CENTRAL L-VALUES AND TORIC PERIODS FOR GL(2)

KIMBALL MARTIN AND DAVID WHITEHOUSE

Abstract. Let π be a cusp form on PGL(2) over a number field F and let E be a quadratic extension of F. We use Jacquet's relative trace formula to prove an explicit identity relating the central L-value of the base change of π to E with a specific toric period integral.

Contents

1. Introduction 1
2. Local normalizations 3
3. Matching functions 5
3.1. E/F unramified 6
3.2. E/F ramified 7
4. Local representation theory 10
4.1. Nonarchimedean fields 10
4.2. Archimedean fields 11
5. Local distributions 11
5.1. Nonarchimedean fields 11
5.2. Archimedean fields 13
6. Global result 14
6.1. Measures 14
6.2. Proof 14
References 18

1. Introduction

Let us begin by recalling relevant results on central L-values for GL(2). Let F be a number field and let E/F be a quadratic extension of F, which we will assume is split at all the infinite places of F. Let π be a cuspidal automorphic representation of $\mathrm{GL}(2, \mathbf{A}_F)$ with trivial central character.

Consider the set of inner forms G of GL(2) such that,

(1) G contains a torus T/F with $T(F) \cong E^\times$, and
(2) π transfers to an automorphic representation π' of $G(\mathbf{A}_F)$.

The L-function under consideration will be that attached to the base change π_E of π to an automorphic representation of $\mathrm{GL}(2, \mathbf{A}_E)$. Let $\varepsilon(s, \pi_E)$ denote the ε-factor associated to π_E. If $\varepsilon(1/2, \pi_E) = -1$ then $L(1/2, \pi_E) = 0$. Moreover,

Date: November 13, 2006.

one knows that for each inner form G as above the space of π' admits no non-zero $T(\mathbf{A}_F)$-invariant linear functional and in particular the map

$$\varphi \mapsto \int_{Z(\mathbf{A}_F)T(F)\backslash T(\mathbf{A}_F)} \varphi(t)\ dt$$

on the space of π' is identically zero. On the other hand, if $\varepsilon(1/2, \pi_E) = +1$, there exists a unique inner form G as above such that the space of π' admits a non-zero $T(\mathbf{A}_F)$-invariant linear functional.

Assume therefore that $\varepsilon(1/2, \pi_E) = +1$. Now fix G and π' so that π' admits such a non-zero form. Waldspurger [Wal85, Proposition 7] obtained a formula of the form

$$L(1/2, \pi_E) = c \left| \int_{Z(\mathbf{A}_F)T(F)\backslash T(\mathbf{A}_F)} \varphi(t)\ dt \right|^2 .$$

However, his formula is not explicit enough for arithmetic purposes. In particular, it is not even clear if one can derive $L(1/2, \pi_E) \geq 0$ from Waldspurger's expression. (The positivity of this central value is predicted by both the Birch and Swinnerton-Dyer Conjecture and the Generalized Riemann Hypothesis. It is immediate from the formula below, and was proven by Guo [Guo96] using the relative trace formula.) Subsequently, explicit formulas for $L(1/2, \pi_E)$ (in fact twists $L(1/2, \pi_E \otimes \chi)$) have been obtained by Gross [Gro87], Zhang [Zha01], Xue [Xue] and Popa [Popa]. The works of Gross, Zhang and Xue are in increasing order of generalization (with respect to this particular result) and cover the case where F is totally real, E/F is imaginary quadratic, and π comes from an even weight Hilbert newform on f (under certain ramification assumptions). Popa's formula is in the case where both E and F are totally real (again, with certain ramification conditions). These results have applications to equidistribution of Heegner points and certain geodesics. They all rely on the theta correspondence and the Rankin-Selberg method.

In this paper, we prove a result which is more general than [Popa] (when χ is trivial) and does not require that F be totally real. Furthermore, we would like to emphasize that we use a completely different approach: Jacquet's relative trace formula [Jac86], following, in part, work of Guo [Guo96]. Precisely, we show the following result.

Theorem 1.1. *Assume that E/F is unramified at the primes of F above 2 and is split at the infinite places of F. Then we have*

$$L(1/2, \pi_E) = C(E, \pi) \frac{[\varphi_0, \varphi_0]}{[\varphi, \varphi]} \left| \int_{T(F)Z(\mathbf{A}_F)\backslash T(\mathbf{A}_F)} \varphi(t)\ dt \right|^2 .$$

for a suitable choice of $\varphi_0 \in \pi$, $\varphi \in \pi'$ and a constant $C(E, \pi)$ made explicit as follows.

Let $d_{E/F}$ be the discriminant of E/F; let q_v be the size of the residue field of F_v; let $S_1(E, \pi)$ be the set of places of F which are unramified and inert in E and at which π is ramified; and let $S_2(E, \pi)$ be the set of places of F for which both E/F and π are ramified. Define $A(\pi_v) = \frac{1}{2}(1 + q_v^{-1})$ if the order of the conductor of π_v is 1, and $A(\pi_v) = \frac{1}{2}$ otherwise. Then

$$C(\pi, E) = \sqrt{d_{E/F}} \prod_{v \in S_1(E,\pi)} \frac{q_v + 1}{q_v - 1} \prod_{v \in S_2(E,\pi)} \left(A(\pi_v) L(1/2, \pi_{E,v}) \right) .$$

See Section 6.2 for how φ_0 and φ are chosen. Note that φ is chosen only up to a non-zero scalar, but the above expression is still well defined.

This formula agrees with that of [Popa] when the hypotheses coincide. We stress that our method is quite general, and we hope to remove the ramification assumptions and generalize the result to include twists by characters in the near future.

2. Local normalizations

For the next few sections we will be working locally. We use F to denote a local non-archimedean field with ring of integers $\mathcal{O}$, prime ideal $\mathfrak{p}$ and units U_F. Define $U_F^0 = U_F$ and $U_F^n = 1 + \mathfrak{p}^n$ for $n > 0$. Let q denote the order of the residue field of F and let $|\cdot|$ (or $|\cdot|_F$ for clarity) denote the multiplicative valuation on F such that $|\varpi| = q^{-1}$ for any uniformizer ϖ in F. We let v denote the additive valuation on F.

Fix an additive character $\psi : F \to \mathbf{C}^\times$. We denote by $n(\psi)$ the conductor of ψ, i.e. ψ is trivial on $\mathfrak{p}^{-n(\psi)}$ but non-trivial when restricted to $\mathfrak{p}^{-n(\psi)-1}$. On F, we take the Haar measure dx which is self dual with respect to the character ψ. On $F^\times$, we take the measure

$$d^\times x = L(1, 1_F)\frac{dx}{|x|}.$$

Set

$$\mathfrak{d} = \mathrm{vol}(\mathcal{O}, dx) = \mathrm{vol}(U_F, d^\times x).$$

For a quadratic extension E/F we use the same notation for E as for F with the addition of a subscript E. We denote by ψ_E the pull back of ψ to E via the trace map to F. We form measures on E and $E^\times$ in the same way relative to this character. We note that the measure on $E^\times$ is

$$d^\times x_E = L(1, 1_E)\frac{dx_E}{|x_E|_E}.$$

We let Δ_E denote the discriminant of E/F. If we write $E = F[\sqrt{D}]$ then for an element $\alpha = a + b\sqrt{D} \in E$ we have $d\alpha = |4D|_F^{\frac{1}{2}}\, da\, db$. Then the following lemma is straightforward.

Lemma 2.1. *We have* $\mathrm{vol}(\mathcal{O}_E, dx_E) = \mathrm{vol}(U_E, dx_E^\times) = \mathfrak{d}^2|\Delta_E|_F^{\frac{1}{2}}$.

We now describe the Haar measure on $\mathrm{GL}(2, F)$. Denote by A the subgroup of $\mathrm{GL}(2)$ of diagonal matrices and N the subgroup of upper triangular matrices. Let K be the maximal compact open subgroup $\mathrm{GL}(2, \mathcal{O})$ in $\mathrm{GL}(2, F)$. Define

$$K_n = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathcal{O}) : c \in \mathfrak{p}^n \right\}.$$

We set

$$w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

and

$$K_n' = w^{-1}K_n w = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathcal{O}) : b \in \mathfrak{p}^n \right\}.$$

Denote the center of $\mathrm{GL}(2)$ by Z.

