MODULI FOR RATIONAL GENUS 2 CURVES WITH REAL MULTIPLICATION FOR DISCRIMINANT 5

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ABSTRACT. Principally polarized abelian surfaces with prescribed real multiplication (RM) are parametrized by certain Hilbert modular surfaces. Thus rational genus 2 curves with RM correspond to rational points on Hilbert modular surfaces via their Jacobians, but the converse is not true. We give a simple generic description of which rational moduli points correspond to rational curves, as well as give associated Weierstrass models, in the case of RM by the ring of integers of $\mathbb{Q}(\sqrt{5})$. To prove this, we provide some techniques for reducing quadratic forms over polynomial rings.

1. INTRODUCTION

We are interested in describing the space of rational genus 2 curves which have certain endomorphism structure on their Jacobians, and will correspond to modular forms.

Let k be a field. Let $D > 0$ be a discriminant, and $\mathcal{O}_D$ the quadratic order of discriminant D. For an abelian surface A/k, if $\mathcal{O}_D$ embeds in $\mathrm{End}_k(A)$, we say A has real multiplication (RM) by $\mathcal{O}_D$, and abbreviate this as RM-D. By extension, if C is a genus 2 curve and $A = \mathrm{Jac}(C)$ has RM-D, we say C has RM-D.

Typically, Jacobians of genus 2 curves, and more generally abelian surfaces, will have endomorphism ring $\mathbb{Z}$. One interest in abelian surfaces A with RM (i.e., RM-D for some D) is that they are of GL(2) type, which by work of Ribet [Rib04] and the proof of Serre's conjecture [KW09], means that abelian surfaces A with RM over $k = \mathbb{Q}$ correspond to elliptic modular forms of weight 2.

Parametrizing genus 2 curves, with or without an RM condition, is essentially understood over $k = \mathbb{C}$, but much less clear over $k = \mathbb{Q}$. Over $\mathbb{C}$, genus 2 curves with RM-5 are parametrized by $\mathbb{C}$-points on the Hilbert modular surface $Y(5)$, which is a rational surface over $\mathbb{Q}$. Thus we may generically parametrize rational points on $Y(5)$ by $(m, n) \in \mathbb{Q}^2$ as in [EK14]. However rational points of $Y(5)$ do not typically correspond to genus 2 curves defined over $\mathbb{C}$. We give a simple generic description of which moduli points correspond to genus 2 curves C with RM-5 over $\mathbb{Q}$.

Theorem 1.1. *The $\mathbb{C}$-isomorphism classes of genus 2 curves $C/\mathbb{Q}$ with RM-5 are generically parametrized by $(m, n) \in \mathbb{Q}^2$ such that $m^2 - 5n^2 - 5$ is a norm from $\mathbb{Q}(\sqrt{5})$.*

We will also describe models for these curves (Proposition 6.1), and be more precise about the meaning of "generically parametrized" here (see Theorem 5.1 and Section 5.1). These results extend to arbitrary subfields k of $\mathbb{C}$, and the models are rather simple when $k \supseteq \mathbb{Q}(\sqrt{5})$.

In order to explain our results more completely, we will first describe moduli for genus 2 curves over $\mathbb{C}$ in more detail. Below, when the field of definition of a curve or variety is not specified, it is assumed to be $\mathbb{C}$.

Let $\mathcal{M}_2$ be the (coarse) moduli space of genus 2 curves and $\mathcal{A}_2$ be the moduli space of principally polarized abelian surfaces. The Torelli map $\mathcal{M}_2 \to \mathcal{A}_2$, corresponding to mapping a genus 2 curve C to its Jacobian $A = \mathrm{Jac}(C)$, is almost surjective—the complement of its image consists of (moduli for) products of 2 elliptic curves. We may identify a point in $\mathcal{M}_2$ corresponding to a genus 2 curve C with Igusa–Clebsch invariants $(I_2 : I_4 : I_6 : I_{10})$ in weighted projective space $\mathbb{P}^3_{1,2,3,5}(\mathbb{C})$. Each $(I_2 : I_4 : I_6 : I_{10})$ with $I_{10} \neq 0$ comes from a genus 2 curve.

The Igusa–Clebsch invariants I_{2j} can be defined as degree $2j$ polynomial functions $I_{2j}(f)$ of the coefficients of a sextic Weierstrass equation $y^2 = f(x)$ for C, and up to projective equivalence do not depend on the model. Consequently, if C has a model over a subfield $k \subseteq \mathbb{C}$, then the Igusa–Clebsch invariants are defined over k (i.e., can all be taken in k after scaling).

However, the converse is not true. (Contrast this to the genus 1 situation: an elliptic curve has a rational model if and only if its j-invariant is rational.) If C is a genus 2 curve without extra automorphisms over $\mathbb{C}$ and its Igusa–Clebsch invariants are defined over k, then Mestre [Mes91b] showed that C is defined over k if and only if a certain conic L/k has a k-rational point. (If C has extra automorphisms, it has a model over k by [CQ05].) The coefficients of the Mestre conic L are polynomials in I_2, I_4, I_6 and I_{10}. Nonetheless, there is no simple characterization of when the Mestre obstruction vanishes, i.e., when L has a k-rational point.

Now we review moduli for genus 2 curves with RM-D. For simplicity, assume D is a fundamental discriminant, so $\mathcal{O}_D$ is the ring of integers of $\mathbb{Q}(\sqrt{D})$. The Hilbert modular surface $Y_-(D)$ is a smooth compactification of the quotient $\mathrm{SL}_2(\mathcal{O}_D)\backslash(\mathfrak{H}^+ \times \mathfrak{H}^-)$, or alternatively $\mathrm{SL}_2(\mathcal{O}_D \oplus \mathcal{O}_D^*)\backslash(\mathfrak{H}^+ \times \mathfrak{H}^+)$, where $\mathcal{O}_D^*$ is the inverse different of $\mathcal{O}_D$ (e.g., see [vdG88]). Then $Y_-(D)$ is a coarse moduli space for principally polarized complex abelian surfaces with real multiplication RM-D, where one fixes an action of $\mathcal{O}_D$ compatible with the polarization.

Suppose $k \subseteq \mathbb{C}$. The moduli interpretation of $Y_-(D)$ allows one to consider $Y_-(D)$ as a surface defined over $\mathbb{Q}$, one can consider its k-points. Then a genus 2 curve C/k with RM-D corresponds to a k-rational point on $Y_-(D)$. However, the converse is not true, even generically. If p is a k-rational point on $Y_-(D)$ which does not correspond to the product of two elliptic curves, then it will correspond to a curve C with RM-D defined over $\bar{k}$ such that C is isomorphic to C^σ for any $\sigma \in \mathrm{Gal}(\bar{k}/k)$.

For p to correspond to a curve over k with RM-D we need both that the Mestre obstruction vanishes, and that some rational model for C has RM-D defined over k. (It can happen that some k-rational models for C have RM defined over k and some do not.) We will see that generically if the Mestre obstruction vanishes, then the RM is defined over k. More precisely, if $\mathrm{End}(\mathrm{Jac}(C))$ is commutative, then a field of definition for C is a field of definition for the RM (Proposition 2.1).

1.1. Strategy of proof.
In the special case of RM-5, the Hilbert modular surface $Y(5) = Y_-(5)$ is a rational surface over $\mathbb{Q}$, i.e., birational to $\mathbb{P}^2_{m,n}(\mathbb{Q})$. Hence to prove Theorem 1.1, it suffices to show that the vanishing of the Mestre obstruction at a rational point (m, n) in $Y(5)$ is generically equivalent to the condition that $m^2 - 5n^2 - 5 = u^2 - 5v^2$ for some

$u, v \in \mathbb{Q}$. This is not at all obvious from the Mestre conic, which is a conic over $\mathbb{Q}[m, n]$ whose coefficients are degree ≤ 14 polynomials in m and n, and whose discriminant is of degree 30. In fact, it was rather surprising to us that there was such a simple characterization of the Mestre obstruction. It was only through computational observations that we were led to believe in Theorem 1.1, and then were able to find a proof after much trial.

The starting point for the proof relies on two birational models for $Y_-(5)$ due to Elkies and Kumar [EK14], which were obtained by studying lattice polarizations of K3 surfaces. The first model is a double cover of $\mathbb{P}^2_{g,h}(\mathbb{Q})$ of the form $z^2 = f(g, h)$, where f is a degree 5 polynomial in g and h. In this model, the norm condition in Theorem 1.1 can be restated as $30g + 4$ being a norm from $\mathbb{Q}(\sqrt{5})$. In particular, the Mestre obstruction only depends on g and not h. (This was our initial computational observation that led to the theorem.) The Igusa–Clebsch invariants now are low-degree expressions in g and h. In terms of g and h, the Mestre conic has coefficients in $\mathbb{Q}[g, h]$ which are of degree ≤ 7 in g and degree ≤ 2 in h, and its discriminant is an integer multiple of $h^2(8h - 9g^2)^2 z^2$.

To our knowledge, there are no general methods to reduce quadratic forms over polynomial rings. The standard technique taught to "simplify" quadratic forms over fields is diagonalization, but unless one is very lucky this is not useful in simplifying quadratic forms over rings. E.g., diagonalizing the conic over $\mathbb{Q}(m, n)$ and clearing denominators gives coefficients which are polynomials of degrees 24, 28 and 32 in m and n.

We will describe a few simple techniques to reduce degrees of polynomial coefficients and remove factors from the discriminant, which we hope may be of use in other situations. In our case, we are able to use these methods to reduce the the Mestre conic in g and h to have polynomial coefficients of degree ≤ 3 and remove the factors of h^2 and $(8h - 9g^2)$ from the discriminant. Then we switch to the (m, n) model and apply our techniques to reduce the Mestre conic over $\mathbb{Q}(m, n)$ to $x_1^2 - 5x_2^2 + (m^2 - 5n^2 - 5)x_3^2 = 0$, which proves Theorem 1.1.

We remark that we needed to use both of these models for $Y_-(5)$ to carry out this reduction of the Mestre conic. While the Mestre conic is simpler in g and h, our final reduced form, which is the same as $x_1^2 - 5x_3^2 + (30g + 4)x_3^2 = 0$, is *not* equivalent to the original Mestre conic over $\mathbb{Q}(g, h)$. That is, these conics are not equivalent over $\mathbb{Q}$ for a generic choice of $g, h \in \mathbb{Q}$—the equivalence requires rational g, h such that $f(g, h)$ is a rational square, i.e., g and h come from a rational point on $Y_-(5)$, and it is not clear how to use the relation $z^2 = f(g, h)$ to carry out this reduction solely in terms of g and h. On the other hand, we were unable to carry out the reduction entirely in terms of m and n because finding suitable changes of variables is more difficult with higher degree polynomial coefficients.

1.2. Moduli of rational curves.

Here we briefly describe to what extent we can make the "generic" aspect of Theorem 1.1 precise. First, our reduction of the Mestre conic L over $\mathbb{Q}(m, n)$ does not give a $\mathbb{Q}$-equivalent conic when specializing to points $(m, n) \in \mathbb{Q}^2$ such that $\operatorname{disc} L = 0$. This happens on a finite number of curves in the moduli space, which we examine separately.

Second, as (m, n) are only affine coordinates for a birational model for $Y_-(5)$, the set of rational (m, n) does not exhaust the rational points on $Y_-(5)$. Fortunately, thanks to work of Wilson [Wil00], we can describe Igusa–Clebsch invariants for the remaining points on $Y_-(5)$ and say explicitly when such points correspond to a genus 2 curve defined over $\mathbb{Q}$.

Consequently, in Theorem 5.1 we give an explicit description of a set $\mathcal{Y}$ of rational moduli in $\mathcal{M}_2$ such that any genus 2 curve $C/\mathbb{Q}$ with RM-5 corresponds to a point on $\mathcal{Y}$. Moreover,

any point in $\mathcal{Y}$ corresponds to a genus 2 curve $C/\mathbb{Q}$ that has potential RM-5, i.e., RM-5 defined over $\bar{\mathbb{Q}}$ but not necessarily $\mathbb{Q}$. We do not know if each such C will always have a twist with RM-5 defined over $\mathbb{Q}$, but we were not able to find any examples to the contrary. At least the collection of such curves generically has RM-5, and we explain two ways in which one can check that the RM-5 is defined over $\mathbb{Q}$.

1.3. **Models of curves.** Several families of rational genus 2 curves $C/\mathbb{Q}$ with RM-5 have been constructed in the literature. For instance, Mestre constructed a 2-parameter family in [Mes91a] and Brumer constructed a 3-parameter family (see [Bru95] for an announcement, and [Has00] for a proof different from Brumer's). For a rational choice of parameters these families generically give rational genus 2 curves C with RM-5 over $\mathbb{Q}$. Moreover, over $\mathbb{C}$ these families are known to exhaust all $\mathbb{C}$-isomorphism classes of genus 2 curves $C/\mathbb{Q}$ with RM-5 (see [HS09] for Brumer's family and [Wil00] or [Sak09] for Mestre's family). However, it is not known how to describe all such rational curves with these families, or how to describe what parameters give $\mathbb{C}$-isomorphic curves.

