LOCAL CONDUCTOR BOUNDS FOR MODULAR ABELIAN VARIETIES

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Abstract. Brumer and Kramer gave bounds on local conductor exponents for an abelian variety $A/\mathbb{Q}$ in terms of the dimension of A and the localization prime p. Here we give improved bounds in the case that A has maximal real multiplication, i.e., A is isogenous to a factor of the Jacobian of a modular curve $X_0(N)$. In many cases, these bounds are sharp. The proof relies on showing that the rationality field of a newform for $\Gamma_0(N)$, and thus the endomorphism algebra of A, contains $\mathbb{Q}(\zeta_{p^r})^+$ when p divides N to a sufficiently high power. We also deduce that certain divisibility conditions on N determine the endomorphism algebra when A is simple.

1. Introduction

Let $A/\mathbb{Q}$ be a d-dimensional abelian variety of conductor N_A. Brumer and Kramer [BK94] gave upper bounds $v_p(N_A) \le B(p,d)$ on local conductor exponents in terms of p and d. Moreover, their bounds are sharp in the sense for any p, d, there exists an abelian variety A with $v_p(N_A) = B(p,d)$. The abelian variety A is said to have real multiplication (RM) if the endomorphism algebra $\mathrm{End}^0(A) := \mathrm{End}(A) \otimes \mathbb{Q}$ contains a totally real number field $K \supsetneq \mathbb{Q}$. Brumer and Kramer suggested that stronger bounds on local conductor exponents may exist when restricting to abelian varieties with RM.

We say A has maximal RM if $\mathrm{End}^0(A)$ contains a totally real field K of degree $d = \dim A$. (Necessarily, $[K : \mathbb{Q}] \mid d$ for any subfield K of $\mathrm{End}^0(A)$.) Abelian varieties with maximal RM are of GL(2)-type, and they are simple over $\mathbb{Q}$ if and only if $\mathrm{End}^0(A) \simeq K$ is a totally real field of degree d [Rib04, Theorem 2.1]. Up to isogeny, the simple abelian varieties with maximal RM are precisely the simple factors of Jacobians $J_0(N) = \mathrm{Jac}(X_0(N))$ of the modular curves $X_0(N)$. If N is minimal such that A is isogenous to a simple factor of $J_0(N)$, then $N_A = N^d$.

We improve the Brumer–Kramer bounds for abelian varieties with maximal RM. Define

$$B_0(p,d) = \begin{cases} 8 + 2v_2(d) & \text{if } p = 2, \\ 5 + 2v_3(d) & \text{if } p = 3, \\ 4 + 2v_p(d) & \text{if } p \ge 5 \text{ and } (p-1) \mid 2d, \\ 2 & \text{else.} \end{cases} \tag{1.1}$$

Theorem 1.1. *Let $A/\mathbb{Q}$ be a d-dimensional simple abelian variety with maximal RM and conductor N^d.*

(1) We have $v_p(N) \le B_0(p,d)$, i.e., $v_p(N_A) \le dB_0(p,d)$.

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(2) *The bounds in (1) are stronger than what the Brumer–Kramer bounds imply. Namely,* $B_0(p,d) \le \lfloor \frac{B(p,d)}{d} \rfloor$ *for all* p, d*, and this is a strict inequality if either (a)* $5 \le p < 2d+1$ *and* $(p-1) \nmid 2d$*, or (b)* $p \le 3$*,* $d > 3$ *and* $p \nmid d$*. It is an equality when* $p \ge 2d+1$*. (In other cases, this is sometimes an equality and sometimes not.)*

(3) *The bound in (1) sharp, i.e.,* $v_p(N) = B_0(p,d)$ *occurs for some such* A*, for all* (p,d) *such that* $d \le 10$*, with the possible exclusion of the following 5 cases:* $B_0(5,10) = 6$*,* $B_0(11,10) = 4$*, and* $B_0(2d+1,d) = 4$ *for* $d = 6,7,8$*.*

Remark 1.2. If A is a d-dimensional simple abelian variety of GL(2)-type, i.e., it is isogenous to a factor of some $J_1(N) = \mathrm{Jac}(X_1(N))$, then one similarly has the bounds $v_p(N_A) \le dB_0(p,d)$ for p odd, and $v_2(N_A) \le d(B_0(2,d)+1)$. This improves the Brumer–Kramer bounds for odd p, and for certain values of d when $p=2$. See Section 3.4.