For a Haar measure on $\mathrm{GL}(2,F)$, we take

$$dg = L(1,1_F)\frac{da\ db\ dc\ dd}{|\det g|^2}$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,F)$.

Lemma 2.2. *We have*

$$\mathrm{vol}(K_n, dg) = \begin{cases} \mathfrak{d}^4 L(2,1_F)^{-1}, & \text{if } n = 0; \\ \mathfrak{d}^4 q^{-n} L(1,1_F)^{-1}, & \text{if } n > 0. \end{cases}$$

Proof. When $n > 0$, it is easy to see that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ lies in K_n if and only if $|a| = 1$, $|d| = 1$, $|b| \leq 1$ and $|c| \leq q^{-n}$. Then the result for positive n follows immediately from our definitions of dg and $\mathfrak{d}$. When $n = 0$, we note that $\#K_0/K_1 = q + 1$. □

Suppose now that E is either a quadratic extension of F or isomorphic to $F \oplus F$. We denote by $X(E/F)$ the set of isomorphism classes $\{(G,T)\}$ where G is an inner form of $\mathrm{GL}(2)$ and T is a torus in G such that $T(F) \cong E^\times$. When $E \cong F \oplus F$ it is evident that $X(E/F)$ contains only $\mathrm{GL}(2)$.

When E/F is a field, we fix $c_1 = 1$ and $c_2 \in F^\times - N_{E/F}E^\times$. Assume that $|c_2| \leq 1$ is chosen to be maximal. Then we can form the algebra

$$D_i = \left\{ \begin{pmatrix} \alpha & c_i\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in E \right\}.$$

Note that $D_1 \cong M_2(F)$. Let $G_i = D_i^\times$. Then we have

$$X(E/F) = \{(G_1,T_1),(G_2,T_2)\}.$$

where T_i denotes the diagonal torus in G_i.

We define orders $R'_{i,n}$ in D_i by

$$R'_{i,n} = \left\{ \begin{pmatrix} \alpha & c_i\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in D_i : \alpha \in \mathcal{O}_E, \beta \in \mathfrak{p}_E^n \right\}.$$

We note that when E/F is unramified $R'_{i,n}$, is an order in D_i of reduced discriminant $\mathfrak{p}^{2n+i-1}$. When E/F is ramified and $q > 2$, $R'_{i,n}$ is an order in D_i of reduced discriminant $\mathfrak{p}^{n+1}$. We define compact open subgroups of $G_i(F)$ by $K_{i,n} = R'^{\times}_{i,n}$. Let Z_i denote the center of G_i.

On $G_i(F)$, take the measure

$$dg_i = L(1,1_F)|c_i|_F \frac{d\alpha\ d\beta}{|\alpha\bar{\alpha} - c_i\beta\bar{\beta}|_F}$$

for

$$g_i = \begin{pmatrix} \alpha & c_i\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}.$$

Lemma 2.3. *When $n > 0$, we have*

$$\mathrm{vol}(K'_{i,n}, dg) = \begin{cases} (1+q^{-1})|c_i|_F q^{-2n}\mathfrak{d}^4, & \text{if } E/F \text{ is unramified;} \\ |c_i|_F q^{-n}\mathfrak{d}^4|\Delta_E|_F, & \text{if } E/F \text{ is ramified.} \end{cases}$$

When $n=0$ and $i=1$,

$$\mathrm{vol}(K'_{1,0},dg)=\begin{cases}(1-q^{-2})\mathfrak{d}^4, & \text{if } E/F \text{ is unramified;}\\ (1-q^{-1})\mathfrak{d}^4|\Delta_E|_F, & \text{if } E/F \text{ is ramified and } q \text{ is odd.}\end{cases}$$

When $n=0$ and $i=2$,

$$\mathrm{vol}(K'_{2,0},dg)=(1+q^{-1})|c_i|_F\mathfrak{d}^4|\Delta_E|_F.$$

Proof. Write an element of $K'_{i,n}$ as $\begin{pmatrix}\alpha & c_i\beta\\ \bar\beta & \bar\alpha\end{pmatrix}$. We first consider the case that $n>0$. Then $\alpha\in U_E$ and $\beta\in\mathfrak{p}_E^n$. The definition of dg implies

$$\mathrm{vol}(K'_{i,n},dg)=L(1,1_F)|c_i|_F\,\mathrm{vol}(U_E,dx)\,\mathrm{vol}(\mathfrak{p}_E^n,dx).$$

Now $\mathrm{vol}(U_E,dx)=L(1,1_E)^{-1}\,\mathrm{vol}(U_E,d^\times x)$ and $\mathrm{vol}(\mathfrak{p}_E^n,dx)=|\varpi_E|_E^n\,\mathrm{vol}(\mathcal{O}_E,dx)$. Hence Lemma 2.1 implies

$$\mathrm{vol}(K'_{i,n},dg)=L(1,1_F)|c_i|_F L(1,1_E)^{-1}|\varpi_E|_E^n\mathfrak{d}^4|\Delta_E|_F,$$

i.e.,

$$\mathrm{vol}(K'_{i,n},dg)=\begin{cases}(1+q^{-1})|c_i|_F q^{-2n}\mathfrak{d}^4, & \text{if } E/F \text{ is unramified;}\\ |c_i|_F q^{-n}\mathfrak{d}^4|\Delta_E|_F, & \text{if } E/F \text{ is ramified.}\end{cases}$$

Assume $n=0$. First consider the case $i=2$. If E/F is unramified, then $|c_i|_F=q^{-1}$, so $\alpha\in U_E$ and $\beta\in\mathcal{O}_E$ and we get

$$\mathrm{vol}(K'_{2,0},dg)=L(1,1_F)|c_i|_F\,\mathrm{vol}(U_E,dx)\,\mathrm{vol}(\mathcal{O}_E,dx)=(1+q^{-1})|c_i|_F\mathfrak{d}^4.$$

If E/F is ramified, then $|c_i|_F=1$. So at least one of α and β is a unit and we have

$$\begin{aligned}\mathrm{vol}(K'_{2,0},dg) &= L(1,1_F)(\mathrm{vol}(U_E,dx)\,\mathrm{vol}(\mathcal{O}_E,dx)+\mathrm{vol}(\mathfrak{p}_E,dx)\,\mathrm{vol}(U_E,dx))\\ &= L(1,1_F)\,\mathrm{vol}(U_E,dx)\,\mathrm{vol}(\mathcal{O}_E,dx)(1+q^{-1})\\ &= (1+q^{-1})\mathfrak{d}^4|\Delta_E|_F.\end{aligned}$$

The case $i=1$ follows similarly using the fact that

$$\mathrm{vol}\left(\{\alpha\in U_E:|\alpha\bar\alpha-1|=1\},dx\right)=\frac{q-2}{q-1}\,\mathrm{vol}(U_E,dx),$$

when E/F is unramified, and

$$\mathrm{vol}\left(\{\alpha\in U_E:|\alpha\bar\alpha-1|=1\},dx\right)=\frac{q-3}{q-1}\,\mathrm{vol}(U_E,dx),$$

when E/F is ramified and q is odd. □

3. Matching functions

Let E be either a quadratic extension of F or else isomorphic to $F\oplus F$. We recall that Jacquet has defined a notion of matching functions between smooth compactly supported functions on $\mathrm{GL}(2,F)$ and tuples of functions

$$\{f_G\in C_c^\infty(G(F)):G\in X(E/F)\}.$$

In the case that E is split over F this matching is trivial. We now recall the notion of matching functions in the case that E/F is a field. Fix representatives G_1 and G_2 for the set $X(E/F)$ of isomorphism classes as in the previous section. Let η denote the quadratic character of $F^\times$ associated to the extension E/F.