Theorem 1.1 generically parametrizes such $C/\mathbb{Q}$. If $(m, n) \in \mathbb{Q}^2$ such that $m^2 - 5n^2 - 5 = u^2 - 5v^2$ with $u, v \in \mathbb{Q}$, we give a generic Weierstrass model $y^2 = f(x)$ for an associated curve in terms of (m, n, u, v). See Proposition 6.1. These results apply arbitrary base fields $k \subseteq \mathbb{C}$. If $k \supseteq \mathbb{Q}(\sqrt{5})$, then the analogous norm condition in Theorem 1.1 is automatically satisfied, and one can write down a model solely in terms of $(m, n) \in k^2$. See Proposition 6.3.

1.4. **Additional remarks.** Our original motivation for this project was to help understand weight 2 elliptic modular forms with rationality field $\mathbb{Q}(\sqrt{5})$. We hope to return to this in the future.

In Section 8, we briefly describe some computational evidence that there are similarly simple descriptions for when the Mestre obstruction vanishes for some other small values of D. However, in these cases, the Mestre conics that arise are more complicated and we have only been partially successful in applying our reduction methods to these cases.

Calculations for this project were carried out in Sage [Sage] and Magma [Magma].

2. Moduli spaces

Henceforth, k denotes a subfield of $\mathbb{C}$.

Let C be a genus 2 curve defined over k. Then it has a rational Weierstrass model of the form $y^2 = f(x)$, where $f(x) \in k[x]$ is a sextic with no repeated irreducible factors. The Igusa–Clebsch invariants I_2, I_4, I_6, I_{10} are polynomial invariants of f of respective degrees $2, 4, 6, 10$ with $I_{10} = \mathrm{disc}(f)$. We view the Igusa–Clebsch invariants as a point $(I_2 : I_4 : I_6 : I_{10})$ in weighted projective space $\mathbb{P}^3_{1,2,3,5}$. (While using weighted projective space $\mathbb{P}^3_{2,4,6,10}$ may be more natural, we use $\mathbb{P}^3_{1,2,3,5}$ because then $\mathbb{Q}^\times$-equivalence of rational points is the same as $\mathbb{C}^\times$-equivalence of rational points, and the latter is what is used in our main result.) In this way, the Igusa–Clebsch invariants in $\mathbb{P}^3_{1,2,3,5}$ depend only on C and not on the choice

4

of the Weierstrass equation. Moreover, the set of $(I_2 : I_4 : I_6 : I_{10})$ with $I_{10} \neq 0$ forms a coarse moduli space $\mathcal{M}_2$ for genus 2 curves.

2.1. Hilbert modular surfaces.
Here we review some facts about certain Hilbert modular surfaces. See [vdG88] and [EK14] for more details.

Let $D > 0$ be a fundamental discriminant. The Hilbert modular surface $Y_-(D)$ is a smooth compactification of the quotient $\mathrm{SL}_2(\mathcal{O}_D \oplus \mathcal{O}_D^*)\backslash \mathfrak{H}^+ \times \mathfrak{H}^+$. When the class number of $K = \mathbb{Q}(\sqrt{D})$ equals its narrow class number, this agrees with the Hilbert modular surface often denoted $Y(D)$.

Fix an embedding $K \subseteq \mathbb{C}$ and denote by τ the nontrivial Galois automorphism of K. One can associate to $(z_1, z_2) \in \mathfrak{H}^+ \times \mathfrak{H}^+$ a lattice

$$L_{(z_1, z_2)} = \{(az_1 + b, a^\tau z_2 + b^\tau) : a \in \mathcal{O}_D, b \in \mathcal{O}_D^*\} \subseteq V = \mathbb{C}^2.$$

Then

$$E((w_1, w_2), (w_1', w_2')) = \frac{\mathrm{Im}\, w_1 \bar{w}_1}{\mathrm{Im}\, z_1} + \frac{\mathrm{Im}\, w_2 \bar{w}_2'}{\mathrm{Im}\, z_2}$$

(with bar denoting complex conjugation) defines a Riemann form on $A = V/L_{(z_1, z_2)}$ such that $L_{(z_1, z_2)}$ is unimodular with respect to this form. This makes A a principally polarized abelian surface (PPAS) with an action of $\mathcal{O}_D$ via $j(\alpha)(w_1, w_2) = (\alpha w_1, \alpha^\tau w_2)$. In fact, one may check that $j : \mathcal{O}_D \hookrightarrow \mathrm{End}(A)^\dagger$, where $\dagger$ denotes the Rosati involution. This construction leads to the fact that $Y_-(D)$ is a moduli space for such pairs (A, j) of PPASs with RM-D.

The Humbert modular surface $\mathcal{H}_D$ is the image of $Y_-(D)$ in $\mathcal{A}_2$, and the map $Y_-(D) \to \mathcal{H}_D$ is generically 2-to-1, corresponding to forgetting the action of $\mathcal{O}_D$. Note that in the above construction, switching z_1 and z_2 corresponds to replacing j with $j \circ \tau$, and for the points (z_1, z_1), the conjugate actions j and $j \circ \tau$ are isomorphic.

If A is a geometrically simple PPAS, then $\mathrm{End}(A)$ is isomorphic to $\mathbb{Z}$, an order in a real quadratic field, an order in a quartic CM field, or an order in an indefinite quaternion algebra. If A is not geometrically simple, but $\mathcal{O}_D$ embeds in $\mathrm{End}(A)$, then $\mathrm{End}(A)$ is an order in either the split quaternion algebra $M_2(\mathbb{Q})$ or in $M_2(F)$ where F is an imaginary quadratic field, according to whether A is isogenous over $\overline{\mathbb{Q}}$ to a product of isogenous elliptic curves without or with CM.

2.2. Fields of definition.
We are interested in fields of definition of curves and endomorphisms. In general, suppose X is a coarse moduli space for a class of varieties V satisfying some property P. If x corresponds to the pair (V, P), then the field of moduli for (V, P) is the field of definition of the point x. If both V and P are defined over k, then the field of moduli contains k, but the converse is not true in general.

In particular, if C is a genus 2 curve over $\mathbb{C}$, then the field of moduli of C is the field of definition of $(I_2 : I_4 : I_6 : I_{10})$, i.e. the minimal field k_0 such that I_2, I_4, I_6, I_{10} can be taken in k_0 after scaling. If C is defined over k, then $k \supseteq k_0$. However, C need not be defined over k_0, i.e., there need not be a curve C'/k_0 such that C' and C are isomorphic over $\mathbb{C}$.

Generically, $\mathrm{Aut}(C)$ is generated by the hyperelliptic involution on C. If $|\mathrm{Aut}(C)| > 2$, then by [CQ05], C is defined over k_0. When $\mathrm{Aut}(C) \simeq C_2$, Mestre [Mes91b] constructed a nonsingular conic L/k_0 such that C is defined over $k \supseteq k_0$ if and only if L has a k-point. The coefficients of L are polynomials in I_2, I_4, I_6 and I_{10}—see Section 4.1 for details. We remark that since L always has a point over a quadratic extension k'/k_0, C is always definable over a (in fact, infinitely many) quadratic extension(s) of k_0.

Now consider a genus 2 curve (C, j) with RM-D, where j is an embedding of $\mathcal{O}_D$ into $\mathrm{End}(A)$, $A = \mathrm{Jac}(C)$, that respects the polarization as above. Then the field of moduli for (C, j) is the minimal field k_0 such that (A, j) corresponds to a k_0-rational point on $Y_-(D)$. This means that the pair (C, j) is defined over $\bar{k}_0$ and is isomorphic to any $\mathrm{Gal}(\bar{k}_0/k_0)$-conjugate of itself. In particular, forgetting the RM, the genus 2 curve $C \simeq C^\sigma$ for any $\sigma \in \mathrm{Gal}(\bar{k}_0/k_0)$. Thus the Igusa-Clebsch invariants for C are defined over k_0, i.e., the field of moduli of C (i.e., the field of definition of the associated point in $\mathcal{A}_2$) contains k_0.

If (C, j) is defined over k, i.e., there is a model for C defined over k such that $j(\mathcal{O}_D) \subseteq \mathrm{End}_k(A)$, then $k \supseteq k_0$. Conversely, given $k \supseteq k_0$, we would like a way to determine whether (C, j) is defined over k. Necessarily, C must be defined over k, i.e., the Mestre conic L must have a k-rational point. The following says that, generically, when the Mestre conic has a point the RM is also defined over k.

Proposition 2.1. *Suppose p is a k-rational point on $Y_-(D)$ corresponding to a PPAS A defined over k with an embedding $j : \mathcal{O}_D \hookrightarrow \mathrm{End}_{\mathbb{C}}(A)$. If $\mathrm{End}_{\mathbb{C}}(A)$ is commutative, then $j(\mathcal{O}_D) \subseteq \mathrm{End}_k(A)$.*

Proof. Let $\sigma \in G_k$, and $\eta = \frac{D+\sqrt{D}}{2}$. Then p being k-rational means there is an isomorphism $\varphi : (A, j) \to (A^\sigma, j^\sigma)$. In particular, φ maps $j(\eta)$ to $j^\sigma(\eta) \in \mathrm{End}_{\mathbb{C}}(A^\sigma)$, which we may identify with $j(\eta)^\sigma \in \mathrm{End}_{\mathbb{C}}(A)$. Consequently, there is an inner automorphism of $\mathrm{End}_{\mathbb{C}}(A)$ taking $j(\eta)$ to $j(\eta)^\sigma$. Hence if $\mathrm{End}_{\mathbb{C}}(A)$ is commutative, this means $j(\eta)^\sigma = j(\eta)$ for all σ, and thus $j(\eta) \in \mathrm{End}_k(A)$. $\qed$

Now we briefly address how to check the field of definition of RM for specific curves C. Suppose C is defined over k, and let $A = \mathrm{Jac}(C)$.

Algorithms for numerically computing $\mathrm{End}_k(A)$ and $\mathrm{End}_{\mathbb{C}}(A)$ have been implemented in Magma, which one can use to provably exhibit RM-D using correspondences—e.g., see [KM16] or [CMSV19].

In the case we consider in this paper, $D = 5$, another criterion which is simpler to provably verify was provided by Wilson:

Proposition 2.2 ([Wil00]). *Let $y^2 = f(x)$ be a sextic Weierstrass model over k for a genus 2 curve C with potential RM-5, i.e., C has RM-5 defined over $\mathbb{C}$. Then C has RM-5 (defined over k) if and only if $\mathrm{Gal}(f) = \mathrm{Gal}(f/k)$ is contained in a transitive copy of A_5 inside S_6.*

It is easy to verify whether C has potential RM-5, because one can check whether it comes from a point on $Y_-(5)$ via its Igusa–Clebsch invariants. In particular, if $C : y^2 = f(x)$ is a genus 2 curve over k with $\deg f = 6$, then C has RM-5 (over k) if and only if its Igusa–Clebsch invariants are of one of the types listed below in Proposition 2.3 and $\mathrm{Gal}(f)$ lies in one of the transitive copies of A_5 inside S_6.

2.3. **Moduli for RM-5.** Elkies and Kumar [EK14] give the following birational model for $Y_-(5)$:

(2.1) $\qquad Y : z^2 = 2(-972g^5 - 324g^4 - 27g^3 - 4500g^2h - 1350gh + 6250h^2 - 108h).$

For (z, g, h) on the surface Y corresponding to a point on $\mathcal{M}_2 \subseteq \mathcal{A}_2$, the Igusa–Clebsch invariants are

$$(I_2 : I_4 : I_6 : I_{10}) = \left(24g + 6 : 9g^2 : 81g^3 + 18g^2 + 36h : 4h^2\right)$$

6

The surface $Y_-(5)$ is rational, and Elkies and Kumar give a birational map between Y and $\mathbb{P}^2$, with affine coordinates (m, n), via

$$30g + 9 = m^2 - 5n^2$$

(2.2)
$$h = m\frac{(30g+9)(15g+2)}{6250} + \frac{9(250g^2 + 75g + 6)}{6250}$$

$$z = n\frac{(30g+9)(15g+2)}{25}.$$

These equations give invertible transformations between the affine coordinates (z, g, h) on Y and (m, n) on $\mathbb{P}^2$ outside of the locus where $g = \frac{m^2 - 5n^2 - 9}{30}$ is $-\frac{3}{10}$ or $-\frac{2}{15}$.