Remark 1.3. If A is not simple, then applying the above bounds to its simple factors yield even stronger bounds in terms of d. E.g., suppose $A \simeq A_1 \times A_2$ where A_1 and A_2 are isogenous simple abelian varieties with endomorphism algebra K_0, a totally real field of degree $d/2$. Then $\mathrm{End}^0(A) \simeq M_2(K_0)$, which contains totally real fields of degree d, and thus A has maximal RM. Since $N_A = N_{A_1}^2$, one sees that $v_p(N_A) \le 2v_p N_{A_1} \le dB_0(p,d/2)$. This is better than the bound $v_p(N_A) \le dB_0(p,d)$ for simple A.

The precise formula for the Brumer–Kramer bounds is slightly complicated—see Section 3.1 for details—but we list the bounds $B'(p,d) := \lfloor \frac{B(p,d)}{d} \rfloor$ (i.e., the Brumer–Kramer bounds applied to $v_p(N)$ with N and A as in Theorem 1.1) for $d \le 10$ in Table 1. For $p > 2d+1$, $B'(p,d) = B_0(p,d) = 2$, and we omit these entries. We write the bound $B_0(p,d)$ from Theorem 1.1 in parentheses when it is smaller.

In this table, we bolded all of the cases where we have checked the upper bound is sharp, by finding associated modular forms. We also starred the cases $B_0(19,9)$ and $B_0(11,10)$ to indicate the upper bounds are at least "almost sharp", in the sense that $v_p(N) = B_0(p,d)-1$ is attained. Note that by quadratic twisting at p, an upper bound of $B_0(p,d) = 2$ is always sharp, provided there exists at least one simple d-dimensional abelian variety $A/\mathbb{Q}$ with maximal RM. It seems plausible that the bound $B_0(p,d)$ is always sharp, but we do not know of constructions of simple abelian varieties with maximal RM (or more generally of GL(2) type) in higher dimensions which would shed light on this.

The proof of part (1) relies on a result about rationality fields of modular forms. Denote by ζ_m a primitive m-th root of unity in $\bar{\mathbb{Q}}$. Let $\mathbb{Q}(\zeta_m)^+ = \mathbb{Q}(\zeta_m + \zeta_m^{-1})$ be the maximal totally real subfield of $\mathbb{Q}(\zeta_m)$. Write $K_f = \mathbb{Q}((a_n)_n)$ for the rationality field of a cuspidal newform $f = \sum a_n q^n$.

Let $f \in S_{2k}(N)$ be a newform. (Here $S_*(N)$ indicates level $\Gamma_0(N)$ with trivial nebentypus). Then Proposition 2.1 states that when $v_p(N)$ is sufficiently large, $K_f \supset \mathbb{Q}(\zeta_{p^r})^+$, where r is a certain function of $v_p(N)$ that grows like $v_p(N)/2$. Consequently, if we fix the (rationality) degree $d := [K_f : \mathbb{Q}]$ of f, then this places bounds on $v_p(N)$, precisely $v_p(N) \le B_0(p,N)$. Applying these bounds for a weight 2 newform f associated to A leads to the first part of the theorem.

Proposition 2.1 is not entirely novel. Brumer [Bru95, Theorem 5.5] proved a version of it in terms of abelian varieties of GL(2)-type, which suffices to deduce Remark 1.2. However, to our knowledge, Remark 1.2 has not appeared in the literature. Moreover, Proposition 2.1 is sharper for $p=2$, and our proof also allows us to say something more about the local representations at $p=3$ (see Corollary 2.3).

	$p=2$	3	5	7	11	13	17	19
$d=1$	**8**	**5**						
2	**10**	**5**	**4**					
3	9 (**8**)	**7**	3 (**2**)	**4**				
4	**12**	6 (**5**)	**4**	3 (**2**)				
5	11 (**8**)	6 (**5**)	4 (**2**)	3 (**2**)	**4**			
6	11 (**10**)	**7**	**4**	**4**	3 (**2**)	4		
7	10 (**8**)	6 (**5**)	4 (**2**)	4 (**2**)	3 (**2**)	3 (**2**)		
8	**14**	6 (**5**)	**4**	3 (**2**)	3 (**2**)	3 (**2**)	4	
9	13 (**8**)	**9**	4 (**2**)	**4**	3 (**2**)	3 (**2**)	3 (**2**)	4*
10	13 (**10**)	8 (**5**)	6	4 (**2**)	4*	3 (**2**)	3 (**2**)	3 (**2**)