For a function $f \in C_c^\infty(\mathrm{GL}(2,F))$ and $a \in F - \{0,1\}$, form the orbital integral

$$H(a; f; \eta) = \int_{(F^\times)^3} f\left(\begin{pmatrix} x & \\ & y \end{pmatrix}\begin{pmatrix} 1 & a \\ 1 & 1 \end{pmatrix}\begin{pmatrix} z & \\ & 1 \end{pmatrix}\right) \eta(z)\, d^\times x\, d^\times y\, d^\times z.$$

For a function $f_i \in C_c^\infty(G_i(F))$ and $\alpha \in E$ such that $c_i\alpha\bar{\alpha} \notin \{0,1\}$, we define

$$H(c_i\alpha\bar{\alpha}; f_i; T_i) = \int_{E^\times/F^\times}\int_{E^\times} f_i\left(\begin{pmatrix} x & \\ & \bar{x} \end{pmatrix}\begin{pmatrix} 1 & c_i\alpha \\ \bar{\alpha} & 1 \end{pmatrix}\begin{pmatrix} y & \\ & \bar{y} \end{pmatrix}\right) d^\times x\, d^\times y,$$

where we take the quotient measure on $E^\times/F^\times$. We note that the integral depends only on $c_i\alpha\bar{\alpha} \in F^\times$.

We say that f and (f_1, f_2) have matching orbital integrals if

$$H(a; f; \eta) = \begin{cases} H(c_1\alpha\bar{\alpha}; f_1; T_1), & \text{if } a = c_1\alpha\bar{\alpha}; \\ H(c_2\alpha\bar{\alpha}; f_2; T_2), & \text{if } a = c_2\alpha\bar{\alpha}. \end{cases}$$

The existence of matching functions is proven in [Jac86].

3.1. E/F **unramified.** Assume that E/F is unramified. We now determine the matching functions for $f_m = \mathbf{1}_{K'_m} \in C_c^\infty(\mathrm{GL}(2,F))$ where K'_m is the compact open subgroup of $\mathrm{GL}(2,F)$ defined above. Let $f_{i,m} \in C_c^\infty(G_i(F))$ be the characteristic function of $K'_{i,m}$.

We begin by computing the orbital integrals of f_m.

Lemma 3.1. *We have*

$$H(a; f_0; \eta) = \begin{cases} 0, & \text{if } v(a) \text{ is odd, or } v(1-a) > 0; \\ \mathrm{vol}(U_F)^3, & \text{otherwise.} \end{cases}$$

And we have

$$H(a; f_m; \eta) = \begin{cases} \mathrm{vol}(U_F)^3, & \text{if } v(a) \geq m \text{ and } v(a) - m \text{ is even;} \\ 0, & \text{otherwise,} \end{cases}$$

when $m > 0$.

Proof. These calculations can be found in [Guo96, Section 2.3]. However, in the case of f_0, those calculations are incorrect, so we include this case here. The integral $H(a; f_0; \eta)$ is equal to the integral of $\eta(yz)$ over the region $(x, y, z) \in F^3$ such that

$$\begin{pmatrix} x & axz/y \\ y & z \end{pmatrix}.$$

Thus we require $|x|, |y|, |z| \leq 1$, $|a||x||z| \leq |y|$ and $|xz||a-1| = 1$.

First, it is clear that the integral vanishes if $v(1-a) > 0$. Next we assume that $v(a) < 0$. Then we need $|y| = 1$ and $|xz| = |a|^{-1}$. Hence we have

$$H(a; f_0; \eta) = \mathrm{vol}(U_F)^2 \int_{|a|^{-1} \leq |z| \leq 1} \eta(z) = \begin{cases} \mathrm{vol}(U_F)^3, & \text{if } v(a) \text{ is even;} \\ 0, & \text{if } v(a) \text{ is odd.} \end{cases}$$

Finally if we assume that $|a| = |a-1| = 1$. Then we clearly have $H(a; f_0; \eta) = \mathrm{vol}(U_F)^3$. $\square$

The calculations of the orbital integrals for the functions $f_{i,m}$ can be extracted from the calculations in the proof of [Guo96, Proposition 2.3].

Lemma 3.2. *We have*

$$H(\alpha\bar{\alpha}; f_{1,0}; T_1) = \begin{cases} 0, & \text{if } v(1-\alpha\bar{\alpha}) > 0; \\ \mathrm{vol}(E^\times/F^\times)\,\mathrm{vol}(U_E), & \text{otherwise.} \end{cases}$$

And we have

$$H(c_i\alpha\bar{\alpha}; f_{i,m}; T_i) = \begin{cases} \mathrm{vol}(E^\times/F^\times)\,\mathrm{vol}(U_E), & \text{if } v_E(\alpha) \geq m; \\ 0, & \text{otherwise,} \end{cases}$$

when $m > 0$, or $m = 0$ and $i = 2$.

We recall our normalization of measures on $E^\times$ and $F^\times$. We have $\mathrm{vol}(U_F) = \mathfrak{d}$, $\mathrm{vol}(U_E) = \mathfrak{d}^2$ and

$$\mathrm{vol}(E^\times/F^\times) = \mathrm{vol}(U_E)/\,\mathrm{vol}(U_F) = \mathfrak{d}.$$

Hence we have the following result.

Lemma 3.3. *The function $f_m \in C_c^\infty(\mathrm{GL}(2,F))$ can be matched with $(\mathbf{1}_{R_{1,m}^\times}, 0)$ when m is even, and with $(0, \mathbf{1}_{R_{2,m}^\times})$ when m is odd; where $R_{i,m}$ denotes an order in D_i containing $\mathcal{O}_E$ and of discriminant $\mathfrak{p}^m$.*

3.2. E/F **ramified.** Now assume that E/F is a ramified quadratic extension. Fix a uniformizer ϖ in F such that $\eta(\varpi) = 1$, and let $\mathfrak{p}^{n(\eta)}$ denote the conductor of η. Having fixed E, we define the function φ_m on $\mathrm{GL}(2,F)$ to be the characteristic function of

$$K_m \begin{pmatrix} 1 & 0 \\ \varpi^{m-n(\eta)} & 1 \end{pmatrix}$$

for $m \geq 0$.

Lemma 3.4. *We have,*

$$H(a; \varphi_{n(\eta)}; \eta) = \begin{cases} 0, & \text{if } v(1-a) > 0; \\ \mathrm{vol}(U_F)^2\,\mathrm{vol}(U_F^{n(\eta)}), & \text{otherwise.} \end{cases}$$

In addition,

$$H(a; \varphi_m; \eta) = \begin{cases} \mathrm{vol}(U_F)^2\,\mathrm{vol}(U_F^{n(\eta)}), & \text{if } v(a) \geq m - n(\eta); \\ 0, & \text{otherwise,} \end{cases}$$

when $m > n(\eta)$.

Proof. We begin with the case that $m = n(\eta)$. We have

$$H(a; \varphi_{n(\eta)}; \eta) = \int_{(F^\times)^3} \varphi_{n(\eta)} \begin{pmatrix} x & ax/y \\ yz & z \end{pmatrix} \eta(y)\; d^\times x\; d^\times y\; d^\times z.$$

Now

$$\begin{pmatrix} x & ax/y \\ yz & z \end{pmatrix} \in K_{n(\eta)} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

if and only if

$$\begin{pmatrix} x(1-a/y) & ax/y \\ z(y-1) & z \end{pmatrix} \in K_{n(\eta)}.$$

This holds if and only if $z \in U_F$, $y \in U_F^{n(\eta)}$ and both $|x(1-a)| = 1$ and $|x| \leq |a|^{-1}$. Hence we deduce that the integral vanishes if $|1-a| < 1$ and otherwise is equal to $\mathrm{vol}(U_F)^2\,\mathrm{vol}(U_F^{n(\eta)})$.

The calculations in the case that $m > n(\eta)$ can be extracted from [Guo96, Section 2.4]. $\square$

The following lemma can be obtained from the proof of [Guo96, Proposition 2.3].

Lemma 3.5. *We have*

$$H(c_i\alpha\bar{\alpha}; f_{i,0}; T_i) = \begin{cases} 0, & \text{if } v(1 - c_i\alpha\bar{\alpha}) > 0; \\ \operatorname{vol}(U_E)\operatorname{vol}(E^\times/F^\times), & \text{otherwise,} \end{cases}$$

and

$$H(c_i\alpha\bar{\alpha}; f_{i,m}; T_i) = \begin{cases} \operatorname{vol}(U_E)\operatorname{vol}(E^\times/F^\times), & \text{if } v_E(\alpha) \geq m; \\ 0, & \text{otherwise,} \end{cases}$$

when $m > 0$.

