 4, \( p^{2kn} \) in RHS should read \( p^{(2k+1)n} \).

Universität Siegen, Fachbereich Mathematik
Walter-Flex-Str. 3, 57068 Siegen, Germany
e-mail: hayashida@math.uni-siegen.de

Department of Mathematics, Graduate School of Science
Osaka University
Machikaneyama 1-16, Toyonaka, Osaka 560-0043, Japan
e-mail: ibukiyam@math.wani.osaka-u.ac.jp

References