TABLE 1. Brumer–Kramer bounds $B'(p,d)$ for simple maximal RM, with the bounds $B_0(p,d)$ in parentheses when different; bounds known to be sharp (resp. almost sharp) are bolded (resp. starred)

Apart from sharpening Brumer's result when $p=2$, what is new here is an explicit formulation of the bounds $B_0(p,d)$, a direct comparison with the Brumer–Kramer bounds, and a computational investigation of whether they are sharp. In fact, Proposition 2.1 tells us more than just the bounds in Theorem 1.1. We explicitly spell out the stronger conclusions one can make for $2 \le d \le 6$.

Proposition 1.4. *Let A be a d-dimensional simple abelian variety with maximal RM of conductor N^d. Set $K = \mathrm{End}^0(A)$, which is a totally real number field of degree d.*

- *(i) Suppose $d=2$. If $2^9 \mid N$, then $K=\mathbb{Q}(\sqrt{2})$. If $5^3 \mid N$, then $K = \mathbb{Q}(\sqrt{5})$. In particular, $2^9 \cdot 5^3 \mid N$ is impossible.*
- *(ii) Suppose $d=3$. If $3^6 \mid N$, then $K=\mathbb{Q}(\zeta_9)^+$. If $7^3 \mid N$, then $K=\mathbb{Q}(\zeta_7)^+$. Hence $3^6 \cdot 7^3 \mid N$ is impossible.*
- *(iii) Suppose $d=4$. If $2^{11} \mid N$, then $K=\mathbb{Q}(\zeta_{16})^+$. If $2^9 \mid N$ (resp $5^3 \mid N$), then K contains $\mathbb{Q}(\sqrt{2})$ (resp. $\mathbb{Q}(\sqrt{5})$). Hence if $2^9 \cdot 5^3 \mid N$, then $K = \mathbb{Q}(\sqrt{2}, \sqrt{5})$. Further, N cannot be divisible by $2^{11} \cdot 5^3$.*
- *(iv) Suppose $d=5$. If $11^3 \mid N$, then $K=\mathbb{Q}(\zeta_{11})^+$.*
- *(v) Suppose $d=6$. If $2^9 \cdot 3^6$, $2^9 \cdot 7^3$, $3^6 \cdot 5^3$, $5^3 \cdot 7^3$, or 13^3 divides N, then $K=\mathbb{Q}(\sqrt{2}, \zeta_9 + \zeta_9^{-1})$, $\mathbb{Q}(\sqrt{2}, \zeta_7+\zeta_7^{-1})$, $\mathbb{Q}(\sqrt{5}, \zeta_9+\zeta_9^{-1})$, $\mathbb{Q}(\sqrt{5}, \zeta_7+\zeta_7^{-1})$, or $\mathbb{Q}(\zeta_{13})^+$, respectively. Further, N cannot be divisible by $2^9 \cdot 5^3$, $2^9 \cdot 13^3$, $3^6 \cdot 7^3$, $3^6 \cdot 13^3$, or $7^3 \cdot 13^3$.*

Corollary 1.5. *Let $C/\mathbb{Q}$ be a genus 2 curve of conductor N. Let A be the Jacobian of C. Suppose C has RM (i.e., A has RM). If N is divisible by 5^6 (resp. 2^{18}), then A is simple and $\mathrm{End}^0(A) \simeq \mathbb{Q}(\sqrt{5})$ (resp. $\mathrm{End}^0(A) \simeq \mathbb{Q}(\sqrt{2})$).*

Parts (i)–(v) of Proposition 1.4 also hold verbatim when $f \in S_{2k}(N)$ is a degree d newform with $K = K_f$. These statements are proved in Section 3.3.

These consequences were initially quite striking to us for the following reason. One can specify a finite number of local discrete series components (and thus local conductors) of automorphic representations in the trace formula, and asymptotically count such representations (e.g., see [Wei09]). Each of these fixed local components merely contribute independent local densities (specified by the Plancherel measure) to this asymptotic. By analogy, one might guess that, when counting weight 2 newforms of a fixed degree d with a finite number of local conductors specified, the asymptotic may be a product of local densities for each of these local conductors. Indeed, the work [SSW21] on counting elliptic curves suggests this is the case for $d = 1$.