In an alternative approach, Wilson [Wil00] constructed a coarse moduli space for genus 2 curves C with RM-5 with coordinates $(z_6 : s_2 : \sigma_5) \in \mathbb{P}^2_{1,2,5}$ with $\sigma_5 \neq 0$ such that

$$(I_2 : I_4 : I_6 : I_{10}) = \left(-2s_2 + 2z_6^2 : \frac{(s_2 + 2z_6^2)^2}{16} : \frac{9z_6\sigma_5 - 4I_4(3s_2 - 2z_6^2)}{16} : \frac{\sigma_5^2}{1024}\right).$$

Moreover if C is defined over k, then so is $(z_6 : s_2 : \sigma_5)$ and the quantity

$$\Delta' = 64z_6^6 s_2^2 + 96z_6^4 s_2^3 + 48z_6^2 s_2^4 - 256z_6^5\sigma_5 + 8s_2^5 - 400z_6^3 s_2\sigma_5 - 1000z_6 s_2^2\sigma_5 + 3125\sigma_5^2$$

must be a square in k.

One can translate Wilson's coordinates to the Elkies–Kumar coordinates via

$$(g, h) = \left(-\frac{2z_6^2 + s_2}{12z_6^2}, \frac{\sigma_5}{64z_6^5}\right).$$

We remark that under this change of coordinates, $\Delta' = 2^{10}z^2$, so the condition that Δ' is a square in k is automatically satisfied when (z, g, h) is a k-rational point on Y.

If $z_6 \neq 0$, we can assume $z_6 = 1$ and this relation gives a one-to-one correspondence between $(g, h) \in \mathbb{C}^2$ and $(s_2, \sigma_5) \in \mathbb{C}^2$. If $z_6 = 0$, then the Igusa–Clebsch invariants of the point $(z_6 : s_2 : \sigma_5)$ must either be $(0 : 0 : 0 : 1)$ if $s_2 = 0$ or

$$(I_2 : I_4 : I_6 : I_{10}) = \left(-8 : 1 : -3 : \frac{\sigma_5^2}{s_2^5}\right)$$

otherwise. Hence any genus 2 curve with RM-5 either corresponds to a point $(g, h) \in \mathbb{C}^2$ or has Igusa–Clebsch invariants of the form $(0 : 0 : 0 : 1)$ or $(8 : 1 : 3 : s)$ for $s \neq 0$. When $z_6 = 0$, $\Delta' = 8s_2^5 + 3125\sigma_5^2$. Thus Δ' being a square in k means either $\sqrt{5} \in k$ if $s_2 = 0$ or $3125s^2 - 8s$ is a square, where $s = -\frac{\sigma_5^2}{s_2^5}$, if $s_2 \neq 0$. It is easy to see that any two of these possibilities are mutually exclusive.

Let us now consider the possibility that (g, h) and (g', h') give the same Igusa–Clebsch invariants, i.e., there exists $\lambda \in \mathbb{C}^\times$ such that

$$\left(24g' + 6 : 9g'^2 : 81g'^3 + 18g'^2 + 36h' : 4h'^2\right) = \lambda \cdot \left(24g + 6 : 9g^2 : 81g^3 + 18g^2 + 36h : 4h^2\right)$$

Since we are interested in genus 2 curves, assume h and h' are both nonzero.

First note if $g = 0$, then $g' = 0$ and we have $h' = \lambda^3 h$ and $h'^2 = \lambda^5 h^2$. Comparing these shows $\lambda = 1$. So assume g, g' are both nonzero. Then comparing I_4's yields $\lambda = \varepsilon\frac{g'}{g}$, where $\varepsilon = \pm 1$. Now comparing I_2's shows $4g' + 1 = \varepsilon(4g' + \frac{g'}{g})$. If $\varepsilon = 1$, then $g = g'$, i.e., $\lambda = 1$

which implies $h = h'$. Thus assume $\varepsilon = -1$. Then $g' = -\frac{g}{8g+1}$ and $\lambda = \frac{1}{8g+1}$. Examining the I_6's and I_{10}'s then gives $h' = \frac{g^3+2h}{2(8g+1)^3}$ and

$$(h')^2 = \frac{(g^3 + 2h)^2}{(8g+1)^6} = \frac{4h^2}{(8g+1)^5}.$$

Using the assumption that $g \neq 0$, the latter equality holds if and only if $32h^2 - 4g^2h - g^5 = 0$, i.e., $h = \frac{g^2}{16}(1 + u)$ where $u^2 = 1 + 8g \neq 0$. Note that if $g' = g$ then $g = -\frac{1}{4}$, $\lambda = -1$ so $h'^2 = -h^2$.

Hence for any $(g, h) = \left(g, \frac{g^2}{16}(1 \pm \sqrt{8g+1})\right)$ with $g \neq 0, -\frac{1}{8}$, the pair $(g', h') = \left(-\frac{g}{8g+1}, \frac{g^3+2h}{(8g+1)^3}\right)$ are distinct coordinates with the same Igusa–Clebsch invariants, and these are the only pairs of distinct (g, h)-coordinates with this property.

Now suppose (g, h) and (g', h') are distinct k-rational pairs giving the same Igusa–Clebsch invariants as above, with $u^2 = 8g + 1$. Expressing g, g', h, h' in terms of u, we see that, for both (g, h) and (g', h'), the right hand side of (2.1) is in the $k^\times$-square class of $-(43u^2 + 22u + 43)$.

The above discussion yields the following.

Proposition 2.3. *Let C be a genus 2 curve with RM-5 defined over k. Then the Igusa–Clebsch invariants of C must be of one of the following types:*

(1) $(I_2 : I_4 : I_6 : I_{10}) = (0 : 0 : 0 : 1)$ when $\sqrt{5} \in k$;
(2) $(I_2 : I_4 : I_6 : I_{10}) = (8 : 1 : 3 : s)$ for some nonzero $s \in k$ such that $3125s^2 - 8s$ is a square; or
(3) $(I_2 : I_4 : I_6 : I_{10}) = \left(24g + 6 : 9g^2 : 81g^3 + 18g^2 + 36h : 4h^2\right)$ for a k-rational solution (z, g, h) to (2.1) with $h \neq 0$.

The above three cases are mutually exclusive. In case (2), s is unique. In case (3), the pair (g, h) is unique except in the case that $(g, h) = \left(\frac{1}{8}(u^2 - 1), \frac{1}{1024}(u - 1)^2(u + 1)^3\right)$ for some $u \in k^\times \setminus \{\pm 1\}$ such that $-(43u^2 + 22u + 43)$ is a square, in which case (g, h) and $(g', h') = \left(-\frac{g}{8g+1}, \frac{g^3+2h}{2(8g+1)^3}\right)$ are distinct elements of k^2 that both correspond to invariants

$$\left(48u^2 + 16 : 36(1 - u)^2(1 + u)^2 : 72(1 - u)^2(1 + u)^2(9u^2 + \frac{2}{u} + 9) : 4(1 - u)^4(1 + u)^6\right).$$

We remark that $-(43u^2 + 22u + 43)$ can be a square in a number field k if and only if every infinite place of k is complex and the completion k_v at every place v above 3 is an extension of $\mathbb{Q}_3$ of even degree. In particular, when $k/\mathbb{Q}$ is quadratic this happens if and only if k is imaginary quadratic and non-split at 3.

Remark 2.4. If we consider the map $\varphi(u) = \left(\frac{1}{8}(u^2 - 1), \frac{1}{1024}(u - 1)^2(u + 1)^3\right)$, then the pairs (g, h) and (g', h') yielding the same Igusa–Clebsch invariants at the end of the proposition are just the points $\varphi(u)$ and $\varphi(\frac{1}{u})$, which both lie on the curve $X_6 : 32h^2 - 4g^2h - g^5 = 0$ on Y. Noam Elkies explained to us how his work in [Elk08] implies that X_6 is the image of the Shimura curve quotient $X(6)/\langle w_6 \rangle$ parametrizing principally polarized abelian surfaces with quaternionic multiplication by the maximal order in the rational quaternion algebra of discriminant 6. Moreover, the involution on X_6 induced from $u \mapsto \frac{1}{u}$ corresponds to the involution $w_2 = w_3$ of $X(6)/\langle w_6 \rangle$.

3. Reduction of quadratic forms over polynomial rings

Here we will explain our approach to reducing quadratic forms over polynomial rings, which we will then apply to Mestre conics. Say $R = k[t_1, \ldots, t_m]$ is a polynomial ring over a field k of characteristic not 2. Let $Q(x_1, \ldots, x_n)$ be a quadratic form over R. Thus we can write Q as

$$Q(x_1, \ldots, x_n) = \sum_{i,j} f_{i,j}(t_1, \ldots, t_m) x_i x_j,$$

where each $A_{i,j} = f_{i,j}(t_1, \ldots, t_m) \in R$ and $A_{j,i} = A_{i,j}$. Then $A = (A_{i,j}) \in M_n(R)$ is the Gram matrix for Q with respect to the standard basis $\{e_1, \ldots, e_n\}$. Define the polynomial degree $\deg_k Q$ of Q to be $\max_{(i,j)} \deg A_{i,j}$.

Consider the following two reduction problems: (i) reduce Q to an equivalent quadratic form Q' over R with minimal polynomial degree; or (ii) reduce Q to a quadratic form Q' over R which is equivalent over the field of fractions F of R with minimal polynomial degree. (By equivalence of quadratic forms, we mean isomorphism up to invertible scaling.) In case (i), specializations of Q and Q' to any $t_1, \ldots, t_m \in k$ will be k-equivalent. In case (ii), specializations of Q and Q' will merely be k-equivalent for generic choices of $t_1, \ldots, t_m \in k$.

It is really reduction problem (ii) that we are interested in, as it allows for much greater possibilities for reducing our quadratic forms. Note that merely diagonalizing Q over F and clearing denominators to obtain a form over R is not typically helpful in reducing the polynomial degree. (Conversely, one cannot always diagonalize and maintain minimal polynomial degree—see Example 3.1, but fortunately for our Mestre conic of interest, our reduction process will also diagonalize the form.) We first describe the types of reduction steps we will use.

(1) Simple degree reduction. By a k-linear change of basis, we may assume the maximal degree of the $f_{i,j}$'s is attained for some of the diagonal terms with $j = i$. Say f_{j_0,j_0} attains the maximal degree of the $f_{i,j}$'s. Write $v = \sum h_i(t_1, \ldots, t_m) e_i$ where each $h_i \in R$. Search for a choice of polynomials h_i such that $\deg Q(v) < \deg f_{j_0,j_0}$ and h_{j_0} has nonzero constant term. Now make the change of variable corresponding to changing basis for the Gram matrix by replacing e_{j_0} in the standard basis with v. The resulting quadratic form will have $Q(v)$ as the coefficient of $x_{j_0}^2$ and so we have reduced the degree of this diagonal term.

   In our Mestre conic case, the degrees of the diagonal terms turn out to control the polynomial degree of Q, so reducing degrees of diagonal terms is sufficient for us. In general, to reduce the degree of the $x_i x_j$ term, one could similarly search for vectors v, v' with polynomial coefficients such that $\deg B(v, v') < \deg_k Q$, and then change bases by replacing e_i with v and e_j with v'.

(2) Discriminant reduction. Let $\Delta = \Delta(t_1, \ldots, t_m) \in R$ be the discriminant of Q. By changing variables over F, one may be able to remove polynomial factors from Δ. For instance, $Q_1 : x_1^2 + t_1 x_1 x_2 + t_1^2 x_2^2$ has $\Delta = -3t_1^2$, and the change of variables $x_2 \mapsto \frac{1}{t_1} x_2$ gives the quadratic form $Q_2 : x_1^2 + x_1 x_2 + x_2^2$ with discriminant -3. In general, since an invertible change of variables preserves the square class of the discriminant, we might hope to remove square factors appearing in Δ.

   First divide out any polynomial factors of the gcd of the coefficients of Q. Now suppose $g(t_1, \ldots, t_m) \in R$ is irreducible over k of positive degree such that $g^2 \mid \Delta$. Then we can attempt the following:

9

(a) Search for a polynomial vector v such that $g^2 \mid Q(v)$, with at least one of the coefficients of v having a nonzero constant term (e.g., one can take $g(t_1) = t_1$ and $v = e_2$ with the above example of Q_1). Then we can try a change of variables corresponding to replacing some basis vector e_i with $\frac{v}{g}$ where the i-th coefficient of v has nonzero constant term. This change of variables could introduce g in the denominator of some $x_i x_j$ coefficients for $j \neq i$. However, if we are fortunate, as always happens in our Mestre conic reduction, then the resulting quadratic form Q' will still have coefficients in R, and we will have removed a factor of g^2 from the discriminant.