We recall our normalization of measures on $E^\times$ and $F^\times$. We write $E = F[\sqrt{D}]$ with $|D| = 1$ or q^{-1}. Then we have have $\operatorname{vol}(U_F) = \mathfrak{d}$, $\operatorname{vol}(U_F^{n(\eta)}) = \mathfrak{d}q^{-n(\eta)}(1 - q^{-1})^{-1}$, $\operatorname{vol}(U_E) = \mathfrak{d}^2|\Delta_E|_F^{\frac{1}{2}}$ and

$$\operatorname{vol}(E^\times/F^\times) = 2\operatorname{vol}(U_E)/\operatorname{vol}(U_F) = 2\mathfrak{d}|\Delta_E|_F^{\frac{1}{2}}.$$

Lemma 3.6. *For $m \geq 0$ the function*

$$2(1 - q^{-1})\varphi_{m+n(\eta)}$$

can be matched with $\left(\mathbf{1}_{R^\times_{1,m+n(\eta)}}, \mathbf{1}_{R^\times_{2,m+n(\eta)}}\right)$; where $R_{i,m+n(\eta)}$ denotes an order in D_i containing $\mathcal{O}_E$ and of discriminant $\mathfrak{p}^{m+n(\eta)}$.

We shall also need the calculation of the orbital integrals when $m = 0$. We begin with the integral on GL(2).

Lemma 3.7. *For $a \neq 0, 1$ we have*

$$H(a; \varphi_0; \eta) = \begin{cases} 2\operatorname{vol}(U_F)^2\operatorname{vol}(U_F^{n(\eta)}), & \text{if } \eta(a) = 1 \text{ and } v(1-a) \leq 0; \\ 2\operatorname{vol}(U_F)^2\operatorname{vol}(U_F^{n(\eta)}), & \text{if } \eta(a) = 1 \text{ and } 0 \leq v(1-a) \leq n(\eta) - 1; \\ \operatorname{vol}(U_F)^2\operatorname{vol}(U_F^{n(\eta)}), & \text{if } v(1-a) = n(\eta); \\ 0, & \text{otherwise;} \end{cases}$$

Proof. For ease of notation we set $n = n(\eta)$. Then we have, as in [Guo96, Section 2.4], that $H(a; \varphi_0; \eta)$ is equal to the integral of $\eta(y)$ over the region $\{(x, y, z) \in F^3\}$ which satisfies

(1) $|z| = |1 - a|^{-1}|x|^{-1}$
(2) $1 \leq |1 - a||x|$
(3) $|\varpi^n||a||x| \leq |y|$
(4) $|y - 1| \leq |\varpi|^n|1 - a||x|$
(5) $|x||y - a| \leq |y|$.

First we consider the case that $|a| > 1$. Then these conditions become

(1) $1 \leq |a||x|$
(2) $|y - 1| \leq |\varpi|^n|a||x| \leq |y|$
(3) $|x||y - a| \leq |y|$.

Thus we have $|y-1| \le |y|$ and hence $|y| \ge 1$. Now if $|y| = 1$, then these conditions reduce to $|ax| = 1$ and $|y-1| \le |\varpi|^n$. On the other hand if $|y| > 1$ then $|y-1| = |y|$ and we deduce that $|y| = |\varpi|^n|ax|$. As we also need $|y-a| \le |\varpi|^n|a|$, it must be that $y = ay_0$ with $y_0 \in U_F^n$.

Next we consider the case that $|a| < 1$. In this case these conditions become

(1) $1 \le |x|$
(2) $|\varpi|^n|a||x| \le |y|$
(3) $|y-1| \le |\varpi|^n|x|$
(4) $|x||y-a| \le |y|$.

We note first that if $|x| = 1$ then all we require is that $y \in U_F^n$. On the other hand if $|x| > 1$ then we require $|y-a| < |y|$ and hence $|y| = |a|$, this implies that $|x| = |\varpi|^{-n}$ and hence that $y = ay_0$ with $y_0 \in U_F^n$.

Now consider the case $|1-a| < 1$. Then our conditions become

(1) $1 \le |1-a||x|$
(2) $|y-1| \le |\varpi|^n|1-a||x| \le |1-a||y|$
(3) $|x||y-a| \le |y|$.

First we note that since $|y-1| \le |1-a||y|$, we require that $|y-1| \le |1-a|$. On the other hand since this forces $|y| = 1$ we also need $|x| \le |\varpi|^{-n}$. Since we also require $|1-a|^{-1} \le |x|$ we deduce that this region is empty unless $|\varpi|^n \le |1-a|$, which we now assume to be the case. We note that the conditions we need to satisfy are

(1) $|1-a|^{-1} \le |x| \le |\varpi|^{-n}$
(2) $|y-1| \le |\varpi|^n|1-a||x|$
(3) $|x||y-a| \le 1$.

We note that if $|x| < |\varpi|^{-n}$, then we have $|y-1| < |1-a|$ and hence $|y-a| = |a-1|$. From this it follows that $|x| = |a-1|^{-1}$ and $y \in U_F^n$. On the other hand if $|x| = |\varpi|^{-n}$ then we just require that $y = ay_0$ with $y_0 \in U_F^n$.

Finally, we consider the case that $|1-a| = |a| = 1$. Then the conditions become

(1) $1 \le |x|$
(2) $|y-1| \le |\varpi|^n|x| \le |y|$
(3) $|x||y-a| \le |y|$.

We note that if $|y-1| < 1$ then $|y-a| = 1$ which forces $|x| = 1$ and we require $y \in U_F^n$. When $1 < |y-1|$ we have $|y| = |y-1| = |y-a|$, this then forces $|x| = 1$ which then implies that $|y-1| \le |\varpi|^n$, a contradiction. So suppose $|y-1| = 1$, then we must also have $|y| = 1$ and $|x| = |\varpi|^{-n}$. Since we need $|y-a| \le |\varpi|^n$, we also need $y = ay_0$ with $y_0 \in U_F^n$. This concludes the proof. □

Let us now consider the integrals on $G_1(F)$ with respect to the non-split torus. We now make the assumption that $p > 2$. In this case a maximal order in D_1 is given by

$$R = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in D_1 : |\alpha|_E, |\beta|_E \le |\varpi_E|_E^{-1}, |(\alpha-\beta) - (\overline{\alpha-\beta})|_E \le 1 \right\}.$$

We define f to be the characteristic function of $R^\times$.

Lemma 3.8. *Assume that $p > 2$. Then we have*

$$H(\alpha\bar{\alpha}; f; T) = \begin{cases} \mathrm{vol}(U_E)\,\mathrm{vol}(E^\times/F^\times), & \text{if } v(1-\alpha\bar{\alpha}) \le 0; \\ \frac{1}{2}\,\mathrm{vol}(U_E)\,\mathrm{vol}(E^\times/F^\times), & \text{if } v(1-\alpha\bar{\alpha}) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. As in [Guo96, Section 2.4] we are reduced to computing the volume of (x, y) with $x \in E^\times$, $y \in E^\times/F^\times$ such that

$$\begin{pmatrix} x & \alpha x y \bar{y}^{-1} \\ \bar{\alpha}\bar{x}\bar{y}y^{-1} & \bar{x} \end{pmatrix} \in K'.$$

Thus we need

(1) $|x\bar{x}(1-\alpha\bar{\alpha})| = 1$
(2) $|x|_E \le |\varpi_E|_E^{-1}$, $|\alpha x|_E \le |\varpi_E|_E^{-1}$
(3) $|x - \bar{x} - \left(\alpha x y \bar{y}^{-1} - \bar{\alpha}\bar{x}\bar{y}y^{-1}\right)|_E \le 1$.

We note first that if $|\alpha|_E > 1$, then $|x|_E = |\alpha|_E^{-1}$ and so the third condition is automatically satisfied. On the other hand if $|1-\alpha\bar{\alpha}|_E = 1$, then we need $|x|_E = 1$ and again the third condition is automatically satisfied.

We are left then to consider the case that $|\alpha|_E = 1$ and $|1-\alpha\bar{\alpha}| < 1$. In this case we need $|x|_E^2 = |1-\alpha\bar{\alpha}|_E^{-1}$. Thus the region is empty unless $v(1-\alpha\bar{\alpha}) = 1$. So suppose $v(1-\alpha\bar{\alpha}) = 1$. We need to check the last condition. We write $x = a + b\sqrt{\varpi}$ with $|a| \le 1$ and $|b| = q^{-1}$.