In contrast, there exist degree 2 newforms (or, if one prefers, genus 2 curves with RM) whose level is divisible by 125, and also degree 2 newforms with level divisible by 512. A naive guess is that each of these sets make up some positive proportions, say δ_{125} and δ_{512}, of all degree 2 newforms, and that the proportion of all degree 2 newforms with level divisible by $125 \cdot 512$ should be $\delta_{125} \cdot \delta_{512} > 0$. But the corollary tells us there are no such forms! (In fact, [CM] suggests 100% of all degree 2 forms may have rationality field $\mathbb{Q}(\sqrt{5})$, so it may be that $\delta_{512} = 0$, but a local-global counting principle would still suggest there should be infinitely many such forms.) Hence Proposition 1.4 and Corollary 1.5 say that, for $d \geq 2$, local conductor behavior is not independent, at least in the case of "sufficiently wild" ramification.

2. Rationality subfields for modular forms

Proposition 2.1. *Let $f \in S_{2k}(N)$ be a newform and fix a prime p. Suppose $p^3 \mid N$ if p is odd and $p^9 \mid N$ if $p = 2$. Set $r = \lceil v_p(N)/2 - v_p(3)/2 \rceil - 1 - v_p(2)$. Then $\mathbb{Q}(\zeta_{p^r})^+ \subset K_f$. In particular, $\frac{1}{2}p^{r-1}(p-1) \mid [K_f : \mathbb{Q}]$.*

The lower bound of 3 or 9 on $v_p(N)$ in the hypothesis is the minimum needed to get a nontrivial conclusion, except when $p = 3$ where one needs $3^6 \mid N$.

This proposition refines earlier results of Saito and Brumer. Namely, [Sai80, Corollary 3.4] uses certain operators on $S_{2k}(N)$ to obtain a weaker version of this proposition with $r = \lfloor v_p(N)/3 \rfloor - v_p(2)$. In the case $2k = 2$, [Bru95, Theorem 5.5] obtains a version of this result in the context of abelian varieties of GL(2)-type with $r = \lceil v_p(N)/2 - 1 - \frac{1}{p-1} \rceil - v_p(2)$.[1] Brumer remarks that his approach also applies to higher weights. This is almost the same as our value of r, but is smaller by 1 when $p = 2$ and $v_2(N)$ is odd. E.g., when $p = 2$, Brumer's result only implies $\mathbb{Q}(\sqrt{2}) \subset K_f$ if $v_2(N) \geq 10$, but this proposition says $v_2(N) \geq 9$ suffices. So this proposition should be viewed as a slight sharpening of Brumer's result when $p = 2$.

We will prove this proposition by examining local Galois types of modular forms, following [DPT21]. In fact, the proof amounts to making some arguments in [DPT21] slightly more precise, and correcting an error therein. The basic idea is the same as Brumer's: show $\zeta_{p^r} + \zeta_{p^r}^{-1}$ appears as a trace of an appropriate representation. However, our proof uses an explicit description of possible representations, and we obtain slightly more information when $p = 2, 3$.

In fact neither Saito's nor Brumer's result, nor the argument we give, requires trivial nebentypus. We assume it for simplicity so that we can directly apply [DPT21].

Proof. Let π_p be the smooth irreducible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to f. Since we assumed $v_p(N) \geq 3$, either π_p is an irreducible principal series representation or a supercuspidal representation (i.e., twisted Steinberg is not possible). For a local representation σ, we write $c(\sigma)$ for its conductor.

First assume $\pi_p = \pi(\mu, \mu^{-1})$ is an irreducible principal series. Then $c(\pi_p) = 2n$, where $n = c(\mu)$. Now μ factors through $\mathbb{Z}_p^\times/(1+p^n\mathbb{Z}_p^\times) \simeq (\mathbb{Z}/p^n\mathbb{Z})^\times$, which is cyclic if p is odd and abstractly isomorphic to $C_2 \times C_{2^{n-2}}$ if $p = 2$. Since $\mu|_{\mathbb{Z}_p^\times}$ does not factor though $1+p^{n-1}\mathbb{Z}_p^\times$,

[1] There is a typo in the $p = 2$ case of [Bru95, Theorem 5.5(i)]—the field in the $p = 2$ case should read as in part (ii) of *loc. cit.*

necessarily the order of $\mu|_{\mathbb{Z}_p^\times}$ is a multiple of $p^{n-1-v_2(p)}$. Now [DPT21, Lemma 3.1] implies that $\zeta_{p^r} + \zeta_{p^r}^{-1} \in K_f$, where $r = n - 1 - v_2(p) = v_p(N)/2 - 1 - v_2(p)$. (This lemma is proved by looking at the images of the associated ℓ-adic Galois representations $\rho_{f,\lambda}$.) This proves the proposition in the principal series case.