(b) Assume $n \geq 3$, and if $n > 3$ that we have the higher divisibility condition $g^r \mid \Delta$ for some $r > \frac{n}{2}$. Then one can look for F-linearly independent vectors $v_1, \ldots, v_r \in R^n$ such that for each $1 \leq i \leq r$, $g \mid Q(v_i)$ but $g \nmid v_i$ (i.e., g does not divide every polynomial coefficient of v_i). Let $j_1, \ldots, j_{n-r}$ be such that that $e_{j_1}, \ldots, e_{j_{n-r}}, v_1, \ldots, v_r$ is a basis of F^n. Then the change of basis $\{e_1, \ldots, e_n\}$ to $\{ge_{j_1}, \ldots, ge_{j_{n-r}}, v_1, \ldots, v_r\}$ transforms Q to a quadratic form Q' with an extra factor of $g^{2(n-r)}$ in its discriminant, but now each coefficient of Q' is divisible by g. Thus the F-equivalent form $g^{-1}Q'$ has coefficients in R, and we will have removed a factor of g^{2r-n} from Δ.

Simple degree reduction preserves R-equivalence, whereas discriminant reduction only preserves F-equivalence. Our strategy is to try simple degree reduction, then discriminant reduction, and repeat until the discriminant is squarefree, and then finish with simple degree reduction.

First we give a baby example of simple degree reduction (1). Below and in the next section, $e_1, \ldots, e_n$ will denote the standard basis of the relevant vector space, and A_i will denote the Gram matrix for Q_i with respect to $\{e_1, \ldots, e_n\}$.

Example 3.1. Let $R = \mathbb{Q}[t]$, and let $\{e_1, e_2\}$ be the standard basis for $M = R^2$. Let $Q_1 = Q$ be the quadratic form on M given by

$$Q_1(x, y) = \left(t^4 + 1\right) x^2 + \left(2t^3 + 2t\right) xy + \left(t^2 - 1\right) y^2.$$

We can perform simple degree reduction as follows. We want to lower the degree of the x^2-coefficient, so let $v = a_1 e_1 + (a_2 + b_2 t) e_2$. Then

$$Q_1(v) = (a_1 + b_2)^2 t^4 + 2a_2(a_1 + b_2)t^3 + (a_2^2 + 2a_1 b_2 - b_2^2)t^2 + 2(a_1 - b_2)a_2 t + (a_1^2 - a_2^2).$$

Hence setting $b_2 = -a_1$ makes $Q_1(v)$ a degree 2 polynomial in t with t^2-coefficient $(a_2^2 - 3a_1^2)$, which we cannot make 0 for nontrivial choices of $a_1, a_2 \in \mathbb{Q}$. However, we can choose to make either the t^1- or t^0-coefficient 0 by taking $a_2 = 0$ or $a_2 = a_1$. Let us take $v_1 = e_1 - te_2$ so $Q_1(v_1) = 1 - 3t^2$, and let A_2 be the Gram matrix for Q_1 with respect to $\{v_1, e_2\}$. Let Q_2 be the associated quadratic form, i.e., the quadratic form which has Gram matrix A_2 with respect to $\{e_1, e_2\}$. In other words, Q_2 is obtained from Q_1 by the change of variables $x \mapsto x$, $y \mapsto -tx + y$. Then

$$Q_2(x, y) = \left(1 - 3t^2\right) x^2 + \left(4t\right) xy + \left(t^2 - 1\right) y^2.$$

Note that Q_2 has discriminant $12t^4 + 4$, so we cannot hope to reduce the degree any further over R.

We remark that straightforward diagonalization of Q_1 gives $\left(t^4 + 1\right) x^2 + \frac{(1 - 3t^4)}{(t^4 + 1)} y^2$ and for Q_2 gives $\left(1 - 3t^2\right) x^2 + \frac{3t^4 + 1}{3t^2 - 1} y^2$. Since the discriminant is irreducible over $\mathbb{Q}$, one cannot diagonalize over R and have polynomial coefficients of degree < 4.

A slightly more interesting example of (1) is given in the reduction of the Mestre conic from Q_1 to Q_2 in Section 4.2. Examples of (2a) are also given by the reductions from Q_2 to Q_3 and Q_3 to Q_4 in the same section. Then the reduction from Q_5 to Q_6 gives an example of (2b).

All of these types of reduction involve finding polynomials $h_i(t_1, \ldots, t_m)e_i$ so that the coefficients of $Q(v)$ satisfy certain conditions (e.g., no coefficients above a certain degree, or whatever relations are imposed upon the coefficients by a divisibility condition). In general, this may be computationally challenging, as it involves finding simultaneous solutions of many quadratic equations in many variables to find suitable h_i's.

As we do not have a general algorithm that will provably minimize the polynomial degree, rather than trying to formulate a precise reduction algorithm, we will just describe a few techniques which can be used to lessen the computational difficulties of these reduction steps in practice. The first two techniques apply to both (1) and (2). The subsequent techniques are just for discriminant reduction.

- *Inductively try more complicated polynomial combinations of basis vectors.* We begin by guessing certain forms for the polynomial coefficients h_i of v. Each term of some h_i with an unknown coefficient adds another variable to solve for in finding a $Q(v)$ satisfying our desired criteria. E.g., in Example 3.1 we need to make certain expressions in the unknown coefficients a_1, a_2, b_2 zero to reduce the degree. To minimize the number of unknowns, we begin by guessing as simple forms for the h_i's as we can hope for, and then try adding more terms as needed.

  In Example 3.1, since we wanted to remove t^4 from the coefficient of x^2, and the coefficient of y^2 is degree 2 in t, it makes sense to consider constant multiples $h_1(t)$ of e_1 plus linear multiples $h_2(t)$ of e_2 for v. In fact, we might have first tried $h_1(t) = a_1$ and $h_2(t) = b_2 t$, and then if this were not sufficient to remove the t^4 term, then we would try including a constant term in $h_2(t)$. If this were still unsuccessful, we could try letting $h_1(t)$ be a linear polynomial, which would necessitate $h_2(t)$ having degree 3. While this is of course not needed in such simple examples as Example 3.1, it may be necessary in the presence of additional variables (both more x_i's and more t_j's).

- *Look for coefficient conditions that factor.* Say for instance that $m = 2$, and we guess linear forms $h_i(t_1, t_2) = a_i + b_i t_1 + c_i t_2$ for each h_i. Then our desired conditions on $Q(v)$ may be something like $\deg Q(v) < 4$ or $(t_1 t_2 + 1)^2 \mid Q(v)$. In the former case, say, we want to make each $t_1^j t_2^{4-j}$ term of $Q(v)$ vanish. That gives 5 quadratic equations in $3n$ unknowns. How can we solve this?

  If our quadratic form is meant to reduce, we might hope it does for algebraically simple reasons. If we are fortunate, then some of these quadratic equations we need to solve may factor, as in the case of the t^4-coefficient of $Q_1(v)$ in Example 3.1. If we are even more fortunate, this forces one of our unknowns to be a certain linear combination of other unknowns, and we can reduce the number of unknowns and repeat. We are fortunate in this way in the case of the Mestre conic we reduce in Section 4.2.

- *Order of discriminant factor removal.* In removing discriminant factors g^r, it may be easier to remove certain factors before others. On one hand, it may help to try to start with factors g^2 where g is of small degree, or g only involves a small number of the variables $t_1, \ldots, t_m$, to more easily find h_i such that $g^2 | Q(v)$ or $g | Q(v)$. For instance, if $m = 2$, $g(t_1, t_2) = t_1$ and we want $g^2 | Q(v)$, then any $t_1^i t_2^j$ term in $Q(v)$

with $i \leq 1$ must vanish. However, the main issue we encountered in reducing our Mestre conic was that, at a given stage, attempting to remove one factor may lead to quadratic coefficient equations which factor, but attempting to remove other factors does not.

Thus for (2) we propose a process roughly of the following form. Try the simplest possible choices for h_i's for removing different factors g^r of the discriminant. Then pursue the ones that lead to linear relations among the unknowns, inductively adding more terms, and repeat until a factor is removed or a bound for the complexity of the h_i's is reached. This approach is what led us the (otherwise unexplained) order of removing discriminant factors we use in Section 4.2.

- *Change variables to remove constant terms.* If we want to remove a factor of say $(t_1 - 3)^2$ from the discriminant, writing down the divisibility conditions is a bit easier in practice if we first change the polynomial variables $t_1 \mapsto t_1 + 3$, so one is asking about removing a factor of t_1^2 from a transformed form Q'. For an example, see the reduction of Q_6 in Section 4.2.
- *Examine minors.* If some factor g^r divides the discriminant of Q, depending on n and r, it may not be clear whether we should try (2a) or (2b). In this case, one can examine the (determinant) minors of the Gram matrix. If some power of g divides sufficiently many minors, this suggest that (2b) may be possible.

Furthermore, if many of the diagonal minors are divisible by g then we can try looking for vectors v_i as in (2b) whose projection to e_j is 0, for each j in a set corresponding to the minors. E.g., if $r = n - 1$ and each diagonal minor is divisible is g, then we can look for vectors $v_1, \ldots, v_{n-1}$ such that the projection of v_i to e_i is 0 for each i. This helps reduce the number of unknowns we need to use, and is used in the reduction of Q_6' in Section 4.2.

4. Reducing the Mestre conic

4.1. **Mestre's construction of genus 2 curves.**

Suppose $k \subseteq \mathbb{C}$, and $(I_2, I_4, I_6, I_{10}) \in k^4$ are Igusa–Clebsch invariants for a genus 2 curve $\mathcal{C}/\mathbb{C}$ without extra automorphisms, i.e., $\mathrm{Aut}_{\mathbb{C}}(\mathcal{C}) \simeq C_2$. (In this section only, we use $\mathcal{C}$ rather than C to denote a genus 2 curve to avoid conflict with the notation for Clebsch invariants.) In [Mes91b], Mestre gave a method to determine whether $\mathcal{C}$ is defined over k, and if so, find a model. Mestre worked in terms of Clebsch invariants (A, B, C, D) rather than Igusa–Clebsch invariants. One can translate between these two sets of invariants via

$$I_2 = -120A, \quad I_4 = 90(-8A^2 + 75B), \quad I_6 = 540(16A^3 - 200AB + 375C)$$
$$I_{10} = -162(384A^5 - 6000A^3B + 18750AB^2 - 10000A^2C + 37500BC + 28125D).$$

Mestre defines two elements L and M of $\mathbb{Q}(A, B, C, D)[x_1, x_2, x_3]$ as

$$L = \sum_{1 \leq i,j \leq 3} L_{ij}x_ix_j \quad \text{and} \quad M = \sum_{1 \leq i,j,k \leq 3} M_{ijk}x_ix_jx_k,$$

with

$$L_{11} = 2C + \tfrac{1}{3}AB \qquad\qquad L_{22} = D$$
$$L_{12} = \tfrac{2}{3}(B^2 + AC) \qquad\qquad L_{23} = \tfrac{1}{3}B(B^2 + AC) + \tfrac{1}{3}C\left(2C + \tfrac{1}{3}AB\right)$$
$$L_{13} = D \qquad\qquad L_{33} = \tfrac{1}{2}BD + \tfrac{2}{9}C(B^2 + AC),$$

$$M_{111} = \tfrac{2}{9}\left(A^2C - 6BC + 9D\right)$$

$$M_{112} = \tfrac{1}{9}\left(2B^3 + 4ABC + 12C^2 + 3AD\right)$$

$$M_{113} = \tfrac{1}{9}\left(AB^3 + \tfrac{4}{3}A^2BC + 4B^2C + 6AC^2 + 3BD\right)$$

$$M_{122} = \tfrac{1}{9}\left(AB^3 + \tfrac{4}{3}A^2BC + 4B^2C + 6AC^2 + 3BD\right)$$

$$M_{123} = \tfrac{1}{18}\left(2B^4 + 4AB^2C + \tfrac{4}{3}A^2C^2 + 4BC^2 + 3ABD + 12CD\right)$$

$$M_{133} = \tfrac{1}{18}\left(AB^4 + \tfrac{4}{3}A^2B^2C + \tfrac{16}{3}B^3C + \tfrac{26}{3}ABC^2 + 8C^3 + 3B^2D + 2ACD\right)$$

$$M_{222} = \tfrac{1}{9}\left(3B^4 + 6AB^2C + \tfrac{8}{3}A^2C^2 + 2BC^2 - 3CD\right)$$

$$M_{223} = \tfrac{1}{18}\left(-\tfrac{2}{3}B^3C - \tfrac{4}{3}ABC^2 - 4C^3 + 9B^2D + 8ACD\right)$$

$$M_{233} = \tfrac{1}{18}\left(B^5 + 2AB^3C + \tfrac{8}{9}A^2BC^2 + \tfrac{2}{3}B^2C^2 - BCD + 9D^2\right)$$

$$M_{333} = \tfrac{1}{36}\left(-2B^4C - 4AB^2C^2 - \tfrac{16}{9}A^2C^3 - \tfrac{4}{3}BC^3 + 9B^3D + 12ABCD + 20C^2D\right),$$

and

$$L_{ij} = L_{ji}, \ M_{ijk} = M_{jik} = M_{ikj}.$$

The *Mestre conic* and the *Mestre cubic* associated to $\mathcal{C}$ (or equivalently, the Clebsch or Igusa–Clebsch invarants) are defined to be the projective varieties $L = 0$ and $M = 0$ over $\mathbb{Q}(A, B, C, D)$. In a slight abuse of terminology, we will occasionally say that L itself is the Mestre conic, and similarly for M.