We write $\alpha = c + d\sqrt{\varpi}$ with $|c| = 1$ and $|d| \le 1$. Then since we are assuming that $v(1-\alpha\bar{\alpha}) = 1$ we have $c \in \pm U_F^1$.

Write $z = y\bar{y}^{-1} = e + f\sqrt{\varpi}$. Since $z\bar{z} = 1$ we have that $e \in \pm\,\mathrm{U}_F^1$.

Now we have

$$\max\{1, |x - \bar{x} - (\alpha x z - \bar{\alpha}\bar{x}\bar{z})\,|\}$$

equal to

$$\max\{1, 2b\sqrt{\varpi} - \left(ceb\sqrt{\varpi} + ceb\sqrt{\varpi}\right)\}$$

which equals

$$\max\{1, (1-ce)\, b\sqrt{\varpi}\},$$

which is equal to 1 if and only if $ce \in U_F^1$. This concludes the calculation. □

Therefore we have the following lemma.

Lemma 3.9. *Assume that $p > 2$. Then the function*

$$(1-q^{-1})\varphi_0$$

can be matched with $\left(\mathbf{1}_{R_{1,0}^\times}, 0\right)$, where $R_{1,0}$ is a maximal order in D_1 which contains $\mathcal{O}_E$.

4. Local representation theory

4.1. **Nonarchimedean fields.** In this section we recall some results from [GP91] on the existence of test vectors in local representations.

We take F to be a non-archimedean local field and we denote by D the (unique) quaternion division algebra over F. We take π to be a unitarizable admissible generic representation of $\mathrm{GL}(2, F)$ with trivial central character, and, when it exists, we denote by π' the Jacquet-Langlands transfer of π to a representation of $D^\times$. Let $n(\pi)$ be the order of the conductor of π, i.e., π has conductor $\mathfrak{p}^{n(\pi)}$. We denote by $Z(F)$ the center of $\mathrm{GL}(2, F)$ and by $Z'(F)$ the center of $D^\times$.

We take E to be a quadratic field extension of F. We fix embeddings of E into $M_2(F)$ and into D. We let η denote the quadratic character of $F^\times$ associated to E/F by class field theory and we denote by π_E the base change of π to $\mathrm{GL}(2, E)$. Also, let $\mathcal{O}_E$ denote the integral closure of $\mathcal{O}$ in E.

We recall the following result due to Tunnell [Tun83] and Waldspurger [Wal85].

Proposition 4.1. *At most one of π and π' admit a non-zero $E^\times$-invariant linear form ℓ. The space of π admits such an ℓ if $\varepsilon(1/2, \pi_E) = \eta(-1)$. Otherwise, i.e., if $\varepsilon(1/2, \pi_E) = -\eta(-1)$, then π' admits such an ℓ. Moreover, the linear form ℓ is unique up to scaling.*

We now assume that E/F is unramified. We note that in this case $\varepsilon(1/2, \pi_E) = (-1)^{n(\pi)}$ and so π admits a non-zero $E^\times$-invariant linear form if and only if $n(\pi)$ is even. Then we have from [GP91, Proposition 2.6] the following.

Proposition 4.2. *If $n(\pi)$ is even then the space $\pi^{K_{1,n(\pi)}}$ is one dimensional and the functional ℓ does not vanish identically on this space. If $n(\pi)$ is odd then the space $(\pi')^{K_{2,n(\pi)}}$ is one dimensional and the functional ℓ does not vanish identically on this space.*

We now assume that the extension E/F is ramified. We have from [GP91, Proposition 2.6] the following.

Proposition 4.3. *Assume that $n(\pi) \leq 1$. If $\varepsilon(1/2, \pi_E) = \eta(-1)$ then the space $\pi^{K_{1,n(\pi)}}$ is one dimensional and the functional ℓ does not vanish identically on this space. If $\varepsilon(1/2, \pi_E) = -\eta(-1)$ then the space $(\pi')^{K_{2,n(\pi)}}$ is one dimensional and the functional ℓ does not vanish identically on this space.*

Next we consider the case that $n(\pi) \geq 2$. We fix a uniformizing element ϖ_E in E. Then from [GP91, Remark 2.7] we have the following propositions.

Proposition 4.4. *Assume that $n(\pi) \geq 2$. When $\varepsilon(1/2, \pi_E) = \delta_{E/F}(-1)$, the space $\pi^{K_{1,n(\pi)}}$ is two dimensional. There is a unique line in this subspace fixed by ϖ_E and the functional ℓ does not vanish identically on this line. When $\varepsilon(1/2, \pi_E) = -\delta_{E/F}(-1)$, the space $(\pi')^{K'_{2,n(\pi)}}$ is two dimensional. There is a unique line in this subspace fixed by ϖ_E and the functional ℓ does not vanish identically on this line.*

4.2. **Archimedean fields.** Suppose now that we have a representation π of $\mathrm{PGL}(2)$ over either $\mathbf{R}$ or $\mathbf{C}$. Fix the additive character ψ of F to be $\psi(x) = e^{2\pi i x}$ or $\psi(z) = e^{4\pi i \Re z}$ according to whether the base field is $\mathbf{R}$ or $\mathbf{C}$. We take T to be the diagonal torus inside $\mathrm{PGL}(2)$. Then we have the following result from [Popb, Proposition 4].

Proposition 4.5. *Let $\mathcal{W}$ be the minimal K-type in the Whittaker model $W(\pi, \psi)$ such that $\dim \mathcal{W}^T = 1$. Then $\zeta(s, W, \psi) = L(s, \pi)$ for some $W \in \mathcal{W}^T$.*

Moreover Popa, [Popb, p. 10], describes the minimal K-type for which $\mathcal{W}^T \neq 0$ for all such π.

5. Local distributions

5.1. **Nonarchimedean fields.** Let F be a nonarchimedean local field. Let E be either a quadratic extension of F or else $F \oplus F$. Let η be the corresponding character of $F^\times$. We denote by $\mathfrak{p}^{n(\eta)}$ the conductor of η. We define

$$\tau(\eta, \psi) = \int_{\mathfrak{p}^{-n(\psi)-n(\eta)}} \eta(x)\ \psi(x)\ d^\times x.$$

We take π to be an irreducible generic unitary representation of $\mathrm{GL}(2, F)$ with trivial central character. We denote by $\mathfrak{p}^{n(\pi)}$ the conductor of π. We denote by

$\mathcal{W}(\pi,\psi)$ the Whittaker model of π with respect to the character ψ. We fix a $\mathrm{GL}(2,F)$-invariant inner product $[\ ,\ ]$ on $\mathcal{W}(\pi,\psi)$. We denote by π_E the base change of π to an irreducible admissible representation of $\mathrm{GL}(2,E)$.

We define, for $s \in \mathbf{C}$ with $\Re s \gg 0$ and $W \in \mathcal{W}(\pi,\psi)$,

$$\zeta(s,W,\psi) = \int_{F^\times} W\begin{pmatrix} x & \\ & 1 \end{pmatrix} |x|^{s-\frac{1}{2}}\ d^\times x$$

and

$$\zeta(s,W,\chi,\psi) = \int_{F^\times} W\begin{pmatrix} x & \\ & 1 \end{pmatrix} \chi(x)\ |x|^{s-\frac{1}{2}}\ d^\times x,$$

where χ is a character $\chi : F^\times \to \mathbf{C}^\times$. As is well known, these integrals converge for $\Re s$ sufficient large and have an analytic continuation to $\mathbf{C}$. We recall that there exists a unique vector $W^0 \in \mathcal{W}(\pi,\psi)^{K_{n(\pi)}}$ such that

$$\zeta(s,\pi(\mathrm{diag}(\varpi^{-n(\psi)},1))W^0,\psi) = L(s,\pi)$$

for any uniformizer ϖ in F.

Having fixed ψ we define, for $W \in \mathcal{W}(\pi,\psi)$,

$$\lambda(W) = \frac{\zeta(1/2,W,\psi)}{L(1/2,\pi)}$$

and

$$\lambda_\eta(W) = \frac{\zeta(1/2,W,\eta,\psi)}{L(1/2,\pi\otimes\eta)}.$$

Having fixed E (and hence η) we define, for $f \in C_c^\infty(G(F))$,

$$\Theta_{\pi,\psi}(f) = \sum_i \lambda(\pi(f)W_i)\overline{\lambda_\eta(W_i)}$$

where $\{W_i\}$ is an orthonormal basis of $\mathcal{W}(\pi,\psi)$.