Now assume π_p is supercuspidal. All the details we will need about supercuspidal representations are recalled or proved in [DPT21, Section 2], with a correction noted below. Necessarily, π_p is dihedral if p is odd. If $p = 2$, there are non-dihedral supercuspidal representations π, but they have conductor exponent $c(\pi) \le 8$ (in fact ≤ 7), in which case the proposition is vacuous. Thus for any p we may assume that π_p is dihedral. This means that $\pi = \pi_p$ corresponds to the induction $\mathrm{Ind}_{W_{\mathbb{Q}_p}}^{W_E} \theta$ of a regular character θ of the Weil group of some quadratic extension $E/\mathbb{Q}_p$. Write $n = c(\theta)$.

First suppose $E/\mathbb{Q}_p$ is unramified. Then $c(\pi) = 2n$. As explained in the proof of [DPT21, Lemma 2.11], the order of θ is a multiple of $p^{n-1-v_2(p)}$.

Next suppose $E/\mathbb{Q}_p$ is ramified with p odd. Then $c(\pi) = n+1$ and $n \ge 2$. *Loc. cit.* states that n must be even and the order of θ is a multiple of $p^{n/2}$. This is correct except in the special case that $E \simeq \mathbb{Q}_3(\sqrt{-3})$, where we can only say that the order of θ is a multiple of $3^{n/2-1}$. The difference in the exponent in this case arises from difference in the structure of $\mathfrak{o}_E/\mathfrak{p}_E^j$ when $E \simeq \mathbb{Q}_3(\sqrt{-3})$ due to the presence of extra roots of unity. The structure of $\mathfrak{o}_E/\mathfrak{p}_E^j$ is described in [DPT21, Theorem 2.10] and implies the above fact about the order of θ. (We remark that there is a typo in this theorem: a and b should be swapped in third line of data in [DPT21, Table 2], which corresponds to $E \ne \mathbb{Q}_3(\sqrt{-3})$ is ramified with $p \ne 2$, and gives the exponent $\frac{n}{2}$ in this case. Neither this typo nor the above correction affect the other proofs or results in [DPT21].)

Finally suppose $p = 2$ and $E/\mathbb{Q}_2$ is ramified. Then $c(\pi) = n + \delta$, where $\delta \in \{2, 3\}$ is the 2-adic valuation of the discriminant of E. As remarked above, when $p = 2$, we may assume $c(\pi) = n + \delta \ge 9$, in which case n is necessarily even. When $\delta = 2$, i.e., $E \simeq \mathbb{Q}_2(\sqrt{-1})$ or $\mathbb{Q}_2(\sqrt{3})$, then θ has order a multiple of $2^{n/2-1}$. If $\delta = 3$, i.e., $E \simeq \mathbb{Q}_2(\sqrt{\pm 2})$ or $\mathbb{Q}_2(\sqrt{\pm 6})$, then θ has order a multiple of $2^{n/2}$.

Thus in all dihedral supercuspidal cases of relevance, the order of θ is a multiple of p^r, where r is as in the statement of the proposition. Now [DPT21, Lemma 3.2] implies $\zeta_{p^r} + \zeta_{p^{-r}} \in K_f$, as desired. $\square$

Corollary 2.2. *Suppose $f \in S_{2k}(N)$ is a newform of degree d. Then $v_p(N) \le B_0(p, d)$, where $B_0(p, d)$ is given in (1.1).*

Proof. Suppose $p \ge 5$. Since $B_0(p, d) \ge 2$, we may assume $v_p(N) \ge 3$. Then Proposition 2.1 implies $\frac{1}{2}p^{r-1}(p-1) \mid d$, where $r = \lceil \frac{v_p(N)}{2} \rceil - 1 \ge 1$. Hence $\frac{p-1}{2} \mid d$ and $\lceil \frac{v_p(N)}{2} \rceil - 2 \le v_p(d)$. Thus $v_p(N) \le B_0(p, d)$.