Theorem 4.1 ([Mes91b]). *Suppose $(A, B, C, D) \in k^4$ are the Clebsch invariants of a genus 2 curve $\mathcal{C}/\mathbb{C}$ without extra automorphisms. Then $\mathcal{C}$ is defined over k if and only if the associated the Mestre conic $L = 0$ in $\mathbb{P}^2(k)$ has a k-rational point.*

If the Mestre conic associated to $\mathcal{C}/\mathbb{C}$ has k-rational points then those rational points are parameterized by a single projective parameter which we will call x. We will write $x_i = x_i(x)$ with $i = 1, 2, 3$ to denote this parametrization.

Theorem 4.2 ([Mes91b]). *Suppose $(A, B, C, D) \in k^4$ are the Clebsch invariants of a genus 2 curve $\mathcal{C}/\mathbb{C}$ without extra automorphisms and the associated Mestre conic $L = 0$ has a k-rational point. Then a model for $\mathcal{C}$ over k is given by*

$$y^2 = M(x_1(x), x_2(x), x_3(x)),$$

where $M = 0$ is the associated Mestre cubic.

Finally, we elaborate on the condition that $\mathcal{C}/\mathbb{C}$ has no extra automorphisms. The possibilities for extra automorphisms of genus 2 curves were determined by Bolza. The reduced automorphism group of $\mathrm{Aut}_{\mathbb{C}}^{\mathrm{red}}(\mathcal{C})$ is $\mathrm{Aut}_{\mathbb{C}}(\mathcal{C})$ modulo the hyperelliptic involution. If $\mathcal{C}$ has extra automorphisms, then $\mathrm{Aut}_{\mathbb{C}}^{\mathrm{red}}(\mathcal{C})$ either contains an involution or has order 5. The latter case happens exactly when the (Clebsch or Igusa–Clebsch invariants) of $\mathcal{C}$ are $(0 : 0 : 0 : 1) \in \mathbb{P}^3_{1,2,3,5}$.

As explained in [Mes91b], the Mestre conic attached to a genus 2 curve $\mathcal{C}/\mathbb{C}$ is singular if and only if the reduced automorphism group of $\mathcal{C}$ contains an involution. Thus the condition that $\mathcal{C}/\mathbb{C}$ has no extra automorphism can be restated as: the Mestre conic $L = 0$ is nonsingular and I_2, I_4 and I_6 are not all 0.

4.2. The general case. Here we study the Mestre conic L associated to a point (z, g, h) of Y, i.e., to Igusa–Clebsch invariants $\left(24g + 6 : 9g^2 : 81g^3 + 18g^2 + 36h : 4h^2\right)$. After scaling by $2^4 \cdot 3^7 \cdot 5^{14}$, the Mestre conic $L : \sum_{i,j=1}^3 L_{ij} x_i x_j = 0$ defined above has coefficients

$$L_{11} = 189843750(-96g^3 - 337g^2 - 108g + 400h - 9)$$

$$L_{12} = -2531250(-144g^4 - 1299g^3 - 754g^2 + 2000gh - 144g + 500h - 9)$$

$$L_{13} = L_{22} = -3750(1944g^5 + 40905g^4 + 36990g^3 - 68400g^2h + 11835g^2 - 43200gh$$
$$+ 50000h^2 + 1620g - 5400h + 81)$$

$$L_{23} = 450(324g^6 + 14931g^5 + 19395g^4 - 25800g^3h + 9105g^3 - 30100g^2h + 2020g^2$$
$$- 8400gh + 10000h^2 + 216g - 700h + 9)$$

$$L_{33} = -(2916g^7 + 283338g^6 + 499041g^5 - 496800g^4h + 319140g^4 - 915300g^3h$$
$$+ 525000g^2h^2 + 101160g^3 - 426300g^2h + 500000gh^2 + 17214g^2 - 76800gh$$
$$+ 100000h^2 + 1512g - 4800h + 54)$$

The discriminant of L, by which we mean the determinant of the Gram matrix, is then

$$\mathrm{disc}(L) = 2^7 \cdot 3^3 \cdot 5^{22} \cdot h^2(8h - 9g^2)^2 z^2.$$

Set $Q_1 = L$ and let A_1 be the Gram matrix of Q_1 with respect to the standard basis $\{e_1, e_2, e_3\}$. We will now perform a series of reductions on the Mestre conic using the techniques described in the previous section.

Note that the x_1^2, x_2^2 and x_3^2 coefficients of $L = Q_1$ are respectively degree 3, 5 and 7 polynomials in g (and degrees 1, 2 and 2 in h). First we want to try to reduce the degree in g of the x_3^2 coefficient. Consider $v_1 = a_1 g^2 e_1 + a_2 g e_2 + e_3$, where a_1, a_2 denote rational variables. The $Q_1(v_1)$ is degree 7 in g, and the g^7-coefficient is $-2916(2500a_1 - 50a_2 + 1)^2$. So set $a_2 = 50a_1 + \frac{1}{50}$. This makes the g^6-coefficient of $Q_1(v_1)$ equal $-\frac{3^5 5^4}{2}(1250a_1 - 1)^2$. Taking $a_1 = \frac{1}{1250}$ gives

$$Q_1(v_1) = -2916g^5 - 24354g^4 + 10800g^3h - 1500000g^2h^2 - 21483g^3 + 78000g^2h$$
$$+ 40000gh^2 - \frac{14259}{2}g^2 + 39000gh - 100000h^2 - 1026g + 4800h - 54,$$

where $v_1 = \frac{1}{1250}g^2 e_1 + \frac{3}{50}g e_2 + e_3$. Thus we now consider the Gram matrix A_2 for Q_1 with respect to the basis $\{e_1, e_2, v_1\}$. Let Q_2 be the resulting quadratic form from this change of variables, i.e., $Q_2(v) = {}^t v A_2 v$. In particular, the x_3^2-coefficient of Q_2 is $Q_1(v_1)$.

The x_1^2, x_2^2 and x_3^2 coefficients of Q_2 are degrees 3, 5 and 5 in g (and no other coefficient has higher degree). We may try to reduce the coefficient degrees for x_2^2 and x_3^2 by replacing e_2 and e_3 with vectors of the form $a_1 g e_1 + e_2$ and $b_1 g e_1 + e_3$. In this way, one to reduce Q_2 to a quadratic form whose coefficients are elements of $\mathbb{Q}[g, h]$ of degree ≤ 4, but there are no obvious ways to further reduce the degree from there, and this reduction does not make the next step any easier, so we will not do this.

Instead, we will next remove a polynomial factor from the discriminant of Q_2, which is a rational multiple of $h^2(8h - 9g^2)^2 z^2$. The h^2 factor has the lowest degree, so we will begin with that. We will find a vector $v_2 = (a_1 + b_1 g)e_1 + a_2 e_2 + a_3 e_3$ such that $Q_2(v_2)$ is divisible by h^2. This is essentially the simplest polynomial combination of standard basis vectors where we can hope to kill off all of the g^j terms in $Q_2(v_2)$, and it turns out to be sufficient.

The constant term of $Q_2(v_2)$ is $-54\left(5625\,a_1 - 75\,a_2 + a_3\right)^2$, so we set $a_3 = 75a_2 - 5625a_1$. Now we kill off the highest degree g^j terms. Then the g^5-coefficient of $Q_2(v_2)$ is $-1822500(225a_1 - a_2 - 100b_1)^2$. Set $a_2 = 225a_1 - 100b_1$. Then the g^4-coefficient is $-118652343750(3a_1 - b_1)^2$. Setting $b_1 = 3a_1$ yields $Q_2(v_2)$ is a multiple of h^2. Specifically, take $a_1 = 2^{-2} \cdot 3^{-2} \cdot 5^{-6}$, and then $Q_2(v_2) = -2(300g^2 + 2g + 3)h^2$, where $v_2 = \frac{1}{562500}((1 + 3g)e_1 - 75e_2 - 11250e_3)$. Let A_3 be the Gram matrix of Q_2 with respect to the basis $\{e_1, e_2, \frac{1}{h}v_2\}$, and Q_3 the associated quadratic form. So Q_3 is not $\mathbb{Q}[g, h]$-equivalent to Q_2, but after specializing to any $g, h \in \mathbb{Q}$ with $h \neq 0$, the forms Q_2 and Q_3 are $\mathbb{Q}$-equivalent.

Now we will remove the $(8h - 9g^2)^2$ factor from the determinant. The degrees in g of the x_1^2, x_2^2 and x_3^2 coefficients of Q_3 are 3, 5 and 3. Let $v_3 = (a_1 + b_1g)e_1 + a_2e_2 + (a_3 + b_3g)e_3$. We want $Q_3(v_3)$ to be a multiple of $(8h - 9g^2)^2$. To kill the constant term of $Q_3(v_3)$, we need to set $a_3 = 225a_2 - 16875a_1$. Then to kill the h-coefficient, we need $a_2 = 75a_1$. Then to kill the g^2-coefficient, $b_3 = 67500a_1 - 16875b_1$. At this point there are only nonzero g^5, g^4 and g_2h and h^2 terms, so for $Q_3(v_3)$ to be a multiple of $(8h - 9g^2)^2$ it needs to be a rational multiple and the g^5 term must vanish. This is accomplished with $b_1 = \frac{3}{2}a_1$. In summary $v_3 = a_1((1 + \frac{3}{2}g)e_1 + 75e_2 + \frac{84375}{2}ge_3)$. Taking $a_1 = 2 \cdot 3^{-1} \cdot 5^{-6}$ then gives $Q_3(v_3) = -30(8h - 9g^2)^2$. Now let A_4 be the Gram matrix of $-Q_3$ with respect to the basis $\{\frac{e_1}{1875}, \frac{v_3}{8h - 9g^2}, e_3\}$, and Q_4 the associated quadratic form,

$$Q_4 : \left(5184g^3 + 18198g^2 + 5832g - 21600h + 486\right) x_1^2 + \left(612g + 108\right) x_1x_2 + 30x_2^2$$
$$+ \left(288g^2 + 684g - 4000h + 108\right) x_1x_3 + (-240g + 12) x_2x_3 + \left(600g^2 + 4g + 6\right) x_3^2.$$

Specializing g, h to any rationals such that $h \neq 0$ and $8h \neq 9g^2$, Q_4 is $\mathbb{Q}$-equivalent to the original Mestre conic L. The discriminant of Q_4 is $-9600z^2$.

Now there is no obvious way to further reduce the degree, and indeed, it seems that there is not much further simplification that can be done over $\mathbb{Q}(g, h)$. The reduction we perform next will not preserve $\mathbb{Q}$-equivalence of quadratic forms (even assuming $z \neq 0$) if g, h are rational but z is not.

Let Q_5 be the quadratic form over $\mathbb{Q}[m, n]$ obtained by converting Q_4 from (g, h) to (m, n) via (2.2). Let A_5 be the Gram matrix of Q_5 with respect to the standard basis. The coefficients of Q_5 are elements of $\mathbb{Q}[m, n]$ of degree ≤ 6, and the discriminant is

$$-\frac{96}{25}n^2(m^2 - 5n^2)^2(m^2 - 5n^2 - 5)^2.$$

Let $v_5 = a_1e_1 + a_2e_2 + a_3e_3$ be a rational linear combination of the standard basis vectors. Then $Q_5(v_5)$ has constant term $\frac{3}{250}(63a_1 - 50a_2 - 70a_3)^2$. Setting $a_3 = \frac{1}{70}(63a_1 - 50a_2)$, then gives that $Q_5(v_5) = p_1(m, n)(m^2 - 5n^2)$ for some polynomial $p_1(m, n)$ with constant term $5(441a_1 - 50a_2)^2$. Hence we set $a_1 = 50$ and $a_2 = 441$ (which makes $a_3 = -270$) to get

$$Q_5(v_5) = 30\left(16m^2 - 80n^2 + 2729\right)\left(m^2 - 5n^2\right)^2.$$

Let A_6 be the Gram matrix of Q_5 with respect to the basis $\{\frac{v_5}{m^2 - 5n^2}, e_2, e_3\}$, and Q_6 the associated quadratic form, which has polynomial degree 4 and discriminant $-9600n^2(m^2 - 5n^2 - 5)^2$.