We now compute $\Theta_{\pi,\psi}(f)$ for certain functions f depending on E/F and π.

5.1.1. *E/F split.* We take $f = \mathrm{vol}(K_{n(\pi)})^{-1}\mathbf{1}_{K_{n(\pi)}}$. Then we clearly have the following.

Lemma 5.1. *For $f = \mathrm{vol}(K_{n(\pi)})^{-1}\mathbf{1}_{K_{n(\pi)}}$ we have*

$$\Theta_{\pi,\psi}(f) = \frac{1}{[W^0,W^0]}.$$

5.1.2. *E/F unramified.* We take $f = \mathrm{vol}(K_{n(\pi)})^{-1}\mathbf{1}_{K'_{n(\pi)}}$. Then we have, as in [Guo96, Proof of Proposition 3.1],

$$\Theta_{\pi,\psi}(f) = \frac{\lambda(\pi(w)W^0)\overline{\lambda_\eta(\pi(w)W^0)}}{[W^0,W^0]},$$

where

$$w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

Now

$$\lambda(\pi(w)W^0) = \varepsilon(1/2,\pi)$$

and

$$\lambda_\eta(\pi(w)W^0) = \varepsilon(1/2,\pi\otimes\eta)\lambda_\eta(W^0) = \varepsilon(1/2,\pi\otimes\eta)\eta(\varpi^{n(\psi)}),$$

which gives the following result.

Lemma 5.2. *For* $f = \mathrm{vol}(K_{n(\pi)})^{-1}\mathbf{1}_{K'_{n(\pi)}}$ *we have*

$$\Theta_{\pi,\psi}(f) = \frac{\varepsilon(1/2, \pi_E)\eta(\varpi^{n(\psi)})}{[W^0, W^0]}.$$

5.1.3. *E/F ramified.* We now assume that E/F is ramified and $p > 2$. We fix a uniformizer ϖ of F such that $\eta(\varpi) = 1$. We wish to compute $\Theta_{\pi,\psi}(\varphi_{n(\pi)})$.

Lemma 5.3. *We have*

$$\Theta_{\pi,\psi}(\varphi_{n(\pi)}) = \frac{1}{[W^0, W^0]} \frac{\varepsilon(1/2, \pi_E)}{L(1/2, \pi \otimes \eta)} \frac{\overline{\tau(\eta, \psi)}}{\mathfrak{d}}.$$

Proof. We follow [Guo96, Section 3.3]. Arguing as in there we get

$$\Theta_{\pi,\psi}(\varphi_{n(\pi)}) = \frac{1}{[W^0, W^0]} \frac{\varepsilon(1/2, \pi_E)}{L(1/2, \pi \otimes \eta)} \overline{\int_{F^\times} \pi\left(\begin{pmatrix} 1 & \varpi^{-n(\eta)} \\ 0 & 1 \end{pmatrix} W^0\right) \begin{pmatrix} a & \\ & 1 \end{pmatrix} \eta(a)\ d^\times a}.$$

We note that we have

$$\begin{aligned}
\left(\begin{pmatrix} 1 & \varpi^{-n(\eta)} \\ 0 & 1 \end{pmatrix} W^0\right) \begin{pmatrix} a & \\ & 1 \end{pmatrix} &= W^0 \begin{pmatrix} a & a\varpi^{-n(\eta)} \\ 0 & 1 \end{pmatrix} \\
&= W^0 \left(\begin{pmatrix} 1 & a\varpi^{-n(\eta)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \\
&= \psi(a\varpi^{-n(\eta)}) W^0 \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

and hence

$$\int_{F^\times} \pi\left(\begin{pmatrix} 1 & \varpi^{-n(\eta)} \\ 0 & 1 \end{pmatrix} W^0\right) \begin{pmatrix} a & \\ & 1 \end{pmatrix} \eta(a)\ d^\times a = \int_{F^\times} W^0 \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \psi(a\varpi^{-n(\eta)})\eta(a)\ d^\times a.$$

Writing this integral as a sum over $F^\times/U_F$ and using the vanishing properties of the integrals

$$\int_{U_F} \psi(a\varpi^{-n})\eta(a)\ d^\times a$$

we get the integral above is equal to

$$W^0 \begin{pmatrix} \varpi^{-n(\psi)} & \\ & 1 \end{pmatrix} \tau(\eta, \psi).$$

Using the fact that

$$W^0 \begin{pmatrix} \varpi^{-n(\psi)} & \\ & 1 \end{pmatrix} = \mathfrak{d}^{-1}$$

gives the result. □

5.2. **Archimedean fields.** We now consider the archimedean case and take $F = \mathbf{R}$ or $\mathbf{C}$. We only consider the case that $E = F \oplus F$. Fix the additive character ψ of F to be $\psi(x) = e^{2\pi i x}$ or $\psi(z) = e^{4\pi i \Re z}$ according to whether $F = \mathbf{R}$ or $\mathbf{C}$. Let $[\ ,\ ]$ be an invariant inner product on $W(\pi, \psi)$. Define ζ and $\Theta_{\pi,\psi}$ as in the non-archimedean setting.

In this case we choose $f \in C_c^\infty(\mathrm{GL}(2,F))$ to project onto a non-zero vector in $\mathcal{W}^T$ where $\mathcal{W}$ is the minimal K-type for which $\mathcal{W}^T \neq 0$ as in [Popb, Proposition 4]. In this case we clearly have

$$\Theta_{\pi,\psi}(f) = \frac{1}{[W^0, W^0]}$$

where $W^0 \in \mathcal{W}^T$ such that $\zeta(s, W^0, \psi) = L(s, \pi)$.

6. Global result

In this section we prove Theorem 1.1. We now take F to be a number field. For a finite place v in F let $\mathfrak{p}_v$ denote the prime ideal in $\mathcal{O}_v$ the ring of integers of F_v. We take E to be a quadratic extension of F which is split at the infinite places of F. For each place v of F we denote by $\mathcal{O}_{E_v}$ the integral closure of $\mathcal{O}_v$ in E_v. We let η denote the quadratic character of $F^\times \backslash \mathbf{A}_F^\times$ associated to E/F by class field theory. For each place v of F we denote by $\mathfrak{p}_v^{n(\eta_v)}$ the conductor of η_v.

We have defined compact open subgroups of the local groups $\mathrm{GL}(2,F)$ in Section 2. We use the same notation here with the addition of a subscript v, for example, $K_v = \mathrm{GL}(2, \mathcal{O}_v)$.

6.1. Measures.

We fix a non-trivial character $\psi : F\backslash \mathbf{A}_F \to \mathbf{C}^\times$ as in [Guo96, Section 4.1]. We write $\psi = \prod_v \psi_v$ where ψ_v is a character of F_v. We use these local characters to form measures on the local groups as in Section 2. Globally we give all discrete subgroups the counting measure. We take the measure on groups over $\mathbf{A}_F$ to be the product of the local measures defined with respect to the characters ψ_v. For groups over $\mathbf{A}_E$ we define measures in the same way relative to the pull back of ψ via the trace map.

We define $\mathfrak{d}_F = \prod_v \mathfrak{d}_v$ where for each v we have

$$\mathfrak{d}_v = \begin{cases} \mathrm{vol}(\mathcal{O}_v), & \text{if } F_v \text{ is non-archimedean;} \\ \mathrm{vol}([0,1]), & \text{if } F_v \text{ is real;} \\ \frac{1}{2}\,\mathrm{vol}\left(\{x+iy : 0 \le x, y \le 1\}\right), & \text{if } F_v \text{ is complex.} \end{cases}$$

We note that $\mathfrak{d}_F = |\Delta_F|^{-\frac{1}{2}}$ where Δ_F is the discriminant of F.

We take the Tamagawa measure on $\mathrm{GL}(2, \mathbf{A}_F)$ and on the adelic points of inner forms of $\mathrm{GL}(2)$.

6.2. Proof.

Fix an irreducible cuspidal automorphic representation π of $\mathrm{GL}(2, \mathbf{A}_F)$ with trivial central character. We consider the set $X(E/F)$ of isomorphism classes of pairs $\{(G,T)\}$ where G is an inner form of $\mathrm{GL}(2)$ and T is a subtorus of G defined over F with $T(F) \xrightarrow{\sim} E^\times$. It is clear from the local results above that we have the following.