The same reasoning gives the result for $p = 2$ and $p = 3$. $\square$

We point out one more consequence of the proof of Proposition 2.1. Given a newform $f \in S_{2k}(N)$, denote by $\pi_p(f)$ the local admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to f. An algorithm for determining $\pi_p(f)$ is given in [LW12]. The main difficulty is distinguishing supercuspidal representations, which [LW12] carries out using the cohomology of the modular curve. Simply knowing $v_p(N)$ places many restrictions on the possibilities for $\pi_p(f)$—in particular if $v_p(N) \ge 3$ is odd, $\pi_p(f)$ must be supercuspidal. The following gives an elementary way to partially distinguish supercuspidal components using K_f when $p = 3$.

Corollary 2.3. *Let $f \in S_{2k}(N)$ be a newform and suppose $3^{2m+1} \parallel N$ with $m \geq 1$. If $\mathbb{Q}(\zeta_{3^m})^+ \not\subset K_f$, then $\pi_3(f)$ is a supercuspidal representation dihedrally induced from $E = \mathbb{Q}_3(\sqrt{-3})$.*

Proof. Since $c(\pi_3(f)) = 2m+1 \geq 3$ is odd, $\pi_3(f)$ must be a dihedral supercuspidal induced from a character θ along a ramified quadratic extension $E/\mathbb{Q}_3$. Now the corollary follows from the difference in exponents for our lower bounds for the order of θ in the cases $E \not\simeq \mathbb{Q}_3(\sqrt{-3})$ and $E \simeq \mathbb{Q}_3(\sqrt{-3})$. □

For instance, up to Galois conjugacy, $S_2(243)$ has 2 rational newforms, 1 newform with rationality field $\mathbb{Q}(\sqrt{3})$, 1 newform with rationality field $\mathbb{Q}(\sqrt{6})$, and 2 newforms with rationality field $\mathbb{Q}(\zeta_9)^+$. The 4 Galois orbits of newforms with rationality fields of degree ≤ 2 must all have local components π_3 being dihedral supercuspidal representations induced from $\mathbb{Q}_3(\sqrt{-3})$.

3. Local conductor bounds for modular abelian varieties

In this section we will prove the results stated in the introduction.

Given a weight 2 newform f for $\Gamma_0(N)$ or $\Gamma_1(N)$, there is an associated simple abelian variety A_f which is a subquotient of $J_0(N)$ or $J_1(N)$. Moreover, $\dim A_f = [K_f : \mathbb{Q}]$, $\mathrm{End}^0(A_f) \simeq K_f$ and $L(s, A) = \prod L(s, f^\sigma)$, where f^σ runs over the Galois conjugates of f. We say an abelian variety $A/\mathbb{Q}$ corresponds to a weight 2 newform f if it is isogenous to A_f (over $\mathbb{Q}$), in which case we say A is modular. Recall that isogenies preserve endomorphism algebras.

Lemma 3.1. *Suppose $A/\mathbb{Q}$ is a simple d-dimensional abelian variety with maximal RM. Then A corresponds to a newform $f \in S_2(N)$, where N^d is the conductor of A.*

This lemma together with Corollary 2.2 implies part (1) of Theorem 1.1.

As we do not know a reference that explicitly concludes the newform f must have trivial nebentypus, we provide a proof.

Proof. Since Serre's conjecture is known, [Rib04, Theorem 4.4] implies that A is a simple factor of $J_1(N)$. Thus A is isogenous to an abelian variety A_f associated to a newform $f \in S_2(M, \varepsilon)$ for some $M \mid N$ and nebentypus ε. Comparing conductors shows $M = N$. Now the rationality field of f equals $K = \mathrm{End}_0(A)$, and thus contains the values of ε. Hence ε is quadratic. If $\varepsilon = 1$, then f is a newform in $S_2(N)$, as desired.

We will use several facts about CM forms, all of which can be found in [Rib77]. Suppose ε is nontrivial. Then f has CM by ε, i.e., $f \otimes \varepsilon = f$. We claim this is impossible because f has weight 2.