Next one might try to remove the n^2 factor from the discriminant, but evaluating Q_6 on simple combinations such as $v = (a_1 + b_1m)e_1 + a_2e_2 + a_3e_3$ leads to polynomials without linear factors in a_1, a_2, a_3, b_1 for the coefficients of powers of m. So it is not immediately clear how to find some v such that $Q_6(v)$ is divisible by n^2. On the other hand, the

diagonal minors of A_6 are divisible by $(m^2 - 5n^2 - 5)$ which suggests we can remove a factor of $(m^2 - 5n^2 - 5)$ from the discriminant by working with combinations of just 2 of the standard basis vectors at a time.

To make it easier to look for multiples of $(m^2 - 5n^2 - 5)$, we first make the change of variables $m = m + 5$, $n = n + 2$. This changes $(m^2 - 5n^2 - 5)$ to $(m^2 - 5n^2 + 10m - 20n)$, which has no constant term. Let Q_6' be the resulting quadratic form. Now one can look for rational linear combinations v of pairs of the basis vectors e_1, e_2 and e_3 such that $Q_6'(v)$ has no constant term. In particular, $u_1 = e_1 - 53e_2$ and $u_2 = 11e_2 - 15e_3$ work and both $Q_6'(u_1)$ and $Q_6'(u_2)$ are divisible by $(m^2 - 5n^2 + 10m - 20n)$. Let A_7' be the Gram matrix for $\frac{1}{6}(m^2 - 5n^2 + 10m - 20n)^{-1}Q_6'$ with respect to the basis $\{\frac{u_1}{4}, \frac{u_2}{5}, (m^2 - 5n^2 + 10m - 20n)e_2\}$, and Q_7' the resulting quadratic form. Let Q_7 and A_7 denote the result of reverting Q_7' and A_7' back to our original variables $m = m - 5$, $n = n - 2$. Then Q_7 is:

$$Q_7 : 5x_1^2 + 2mx_1x_2 + \left(m^2 - 5n^2 - 4\right)x_2^2$$
$$+ \left(4m^2 - 20n^2 - 20\right)x_2x_3 + \left(5m^2 - 25n^2 - 25\right)x_3^2.$$

The discriminant of Q_7 is $-25n^2(m^2 - 5n^2 - 5)$.

Now we can use a vector of the form $v = (a_1 + b_1m)e_1 + a_2e_2 + a_3e_3$ to removed the n^2 factor from the determinant. Explicitly, by zeroing out of coefficients of powers of m in $Q_7(v)$, we find that $Q_7(v_7) = -25n^2$ where $v_7 = me_1 - 5e_2 + 2e_3$. Let A_8 be the Gram matrix for $\frac{1}{5}Q_7$ with respect to the basis $\{e_1, \frac{v_7}{n}, e_3\}$. Then the associated quadratic form is

$$Q_8 : x_1^2 - 5x_2^2 + (m^2 - 5n^2 - 5)x_3^2.$$

For any $m, n \in \mathbb{Q}$ such that $\operatorname{disc} L \neq 0$ the form $Q_8 \in \mathbb{Q}[x_1, x_2, x_3]$ is similar to Q_1. Thus for such m, n, the Mestre conic L has a rational point if and only if $\pm(m^2 - 5n^2 - 5)$ is a norm from $\mathbb{Q}(\sqrt{5})$. (Note that -1 is a norm from $\mathbb{Q}(\sqrt{5})$.) Consequently, the analogue holds for any extension $k \supseteq \mathbb{Q}$.

4.3. Points at infinity.

Here we consider k-rational points on $Y_-(5)$ not coming from affine coordinates $(z, g, h) \in Y$. By Proposition 2.3, if such a point corresponds to a genus 2 curve C, there are two possibilities for the Igusa–Clebsch invariants: (1) $(0 : 0 : 0 : 1)$ when $\sqrt{5} \in k$, and (2) $(8 : 1 : 3 : s)$ where $s \in k^\times$ and $3125s^2 - 8s$ is a square in k.

We wish to determine when C can be defined over k in these cases. In case (1), C is already defined over $\mathbb{Q}$ with a model $y^2 = x^5 - 1$. So we only need to analyze case (2).

Let us consider the Mestre conic for Igusa–Clebsch invariants as in (2). After replacing x_1 with $2^{-1} \cdot 3^2 \cdot 5^3 x_1$, x_2 with $2 \cdot 3^3 \cdot 5^5 x_2$ and x_3 with $2^2 \cdot 3^4 \cdot 5^7 x_3$, the Gram matrix A_1 for the Mestre conic $Q_1 = L$ is

$$A_1 = \begin{pmatrix} -1 & 2 & -3125s + 2 \\ 2 & -6250s + 4 & 4 \\ -3125s + 2 & 4 & -43750s + 4 \end{pmatrix}$$

Then

$$\det A = 2 \cdot 5^{10}(3125s - 8)s^2.$$

Then $Q_1(a_1e_1 + a_2e_2 + a_3e_3)$ has constant term $-(a_1 - 2a_2 + 2a_3)^2$. Now letting A_2 be the Gram matrix with respect to the basis $\{e_1, \frac{1}{25}(2e_1 + e_2), \frac{1}{125}(2e_1 - e_3)\}$, we see

$$A_2 = \begin{pmatrix} 1 & 0 & 25s \\ 0 & -10s & 2s \\ 25s & 2s & -2s \end{pmatrix}.$$

16

Scale A_2 by s and replace x_2 and x_3 with x_2/s and x_3/s, respectively, to get the equivalent Gram matrix

$$A_3 = \begin{pmatrix} -s & 0 & 25s \\ 0 & -10 & 2 \\ 25s & 2 & -2 \end{pmatrix}$$

with quadratic form

$$Q_3 : -sx_1^2 - 10x_2^2 + 50x_1x_3 + 4x_2x_3 - 2x_3^2$$

This has determinant $2s(3125s - 8)$, and the associated quadratic form is $\mathbb{Q}$-equivalent to the diagonal form

$$2x_1^2 + 5sx_2^2 - (3125s - 8)x_3^2.$$

Assuming that $s(3125s - 8)$ is a square in $k^\times$, this form is k-equivalent to the forms

$$2x_1^2 + 5sx_2^2 - sx_3^2 \sim 2sx_1^2 - (x_3^2 - 5x_2^2).$$

Clearly, this has a rational point if and only if $2s$ is a norm from $k(\sqrt{5})$ (which is automatic if $\sqrt{5} \in k$).

5. Moduli for rational curves

Here we state our main result and complete the proof.

Theorem 5.1. *Let C be a genus 2 curve with RM-5 defined over k. Then the Igusa–Clebsch invariants $(I_2 : I_4 : I_6 : I_{10}) \in \mathbb{P}^3_{1,2,3,5}$ are of one of the following forms:*

(1) $\left(24g + 6 : 9g^2 : 81g^3 + 18g^2 + 36h : 4h^2\right)$ *for a k-rational solution (z, g, h) to* (2.1) *such that such that $30g + 4$ is a norm from $k(\sqrt{5})$ and $hz(8h - 9g^2) \neq 0$;*

(2) $(8 : 1 : 3 : s)$ *where $s \in k^\times$ such that $s(3125s - 8)$ is a square in $k^\times$ and $2s$ is a norm from $k(\sqrt{5})$;*

(3) $(12(4g + 1) : 36g^2 : 36(18g + 13)g^2 : 162g^4)$ *where $g \in k^\times$ such that $-3(128g + 9)$ is a square in k;*

(4)

$$\left(20(2m^2 - 3) : 25(m - 3)^2(m + 3)^2 : \right.$$
$$\left. 5(m + 3)^2(75m^4 - 378m^3 + 428m^2 + 474m - 711) : 8(m - 2)^4(m + 3)^6)\right)$$

where $m \in k$ or $m = \sqrt{5}$;

(5) $(8 : 1 : 3 : \frac{8}{3125})$;

(6) $(0 : 0 : 0 : 1)$ *if $\sqrt{5} \in k$.*

Cases (1) and (2) correspond to $\mathrm{Aut}^{\mathrm{red}}_{\mathbb{C}}(C) = \{1\}$. Cases (3)–(5) correspond to $\mathrm{Aut}^{\mathrm{red}}_{\mathbb{C}}(C)$ containing an involution, and case (6) corresponds to $\#\mathrm{Aut}^{\mathrm{red}}_{\mathbb{C}}(C) = 5$.

Conversely, if C is a genus 2 curve over $\mathbb{C}$ with Igusa–Clebsch invariants in one of the forms (1)–(6), then C can be defined over k and C has potential RM-5. Moreover, in case (1), if $\mathrm{End}_{\mathbb{C}}(\mathrm{Jac}(C))$ is commutative, then C has RM-5 defined over k.

In this theorem, by "a norm from $k(\sqrt{5})$" we mean in the image of the relative norm map from $k(\sqrt{5})$ to k. Thus such a norm condition is automatically satisfied if $\sqrt{5} \in k$.

In Section 5.1, we reformulate condition (1) in terms of (m, n), which removes the need to check (2.1) to determine the existence of a k-rational point $(z, g, h) \in Y$ given $g, h \in k$.

17

Remark 5.2. Suppose $k = \mathbb{Q}$ now, and that C is a genus 2 curve over $\mathbb{Q}$ with Igusa–Clebsch invariants of one of the forms (1)–(5) in the theorem. We would like to be able to say when C (or a twist) actually has RM-5 defined over k. Write $A = \text{Jac}(C)$. Generically, $\text{End}_{\mathbb{C}}(A) \simeq \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$ in case (1) so the RM-5 will be defined over $\mathbb{Q}$, but there seems to be no simple way to describe the moduli points where A has (split or non-split) quaternionic multiplication over $\mathbb{C}$. We do not know whether the RM-5 must be defined over $\mathbb{Q}$ if $\text{End}_{\mathbb{C}}(A)$ is not commutative.

In case (2), we also expect that generically $\text{End}_{\mathbb{C}}(A) \simeq \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$, and the RM will be defined over $\mathbb{Q}$—however we have not checked that the points satisfying (2) always correspond rational points on $Y_-(5)$ so cannot apply Proposition 2.1. Still, one can check in examples for case (2), e.g., $s = \frac{2}{25}$, that one gets a genus 2 curve with RM-5 defined over $\mathbb{Q}$.

In cases (3)–(5), $\text{End}_{\mathbb{C}}(A)$ is an order in a 2×2 matrix algebra, hence not commutative, and Proposition 2.1 does not apply. Here C has more than just quadratic twists, and one twist may have RM-5 defined over $\mathbb{Q}$, and another may not. For instance if $C : y^2 = x^6 - x^4 + 4x^2 - 1$, which has Igusa–Clebsch invariants $(88 : 169 : 28561 : 57122)$ corresponding to $g = -\frac{13}{96}$ in case (3), then C has RM-5 defined over $\mathbb{Q}$, but the twist corresponding to $x \mapsto \sqrt{-1}x$ does not. This may be checked, for instance, by computing Galois groups and using Proposition 2.2. We do not know whether there will always be some twist with RM-5 defined over $\mathbb{Q}$ in these cases.

Proof. Suppose C is a genus 2 curve over $\mathbb{C}$ with RM-5 and Igusa–Clebsch invariants $(I_2 : I_4 : I_6 : I_{10})$ defined over k. If C as well as the RM-5 is defined over k, then by Proposition 2.3, we know that the Igusa–Clebsch invariants must be of the form $(0 : 0 : 0 : 1)$ (only if $\sqrt{5} \in k$), $(8 : 1 : 3 : s)$ for $s \in k^{\times}$ such that $3125s^2 - 8s$ is a square in k, or they correspond to a k-rational point $(z, g, h) \in Y$ with $h \neq 0$, so we may assume our Igusa–Clebsch invariants take one of these forms.

As explained earlier, C has a model over k if and only if the Mestre conic has a k-rational point or C has extra automorphisms. Thus, to prove both directions of the theorem, it will suffice to show that: (i) when C has no extra automorphisms, the Mestre conic has a k-rational point exactly in cases (1) and (2); and (ii) the Igusa–Clebsch invariants from Proposition 2.3 corresponding to curves with extra automorphisms are described exactly by cases (3)–(6).

If the Igusa–Clebsch invariants are $(0 : 0 : 0 : 1)$, then C has a model over $\mathbb{Q}$ given by $y^2 = x^5 - 1$, and the RM-5 is defined over $\mathbb{Q}(\sqrt{5})$. This verifies the theorem (in both directions) for these Igusa–Clebsch invariants, i.e., case (6).