Proposition 6.1. *Suppose that $\varepsilon(1/2, \pi_E) = -1$. Let $(G,T) \in X(E/F)$ such that π transfers to a representation π' of $G(\mathbf{A}_F)$. Then the only $T(\mathbf{A}_F)$-invariant linear form on π' is zero.*

Suppose that $\varepsilon(1/2, \pi_E) = 1$. Then there exists a unique pair $(G,T) \in X(E/F)$ such that π transfers to a representation π' of $G(\mathbf{A}_F)$ and such that the space of π' admits a non-zero $T(\mathbf{A}_F)$-invariant linear form. Moreover such a linear form is unique up to scaling.

We assume from now on that $\varepsilon(1/2, \pi_E) = 1$ and fix (G, T) as in the proposition. We identify $G(F)$ as $D^\times$, where D is either $M_2(F)$ or a quaternion algebra over F, and we fix an embedding $E \hookrightarrow D$ which induces $E^\times \xrightarrow{\sim} T(F)$. We denote by π' the Jacquet-Langlands transfer of π to $G(\mathbf{A}_F)$.

For $n \in \mathbf{Z}_{\geq 0}$ we denote by $R^D_{v,n}$ an order of reduced discriminant $\mathfrak{p}_v^n$ in $D_v = D \otimes_F F_v$ which contains $\mathcal{O}_{E_v}$. We denote by $K^D_{v,n}$ the compact open subgroup $(R^D_{v,n})^\times$ of $G(F_v)$.

For each finite place v of F we denote by $\mathfrak{p}_v^{n(\pi_v)}$ the conductor of π_v. We form $\mathcal{W}(\pi, \psi)$, the Whittaker model of π with respect to ψ, of functions

$$W_\varphi(g) = \int_{F\backslash \mathbf{A}_F} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \overline{\psi(x)}\, dx.$$

We fix inner products on the local Whittaker models, $\mathcal{W}(\pi_v, \psi_v)$, compatible with the L^2-inner product on the space of π.

At each place v we fix an element $W_v \in \mathcal{W}(\pi_v, \psi_v)$. If v is non-archimedean then $W_v \in \mathcal{W}(\pi_v, \psi_v)^{K_{n(\pi_v)}}$ is such that $\zeta(s, \pi(\mathrm{diag}(\varpi^{-n(\psi_v)}, 1))W_v, \psi_v) = L(s, \pi_v)$. If v is archimedean then we take $W_v \in \mathcal{W}(\pi_v, \psi_v)$ as in [Popb, Proposition 4]. We take $\varphi_0 \in \pi$ such that $W_{\varphi_0} = \prod_v W_v$.

Let S denote the set of places v of F such that v satisfies one of the following conditions,

(1) v is archimedean,
(2) v is ramified in E,
(3) $n(\psi_v) \neq 0$, or
(4) π_v is ramified.

We let $\{\varphi_i\}$ denote an orthonormal basis of π^{K^S} and $\{\varphi'_i\}$ an orthonormal basis of $(\pi')^{K^S}$. We take $f_S = \prod_{v \in S} f_v$ to be a smooth function on $\mathrm{GL}(2, \mathbf{A}_S)$. Let A denote the diagonal torus in GL(2). Then the fundamental identity obtained from the relative trace formula [Jac86] is that

$$\sum_{\{\varphi_i\}} \int_{Z(\mathbf{A}_F)A(F)\backslash A(\mathbf{A}_F)} (\pi(f_S)\varphi_i)(a)\, da \int_{Z(\mathbf{A}_F)A(F)\backslash A(\mathbf{A}_F)} \overline{\varphi_i(b)} \eta(\det b)\, db \tag{1}$$

is equal to

$$\sum_{\{\varphi'_i\}} \int_{Z'(\mathbf{A}_F)T(F)\backslash T(\mathbf{A}_F)} (\pi'(f'_S)\varphi'_i)(s)\, ds \int_{Z'(\mathbf{A}_F)T(F)\backslash T(\mathbf{A}_F)} \overline{\varphi'_i(t)}\, dt. \tag{2}$$

For each $v \in S$ we fix an orthonormal basis $\{W_{v,i}\}$ of $\mathcal{W}(\pi_v, \psi_v)$. For $v \notin S$ we take W_v to be the essential vector in $\mathcal{W}(\pi_v, \psi_v)$, i.e. W_v is the vector in $\mathcal{W}(\pi_v, \psi_v)$ which is invariant under $\mathrm{GL}(2, \mathcal{O}_v)$ and $W_v(k_v) = 1$ for all $k_v \in \mathrm{GL}(2, \mathcal{O}_v)$. Tensoring these up gives a basis $\{\varphi_i\}$ of π^{K^S}. Then the left hand side of the identity above is equal to

$$\frac{L(1/2, \pi_E)}{\prod_{v \notin S} [W_v, W_v]} \prod_{v \in S} \Theta_{\pi_v, \psi_v}(f_v),$$

where for $v \in S$ we have

$$\Theta_{\pi_v, \psi_v}(f_v) = \sum_i \lambda_v(\pi_v(f_v) W_{v,i}) \overline{\lambda_{\eta_v}(W_{v,i})}.$$

We now proceed to choose suitable test functions f_v, for $v \in S$, to plug into the trace formula identity. We write $S = S_0 \amalg S_1 \amalg S_2$ where

$$S_0 = \{v \in S : v \text{ is split in } E\},$$
$$S_1 = \{v \in S : v \text{ is inert in } E\},$$

and

$$S_2 = \{v \in S : v \text{ is ramified in } E\}.$$

We further write $S_0 = S_0' \amalg S_0''$ where

$$S_0' = \{v \in S_0 : v \text{ is non-archimedean}\},$$

and

$$S_0'' = \{v \in S_0 : v \text{ is archimedean}\}.$$

For $v \in S_0'$, we take

$$f_v = \mathrm{vol}(K_{n(\pi_v)})^{-1} \mathbf{1}_{K_{v,n(\pi_v)}}.$$

Hence we can take $f_v' = \mathrm{vol}(K^D_{v,n(\pi_v)})^{-1} \mathbf{1}_{K^D_{v,n(\pi_v)}}$. By Lemma 5.1 we have

$$\Theta_{\pi_v,\psi_v}(f_v) = \frac{1}{[W_v, W_v]}.$$

For $v \in S_0''$, we take f_v as in Section 5.2. Then

$$\Theta_{\pi_v,\psi_v}(f_v) = \frac{1}{[W_v, W_v]}.$$

For $v \in S_1$, we take

$$f_v = \mathrm{vol}(K_{v,n(\pi_v)})^{-1} \mathbf{1}_{K'_{v,n(\pi_v)}}.$$

We note that, by Lemma 2.2,

$$\mathrm{vol}(K_{v,n(\pi_v)}) = \begin{cases} \mathfrak{d}_v^4(1 - q_v^{-2}), & \text{if } n(\pi_v) = 0; \\ \mathfrak{d}_v^4 q_v^{-n(\pi_v)}(1 - q_v^{-1}), & \text{if } n(\pi_v) > 0. \end{cases}$$

On the other hand, by Lemma 2.3,

$$\mathrm{vol}(K^D_{v,n(\pi_v)}) = \begin{cases} \mathfrak{d}_v^4(1 - q_v^{-2}), & \text{if } n(\pi_v) = 0; \\ \mathfrak{d}_v^4 q_v^{-n(\pi_v)}(1 + q_v^{-1}), & \text{if } n(\pi_v) > 0. \end{cases}$$

Hence, by Lemma 3.3, we can take

$$f_v' = \frac{C(\pi_v)}{\mathrm{vol}(K^D_{v,n(\pi_v)})} 1_{K^D_{v,n(\pi_v)}},$$

where

$$C(\pi_v) = \begin{cases} 1, & \text{if } n(\pi_v) = 0; \\ \frac{q+1}{q-1}, & \text{if } n(\pi_v) > 0. \end{cases}$$

Thus by Lemma 5.2 we have

$$\Theta_{\pi_v,\psi_v}(f_v) = \frac{\varepsilon(1/2, \pi_{E,v})\eta_v(\varpi)^{n(\psi_v)}}{[W_v, W_v]}.$$

For $v \in S_2$, we take

$$f_v = \mathrm{vol}(K_{v,n(\pi_v)})^{-1} \varphi_{v,n(\pi_v)}.$$

By Lemma 2.2,

$$\mathrm{vol}(K_{v,n(\pi_v)}) = \begin{cases} \mathfrak{d}_v^4(1 - q_v^{-2}), & \text{if } n(\pi_v) = 0; \\ \mathfrak{d}_v^4(1 - q_v^{-1}) q_v^{-n(\pi_v)}, & \text{if } n(\pi_v) > 0. \end{cases}$$

And by Lemma 2.3

$$\mathrm{vol}(K^D_{v,n(\pi_v)}) = \begin{cases} \mathfrak{d}_v^4(1-q_v^{-2}), & \text{if } n(\pi_v)=0; \\ \mathfrak{d}_v^4(1-q_v^{-1})q_v^{-1}, & \text{if } n(\pi_v)=1 \text{ and } D_v = M_2(F_v); \\ \mathfrak{d}_v^4(1+q_v^{-1})q_v^{-1}, & \text{if } n(\pi_v)=1 \text{ and } D_v \neq M_2(F_v); \\ \mathfrak{d}_v^4(1-q_v^{-1})q_v^{-n(\pi_v)}, & \text{if } n(\pi_v)>1. \end{cases}$$

Hence we can take, by Lemma 3.6 and Lemma 3.9,

$$f'_v = \frac{C(\pi_v)}{\mathrm{vol}(K^D_{v,n(\pi_v)})}\mathbf{1}_{K^D_{v,n(\pi_v)}},$$

where

$$C(\pi_v) = \begin{cases} (1-q_v^{-1})^{-1}, & \text{if } n(\pi_v)=0; \\ \frac{1}{2}(1-q_v^{-1})^{-1}, & \text{if } n(\pi_v)=1 \text{ and } D_v = M_2(F_v); \\ \frac{1}{2}(1+q_v^{-1})(1-q_v^{-1})^{-2}, & \text{if } n(\pi_v)=1 \text{ and } D_v \neq M_2(F_v); \\ \frac{1}{2}(1-q_v^{-1})^{-1}, & \text{if } n(\pi_v)>1. \end{cases}$$

Then by Lemma 5.3 we have

$$\Theta_{\pi_v,\psi_v}(f_v) = \frac{1}{[W_v,W_v]}\frac{\varepsilon(1/2,\pi_{E,v})}{L(1/2,\pi_v\otimes\eta_v)}\frac{\overline{\tau(\eta_v,\psi_v)}}{\mathfrak{d}_v}.$$

We note that when $n(\pi_v)=0$ we have

$$\Theta_{\pi_v,\psi_v}(f_v) = \frac{1}{[W_v,W_v]}\varepsilon(1/2,\pi_{E,v})\frac{\overline{\tau(\eta_v,\psi_v)}}{\mathfrak{d}_v}.$$

We use these choices for f_v in our trace formula identity, together with the facts that $\varepsilon(1/2,\pi_E)=1$,

$$\prod_v \frac{\tau(\eta_v,\psi_v)}{|\tau(\eta_v,\psi_v)|} = W(\eta) = 1$$

and that at the places v of F which ramify in E we have

$$|\tau(\eta_v,\psi_v)| = \mathfrak{d}_v q_v^{-\frac{n(\eta_v)}{2}}(1-q_v^{-1})^{-1}.$$

For (1), this yields

$$\frac{L(1/2,\pi_E)}{[\varphi_0,\varphi_0]}\prod_{v\in S(E)} q_v^{-\frac{n(\eta_v)}{2}}(1-q_v^{-1})^{-1}\prod_{v\in S_2(E,\pi)}\frac{1}{L(1/2,\pi_v\otimes\eta_v)},$$

where $S(E)$ denotes the set of places of F which ramify in E and $S_2(E,\pi)$ denotes the set of places of F which are ramified for both E and π.

On the other hand, (2) becomes

$$\frac{|\int_T \varphi(t)\ dt|^2}{[\varphi,\varphi]}\prod_{v\in S_1(E,\pi)}\frac{q_v+1}{q_v-1}\prod_{v\in S(E)} C(\pi_v),$$

where $S_1(E,\pi)$ denotes the set of places of F which are unramified, inert in E and at which π is ramified, and $S(E)$ is the of places of F ramified in E and $C(\pi_v)$ is defined as above. Here $\varphi\in\pi'$ is a non-zero vector such that,

(1) at each finite place v, φ is fixed by $K^D_{v,n(\pi_v)}$,
(2) at each finite place v which is ramified in E, φ is fixed by $T(F_v)$, and
(3) at each archimedean place v φ lies in $\mathcal{W}^T$ as in [Popb, Proposition 4].

These conditions determine $\varphi \in \pi'$ up to a non-zero scalar.

We note that if v is a place such that $n(\pi_v) = 1$ then we have

$$L(s, \pi_v) = (1 + q_v^{-s-\frac{1}{2}})^{-1}$$

if $D_v = \mathrm{GL}(2, F_v)$ and

$$L(s, \pi_v) = (1 - q_v^{-s-\frac{1}{2}})^{-1}$$

if $D_v \neq \mathrm{GL}(2, F_v)$. And, of course, $L(s, \pi_v) \equiv 1$ if $n(\pi_v) > 1$. Hence we deduce that

$$L(1/2, \pi_E) = C(\pi, E) \frac{[\varphi_0, \varphi_0]}{[\varphi, \varphi]} \left| \int_T \varphi(t)\ dt \right|^2$$

where

$$C(\pi, E) = \sqrt{d_{E/F}} \prod_{v \in S_1(E,\pi)} \frac{q_v + 1}{q_v - 1} \prod_{v \in S_2(E,\pi)} \left(A(\pi_v) L(1/2, \pi_{E,v})\right).$$

Here $d_{E/F}$ denotes the discriminant of E/F, $S_2(E, \pi)$ denotes the set of places of F which are ramified in E and at which π is ramified, and we define

$$A(\pi_v) = \begin{cases} \frac{1}{2}(1 + q_v^{-1}), & \text{if } n(\pi_v) = 1; \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

References

[GP91] Benedict H. Gross and Dipendra Prasad, *Test vectors for linear forms*, Math. Ann. **291** (1991), no. 2, 343–355. MR **MR1129372 (92k:**22028)

[Gro87] Benedict H. Gross, *Heights and the special values of L-series*, Number theory (Montreal, Que., 1985), CMS Conf. Proc., vol. 7, Amer. Math. Soc., Providence, RI, 1987, pp. 115–187. MR **MR894322 (89c:**11082)

[Guo96] Jiandong Guo, *On the positivity of the central critical values of automorphic L-functions for* GL(2), Duke Math. J. **83** (1996), no. 1, 157–190. MR **MR1388847 (97h:**11055)

[Jac86] Hervé Jacquet, *Sur un résultat de Waldspurger*, Ann. Sci. Éc. Norm. Supér., IV. **19** (1986), no. 2, 185–229.

[Popa] Alexandru Popa, *Central values of Rankin L-series over real quadratic fields*, preprint.

[Popb] ———, *Whittaker newforms for local representations of GL(2)*, preprint.

[Tun83] Jerrold B. Tunnell, *Local ϵ-factors and characters of* GL(2), Amer. J. Math. **105** (1983), no. 6, 1277–1307. MR **MR721997 (86a:**22018)

[Wal85] Jean-Loup Waldspurger, *Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie*, Compositio Math. **54** (1985), no. 2, 173–242. MR **MR783511 (87g:**11061b)

[Xue] Hui Xue, *Central values of L-functions over CM fields*, preprint.

[Zha01] Shou-Wu Zhang, *Gross-Zagier formula for* GL_2, Asian J. Math. **5** (2001), no. 2, 183–290. MR **MR1868935 (2003k:**11101)

(KM) Columbia University, Department of Mathematics, 2990 Broadway, Mailcode 4406, New York, NY 10027, USA

E-mail address, KM: `kimball@math.columbia.edu`

(DW) School of Mathematics, Institute for Advanced Study, Einstein Drive, Princeton, NJ 08540, USA

E-mail address, DW: `dw@ias.edu`