Assume now that the Igusa–Clebsch invariants come from a k-rational point $(z, g, h) \in Y$ with $h \neq 0$.

First suppose $z(8h - 9g^2) \neq 0$, so that the Mestre conic is nonsingular and C has no extra automorphisms. Then the reduction we performed over $\mathbb{Q}$ in Section 4.2 implies that the Mestre conic has a k-rational point if and only if $30g + 4 = m^2 - 5n^2 - 5$ is a norm from $k(\sqrt{5})$, except in the two special cases $g \in \{-\frac{3}{10}, -\frac{2}{15}\}$, where there is not a one-to-one correspondence between the (z, g, h) and (m, n) coordinates. In Section 5.1 below, we check that one has a k-rational $(z, g, h) \in Y$ for $g \in \{-\frac{3}{10}, -\frac{2}{15}\}$ if and only if $\sqrt{5} \in k$, and in this case the Mestre conic always has a k-rational point. This, together with Proposition 2.1, proves (both directions of) the theorem in case (1).

For the cases where the Mestre conic is singular, the k-rational $(z, g, h) \in Y$ with $z(8h - 9g^2) = 0$ correspond to Igusa–Clebsch invariants of the forms in cases (3) and (4). The details are given in Section 5.2.

Finally, suppose the Igusa–Clebsch invariants are of the form $(8 : 1 : 3 : s)$, where $s \in k^\times$ and $3125s^2 - 8s$ is a square in k. Both directions of the theorem in case (2) follows from the reduction of the Mestre conic in Section 4.3. Case (5) follows from Section 5.2. $\square$

5.1. Translation to (m, n)-coordinates.
Here we explain how to translate Theorem 5.1 into the rational model $\mathbb{P}^2_{m,n}$ for $Y_-(5)$, and treat the exceptional cases $g \in \{-\frac{3}{10}, -\frac{2}{15}\}$ in the proof of Theorem 5.1.

Recall that there is a one-to-one correspondence between k-rational coordinates $(z, g, h) \in Y$ and k-rational coordinates $(m, n) \in \mathbb{A}^2$ such that $g = \frac{m^2 - 5n^2 - 9}{30} \notin \{-\frac{3}{10}, -\frac{2}{15}\}$.

If $g = -\frac{3}{10}$, the equation for Y becomes $z^2 = \frac{4}{3125}(3125h - 27)^2$. Hence there are no k-rational points $(z, -\frac{3}{10}, h)$ on Y if $\sqrt{5} \notin k$. If $\sqrt{5} \in k$, then for all $h \in k$, there is a k-rational $(z, -\frac{3}{10}, h) \in Y$. Here the associated Mestre conic is nonsingular if $h \notin \{0, \frac{81}{800}, \frac{27}{3125}\}$, and always has a k-rational point. For instance, in Sage we find the k-rational point $(\frac{64000}{81}h^2 - \frac{2368}{75}h + \frac{284}{3125} : \frac{128}{15}h - \frac{51}{1250} : h + \frac{9}{1000})$.

If $g = -\frac{2}{15}$, the equation for Y becomes $z^2 = \frac{4}{3125}(3125h - 2)^2$. Similarly, there are no k-rational points $(z, -\frac{2}{15}, h)$ on Y if $\sqrt{5} \notin k$, but if $\sqrt{5} \in k$, then there is a k-rational $(z, -\frac{2}{15}, h) \in Y$ for all $h \in k$. The associated Mestre conic is nonsingular if $h \notin \{0, \frac{1}{50}, \frac{2}{3125}\}$. Again, one may check in Sage that the Mestre conic always has a rational point.

These calculations complete the proof of Theorem 5.1 in case (1). Consequently, we may alternatively formulate case (1) of the theorem as saying that $(I_2 : I_4 : I_6 : I_{10})$ is of one of the following forms:

(1a)

$$(5.1) \quad (-20(-2m^2 + 10n^2 + 3) : 25(-m^2 + 5n^2 + 9)^2 :$$

$$- 5(-75m^6 + 1125m^4n^2 - 5625m^2n^4 + 9375n^6 - 72m^5 + 720m^3n^2 - 1800mn^4$$

$$+ 1165m^4 - 11650m^2n^2 + 29125n^4 + 360m^3 - 1800mn^2 - 5985m^2 + 29925n^2 + 6399) :$$

$$8(m^5 - 10m^3n^2 + 25mn^4 + 5m^4 - 50m^2n^2 + 125n^4 - 5m^3 + 25mn^2 - 45m^2 + 225n^2 + 108)^2)$$

where $(m, n) \in k^2$ such that $m^2 - 5n^2 - 5$ is a norm from $k(\sqrt{5})$, $n(m^2 - 5n^2)(m^2 - 5n^2 - 5) \neq 0$ and $8m^5 - 80m^3n^2 + 200mn^4 - 85m^4 + 850m^2n^2 - 2125n^4 - 40m^3 + 200mn^2 + 1890m^2 - 9450n^2 - 9261 \neq 0$;

(1b) $(-12 : 81 : 36000h - 567 : 400000h^2)$ if $\sqrt{5} \in k$ and $h \in k \setminus \{0, \frac{81}{800}, \frac{27}{3125}\}$; or

(1c) $(14 : 4 : 4500h + 16 : 12500h^2)$ if $\sqrt{5} \in k$ and $h \in k \setminus \{0, \frac{1}{50}, \frac{2}{3125}\}$.

In particular, when $k = \mathbb{Q}$, we can deduce the following precise interpretation of Theorem 1.1 from Theorem 5.1: The set of all genus 2 curves C with RM-5 over $\mathbb{Q}$ up to $\mathbb{C}$-isomorphism such that $\text{Aut}^{\text{red}}_{\mathbb{C}}(C) = \{1\}$ excluding the 1-parameter family in case (2) correspond to points (m, n) with Igusa–Clebsch invariants as in (1a). Moreover, each tuple of Igusa–Clebsch invariants as in (1a) comes from such a curve, except possibly when these Igusa–Clebsch invariants lead to a non-commutative endomorphism algebra, in which case we only know that such (m, n) corresponds to a curve defined over $\mathbb{Q}$ with potential RM-5. Further, distinct points (m, n) as in (1a) correspond to distinct $\mathbb{C}$-isomorphism classes of genus 2 curves by Proposition 2.3.

19

5.2. Singularities of the Mestre conic.

Here we describe the k-rational Igusa–Clebsch invariants of types (2) and (3) in Proposition 2.3 for which the the Mestre conic is singular. By [CQ05], these invariants always yield a genus 2 curve defined over k. This will complete the proof of Theorem 5.1.

First, as in Section 4.2, let L be the Mestre conic associated to a $(z, g, h) \in Y$. Then the Mestre conic is singular if and only if $h = 0$, $h = \frac{9g^2}{8}$ or $z = 0$.

We remark that the curve $h = 0$ on Y is given by $z^2 = -54(6g+1)^2 g^3$, and the k-rational points are parametrized by $(g, 0)$ where $-6g$ is a square in k. However, $h = 0$ means that $I_{10} = 0$, so these points do not correspond to genus 2 curves.

The curve $h = \frac{9g^2}{8}$ on Y is given by $z^2 = -\frac{27}{16}(128g + 9)g^2(3g - 4)^2$, and the k-rational points are parametrized by $(g, \frac{9g^2}{8})$ where $-3(128g + 9)$ is a square in k. This completes case (3) of the theorem.

Now suppose $z = 0$, which means that either $g \in \{-\frac{3}{10}, -\frac{2}{15}\}$ or $n = 0$. If $g = -\frac{3}{10}$ or $g = -\frac{2}{15}$, then $h = \frac{27}{3125}$ or $h = \frac{2}{3125}$, respectively, and these are clearly k-rational points on Y. The corresponding Igusa–Clebsch invariants are $(20 : 225 : 1185 : -384)$ and $(70 : 100 : 2360 : 16)$, respectively. As noted in [EK14], the k-rational points on Y with $n = 0$ are given by

$$(z, g, h) = \left(0, \frac{m^2 - 9}{30}, \frac{(m - 2)^2(m + 3)^3}{12500} \right), \quad m \in k.$$

Viewing this as a map from points $(m, 0)$ to $(0, g, h)$, note that $m = 0$ and $m = \pm\sqrt{5}$ respectively map to $(g, h) = (-\frac{3}{10}, \frac{27}{3125})$ and $(-\frac{2}{15}, \frac{2}{3125})$. This gives case (4) of the theorem.

Now we consider the "points at infinity" discussed in Section 4.3. The Mestre conic associated to Igusa–Clebsch invariants $(8 : 1 : 3 : s)$ for $s \in k^\times$ is singular if and only if $s = \frac{8}{3125}$. In terms of Wilson's moduli, this point corresponds to $(z_6, s_2, \sigma_5) = (0, -\frac{5}{2}, \frac{1}{2})$. Here Wilson's discriminant Δ' is 0. Using Magma, we can construct a rational genus 2 curve with invariants $(8 : 1 : 3 : \frac{8}{3125})$, namely

$$(5.2) \qquad y^2 = f(x) = (2x^3 - 2x^2 - x - 1)(x^3 - x^2 + 2x + 2).$$

This yields case (5) of the theorem.

Remark 5.3. Calculations in Magma indicate that the curve in (5.2) has conductor 800^2, and corresponds to the weight 2 modular form $f(z) = q - \sqrt{5}q^3 - 2\sqrt{5}q^7 + 2q^9 - \sqrt{5}q^{11} + \cdots$ with Fourier coefficient ring $\mathbb{Z}[\sqrt{5}]$ and LMFDB label 800.2.a.1.

6. Generic models

In this section we give explicit rational Weierstrass models for (m, n) in the birational model $\mathbb{P}^2_{m,n}$ for $Y_-(5)$.

Proposition 6.1. *Let $k \subseteq \mathbb{C}$ be a field which does not contain $\sqrt{5}$. For any $m, n \in k$ such that $-(m^2 - 5n^2 - 5)$ is the norm of some nonzero element $\eta \in k(\sqrt{5})/k$, let $\mu := m + n\sqrt{5}$ and define C/k be the curve with Weierstrass model*

$$y^2 = \mathrm{Tr}\left(\mu^2 \eta^3 \left(\frac{1 - x\sqrt{5}}{1 + x\sqrt{5}} \right)^3 - 2N(\mu)\mu\eta^2 \left(\frac{1 - x\sqrt{5}}{1 + x\sqrt{5}} \right)^2 - 5N(\mu)(N(\mu) - 5) \right)(1 - 5x^2)^3,$$

where N and Tr denote the norm and trace from $k(\sqrt{5})$ to k respectively. When C is a genus 2 curve, the Igusa–Clebsch invariants of C are as in (5.1), i.e. C corresponds to the point (m, n) in the $\mathbb{P}^2_{m,n}$ birational model for $Y_-(5)$.

Note that the right hand side of the Weierstrass equation given in Proposition 6.1 is indeed a sextic in x; the factor of $(1 - 5x^2)^3$ clears denominators.

Remark 6.2. Since $m^2 - 5n^2 - 5 + u^2 - 5v^2 = 0$ is a quadric in $\mathbb{P}^4$ it is birational to $\mathbb{P}^3$, so one may generically express the family of curves in Proposition 6.1 in terms of a 3-parameter (a, b, c). For instance, one may generically write

$$v = (4a + 2c)/(5a^2 - b^2 + 5c^2 - 1), \quad m = 5av - 2, \quad n = -bv, \quad u = 5cv - 1$$

to get a 3-parameter family of genus 2 curves with RM-5. However, the resulting models are rather complicated and we omit them.

Proposition 6.3. *Let $k \subseteq \mathbb{C}$ be a field containing $\sqrt{5}$. For any $m, n \in k$, define C/k to be the curve with Weierstrass model*

$$
\begin{aligned}
y^2 = \ & (m - n\sqrt{5})^2 x^6 - 2(m^2 - 5n^2)(m - n\sqrt{5})x^5 - 10(m^2 - 5n^2)(m^2 - 5n^2 - 5)x^3 \\
& -2(m^2 - 5n^2)(m^2 - 5n^2 - 5)^2(m + n\sqrt{5})x - (m^2 - 5n^2 - 5)^3(m + n\sqrt{5})^2.
\end{aligned}
$$

When C is a genus 2 curve, the Igusa–Clebsch invariants of C are as in (5.1), i.e. C corresponds to the point (m, n) in the $\mathbb{P}^2_{m,n}$ birational model for $Y_-(5)$.

Note that both the x^4- and x^2-coefficients are zero in this model.

Kumar and Mukamel also gave a different simple model for RM-5 curves over $\mathbb{Q}(\sqrt{5})$ in terms of (m, n) in Section 6 of [KM16], but their method does not lead to generic models over $\mathbb{Q}$.

Proof of Propositions 6.1 and 6.3. In Section 4.2, we found a linear transformation T defined over $\mathbb{Q}(m, n)$ and a scaling factor $c \in \mathbb{Q}(m, n)$ such that

$$cL_0(T(x_1, x_2, x_3)) = x_1^2 - 5x_2^2 + (m^2 - 5n^2 - 5)x_3^2,$$

whenever disc $L_0 \neq 0$, where L_0 is the Mestre conic associated to the Igusa–Clebsch invariants in (5.1). Applying the same transformation T to the Mestre cubic M_0 and rescaling by some $c' \in \mathbb{Q}(m, n)$ yields

$$
\begin{aligned}
c'M_0(T(x_1, x_2, x_3)) = \ & (m + 5n^2)x_1^3 + 30mnx_1^2x_2 + 15(m^2 + 5n^2)x_1x_2^2 + 50mnx_2^3 \\
& - (2m - 3)(m^2 - 5n^2)x_1^2x_3 - 20n(m^2 - 5n^2)x_1x_2x_3 \\
& - 5(2m + 3)(m^2 - 5n^2)x_2^2x_3 - 2(m^2 - 5n^2 - 5)(m^2 - 5n^2)x_3^3.
\end{aligned}
$$

Define L and M to be these reduced forms of L_0 and M_0 respectively.

We first consider the case where $\sqrt{5} \notin k$. By inspection, $L(k)$ has no points with $x_3 = 0$. Suppose that $(u_0 : v_0 : 1) \in L(k)$. Parametrizing $L(k)$ in the usual way using this point gives

$$\{(x_1 : x_2 : 1) \in L(k)\} = \left\{ \left((1 + 5x^2)u_0 - 10xv_0 : (1 + 5x^2)v_0 - 2xu_0 : 1 - 5x^2 \right) : x \in \mathbb{P}(k) \right\}.$$

It will be convenient for us to write this parametrization in terms of elements of $k(\sqrt{5})$. Define $\eta = u_0 + v_0\sqrt{5} \in k(\sqrt{5})/k$. The parametrization above can then be expressed as

$$\{(x_1 : x_2 : 1) \in L(k)\} = \left\{ (u : v : 1) : u + \sqrt{5}v = \eta\frac{1 - x\sqrt{5}}{1 + x\sqrt{5}}, \ x \in \mathbb{P}(k) \right\}.$$

Let $\mu = m + n\sqrt{5} \in k(\sqrt{5})/k$. Then one verifies that, when $x_3 = 1$, the reduced Mestre cubic M can be written as

$$\tfrac{1}{2}\mathrm{Tr}(\mu^2(x_1 + x_2\sqrt{5})^3) + 3N(\mu)N(x_1 + x_2\sqrt{5}) - N(\mu)\mathrm{Tr}(\mu(x_1 + x_2\sqrt{5})^2) - 2N(\mu)(N(\mu) - 5),$$

where N and Tr denote the norm and trace from $k(\sqrt{5})$ to k. We can now substitute the parametrization of the k-rational points of L into M to obtain a k-rational Weierstrass model for the associated genus 2 curve C, as described in Theorem 4.2. This gives the k-rational model from Proposition 6.1.

Now suppose that $\sqrt{5} \in k$. It is possible to mimic the calculations from the case where $\sqrt{5} \notin k$ by taking

$$(u_0 : v_0 : 1) := \left(\left(m^2 - 5n^2 - 5 - \tfrac{1}{20}\right)\sqrt{5} : m^2 - 5n^2 - 5 + \tfrac{1}{20} : 1\right)$$

and working in the ring $k[t]/(t^2 - 5)$. However, we get a tidier Weierstrass model by instead using the point $(\sqrt{5} : 1 : 0)$ to parameterize $L(k)$. A straightforward calculation yields the k-rational model given in Proposition 6.3. $\qquad\square$

Remark 6.4. Note that if we take $k = \mathbb{Q}(\sqrt{5})$ in Proposition 6.3, and $m \in \mathbb{Q}$, $n \in \sqrt{5}\mathbb{Q}\setminus\{0\}$, we get a 2-parameter family of genus 2 curves defined over $\mathbb{Q}$ which have potential RM-5, but not RM-5 defined over $\mathbb{Q}$. To see the RM-5 is not actually defined over $\mathbb{Q}$, one can check that for $m \in \mathbb{Q}$, $n \in \sqrt{5}\mathbb{Q}\setminus\{0\}$, Wilson's discriminant Δ' is in the square class of n^2, which is a non-square. Hence the Igusa–Clebsch invariants are rational, but the moduli points on $Y_-(5)$ are not rational, and so the RM-5 cannot be defined rationally. (The irrationality of these moduli points on $Y_-(5)$ is suggested by the fact that $(m, n) \notin \mathbb{Q}^2$ but is not *a priori* implied by this as (m, n) are only coordinates for a birational model of $Y_-(5)$, and we have not determined an explicit birational map from $\mathbb{P}^2_{m,n}$ to $Y_-(5)$).

7. Comparisons with known families

7.1. Mestre's family. Let f be the polynomial

$$f(a, b, x) = x^5 + (a - 3)x^4 + (-a + b + 3)x^3 + (a^2 - a - 2b - 1)x^2 + bx + a,$$

and let $X(a, b)$ be the genus 2 curve

$$X(a, b) : y^2 = xf(a, b, x).$$

In [Mes91b], Mestre proves that $X(a, b)$ has RM-5 for every a, b in $\mathbb{C}$ such that $xf(a, b, x)$ has six distinct zeroes, and that the RM is defined over $k = \mathbb{Q}(a, b)$. Using Humbert's criterion for RM-5, Wilson [Wil00] showed that this family of curves over k gives all genus 2 curves with RM-5 over k which have a Weierstrass point in k, up to k-isomorphism. In particular, for any genus 2 curve C with RM-5, there exist $a, b \in \mathbb{C}$ such that C is $\mathbb{C}$-isomorphic to $X(a, b)$. See also [Sak09] for an alternative proof of this last result.

Define $g_{a,b}$ and $h_{a,b}$ as

$$g_{a,b} = \frac{2(3a^3 - 8a^2 - 5ab - b^2 - 3a)}{3(a^2 - 5a - 2b + 1)^2}$$

and

$$h_{a,b} = \frac{-a^2(4a^5 - 4a^4 - 24a^3b - a^2b^2 - 40a^3 + 34a^2b + 30ab^2 + 4b^3 + 91a^2 + 14ab - b^2 - 4a)}{2(a^2 - 5a - 2b + 1)^5}.$$

Then, by comparing Igusa–Clebsch invariants, one can verify that $X(a,b)$ is $\mathbb{C}$-isomorphic to a genus 2 curve associated to $(g,h) = (g_{a,b}, h_{a,b})$ in the Elkies–Kumar model (2.1), assuming $a^2 - 5a - 2b + 1 \neq 0$ and $h_{a,b} \neq 0$.

Since choices of $a, b \in k$ can only yield genus 2 curves with RM-5 over k which have a k-rational Weierstrass point, one cannot easily describe all genus 2 curves with RM-5 over k using Mestre's family. For example, there are no rational values of a and b for which $X(a,b)$ is $\mathbb{C}$-isomorphic to the genus 2 curve associated to $(g,h) = (-\frac{4}{15}, \frac{16}{3125})$.

7.2. Brumer's family.
Brumer constructed a family of curves $C_{b,c,d}$ defined by

$$C_{b,c,d} : y^2 + (x^3 + x + 1 + c(x^2 + x))y = b + (1 + 3b)x + (1 - bd + 3b)x^2$$
$$+ (b - 2bd - d)x^3 - bdx^4,$$

and showed that if $C_{b,c,d}$ is nonsingular, then it is a genus 2 curve with RM-5 over $\mathbb{Q}(b,c,d)$. Moreover, every genus 2 curve with RM-5 is $\mathbb{C}$-isomorphic to $C_{b,c,d}$ for some $b,c,d \in \mathbb{C}$. Brumer did not publish the details of his proof (see [Bru95] for an announcement), but the above statements were reproved by different methods in [Has00] and [HS09].

Define $g_{b,c,d}$ and $h_{b,c,d}$ as

$$g_{b,c,d} = \frac{-c^4 + 8bc^2d - 16b^2d^2 + 6c^3 - 24bcd + 24bc + c^2 - 68bd - 24cd - 108b - 30c - 36d - 61}{6(c^2 - 4bd - 2b - 3c - 2d - 5)^2}$$

and

$$\begin{aligned}
h_{b,c,d} = (c^2 &- 4bd - 2b - 3c - 2d - 5)^{-5} \big(bc^6d - 12b^2c^4d^2 + 48b^3c^2d^3 - 64b^4d^4 - b^2c^4d - 9bc^5d \\
&+ 8b^3c^2d^2 + 72b^2c^3d^2 - bc^4d^2 - 16b^4d^3 - 144b^3cd^3 + 8b^2c^2d^3 - 16b^3d^4 + bc^5 - 40b^2c^3d \\
&+ 12bc^4d - c^5d + 144b^3cd^2 - 152b^2c^2d^2 + 52bc^3d^2 + 416b^3d^3 - 192b^2cd^3 - b^2c^3 - 9bc^4 \\
&+ 36b^3cd + 334b^2c^2d + 63bc^3d + 6c^4d + 24b^3d^2 + 132b^2cd^2 - 80bc^2d^2 + c^3d^2 + 528b^2d^3 \\
&- 36bcd^3 - 27b^2c^2 + 13bc^3 - c^4 + 108b^3d - 720b^2cd + 74bc^2d + 5c^3d - 456b^2d^2 - 96bcd^2 \\
&- 36c^2d^2 + 216bd^3 + 27b^3 + 252b^2c + 56bc^2 + 6c^3 - 66b^2d - 627bcd - 43c^2d - 381bd^2 \\
&- 63cd^2 + 27d^3 - 567b^2 + 27bc + 4c^2 - 121bd - 147cd - 81d^2 - 484b - 39c - 34d - 103\big).
\end{aligned}$$

Then, by comparing Igusa–Clebsch invariants, one can verify that $C_{b,c,d}$ is $\mathbb{C}$-isomorphic to the genus 2 curve associated to $(g,h) = (g_{b,c,d}, h_{b,c,d})$ in the Elkies–Kumar model (2.1) when $c^2 - 4bd - 2b - 3c - 2d - 5 \neq 0$ and $h_{b,c,d} \neq 0$.

One can ask if Brumer's family provides a way to describe all genus 2 curves C with RM-5 defined over k. However, it is not clear whether these will all come from a k-rational choice of parameters b, c, d. E.g., if (z, g, h) is a generic rational point on Y such that $30g + 4$ is a norm from $\mathbb{Q}(\sqrt{5})$, it is not clear if we can write $(g,h) = (g_{b,c,d}, h_{b,c,d})$ for some $b, c, d \in \mathbb{Q}$.

While Brumer's models are simpler than what we give in Section 6, over $\mathbb{Q}$ they might not comprise all rational curves C with RM-5, even generically. Moreover there is no simple description of which choices of b, c, d will give $\mathbb{C}$-isomorphic curves.

8. Beyond RM-5

The Hilbert modular surface $Y_-(D)$ is rational if and only if D is one of $5, 8, 12, 13,$ or 17. One might wonder if there are analogues of Theorem 1.1 for each of these discriminants. Numerical experimentation suggests that the answer is yes.

Define

$$p_5(m,n) = -m^2 + 5n^2 + 5,$$
$$p_8(m,n) = m + 1,$$
$$p_{12}(m,n) = -27m^2 + n^2 + 27,$$
$$p_{13}(m,n) = 1803m^2 - 72mn + n^2 + 3168m - 1440n - 768, \text{ and}$$
$$p_{17}(m,n) = 1.$$

In [EK14], Elkies and Kumar give rational models of $Y_-(D)$ for all fundamental discriminants between 1 and 100. The polynomials $p_D(m,n)$ above are all factors of the discriminant of the Mestre conic one obtains when using Igusa–Clebsch invariants from [EK14] in the construction given in Section 4.1. We chose several thousand values of $(m,n) \in \mathbb{Q}^2$ at random, and for each of these the associated Mestre conic was equivalent to $x_1^2 - Dx_2^2 - p_D(m,n)x_3^2 = 0$ over $\mathbb{Q}$ whenever it was nonsingular. In particular, the Mestre obstruction appears to vanish generically for $D = 17$, which is quite surprising.

We have attempted using the methods from in Section 3 to reduce the Mestre conics for these other values of D, but thus far have only been partially successful in removing the other polynomial factors from the discriminant.

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