More generally, take a newform $f \in S_k(N, \varepsilon)$ of any weight $k \geq 2$, and assume f has CM by ε. Then f is induced from a Grossencharacter ψ of an imaginary quadratic field F, and f has CM by the quadratic Dirichlet character χ_F attached to f. Since the character of CM is unique, $\varepsilon = \chi_F$. This forces ε to be odd, as F is imaginary quadratic. But because ε is also the nebentypus character, we must have $\varepsilon(-1) = (-1)^k$. Hence k is odd, proving the claim. □

3.1. **Comparison with Brumer–Kramer bounds.** Suppose $A/\mathbb{Q}$ is a d-dimensional simple factor of the new part of $J_0(N)$. Thus A has conductor N^d. Put $e = v_p(N)$. Brumer and Kramer's conductor exponent bounds for abelian varieties [BK94, Theorem 6.2] states that

$$dv_p(N) \le B(p,d) := 2d + pt + (p-1)\lambda_p(t),$$

where $t = \lfloor 2d/(p-1) \rfloor$ and λ_p is defined by

$$\lambda_p(m) = \sum_{i=0}^{s} i c_i p^i, \quad \text{for } m = \sum_{i=0}^{s} c_i p^i, \quad 0 \le c_i < p.$$

Let us rewrite this as

$$v_p(N) \le B'(p,d) := 2 + \lfloor \frac{pt + (p-1)\lambda_p(t)}{d} \rfloor.$$

Thus $B'(p,d)$ can be thought of as the Brumer–Kramer bound for abelian varieties of GL(2)-type. We want to compare $B'(p,d)$ with $B_0(p,d)$.

Lemma 3.2. *First suppose $p \ge 5$. When $p \ge d$, $B'(p,d)$ is given by*

$$B'(p,d) = \begin{cases} 2 & p > 2d+1 \\ 4 & p = 2d+1 \text{ or } p = d \\ 3 & d+1 < p < 2d+1 \end{cases}$$

If $5 \le p \le d$, then $B'(p,d) \ge 3$.

When $p = 3$, we have $B'(3,1) = B'(3,2) = 5$ and $B'(3,d) \ge 6$ for $d \ge 3$.

When $p = 2$, we have $B'(2,1) = 8$, $B'(2,2) = 10$, $B'(2,3) = 9$ and $B'(2,d) \ge 9$ for $d \ge 4$.

For any p, if $(p-1) \mid 2d$, then $B'(p,d) \ge 4 + 2v_p(d) + 4v_2(p) + v_2(3)$, with equality when $\frac{2d}{p^{v_p(2d)}(p-1)} < p$.

Proof. Note that $\lambda_p(t) = 0$ if and only if $t < p$, i.e., $2d < p(p-1)$.

First suppose $p \ge 5$. Since $t = 0 \iff p > 2d+1$, we see $B'(p,d) = 2 = B_0(p,d)$ when $p > 2d+1$. We also note that if $p = 2d+1$, so $t = 1$, then $B'(p,d) = 4 = B_0(p,d)$. If $d+1 < p < 2d+1$, so $t = 1$, then $B'(p,d) = 3 > B_0(p,d) = 2$. When $p = d$, so $t = 2$, $B'(p,d) = 4 > B_0(p,d) = 2$.

Next suppose $5 \le p < d$. Then $t \ge 2$. Since $t + 1 > \frac{2d}{p-1}$, we have $(p-1)t + (p-1) > 2d$, and thus $pt > 2d + 1 + t - p > d + 1$. Hence $\frac{pt}{d} > 1$ and $B'(p,d) \ge 3$.

For $p = 3$, note that $t = d$ so $B'(p,d) = 5 + \lfloor \frac{2\lambda_3(d)}{d} \rfloor$. Since we always have $\lambda_p(m) \ge m - p + 1$, one gets $2\lambda_3(d) \ge 3$ for all $d \ge 3$. This gives the $p = 3$ statement.

When $p = 2$, we have $t = 2d$ and $B'(p,d) = 6 + \lfloor \frac{\lambda_2(2d)}{d} \rfloor$. One can compute $B'(2,d)$ case-by-case for $d \le 3$. If $s = \lfloor \log_2(d) \rfloor$, then $2^{s+1} > d$ so $\lambda_2(2d) \ge (s+1)2^{s+1} > (s+1)d \ge 3d$. This implies the $p = 2$ lower bound.

Lastly, consider any $p \ge 2$. Suppose $2d = ap^m(p-1)$ for some $m \ge 0$ and integer a coprime to p. Then $t = ap^m$ so $\frac{pt}{d} = 2 + \frac{2}{p-1}$. If $a < p$, then $\lambda_p(t) = map^m$, and thus $\frac{(p-1)\lambda_p(t)}{d} = 2m = 2(v_p(d) + v_2(p))$ If $a > p$, then $\lambda_p(t) > map^m$. This proves the assertion when $(p-1) \mid 2d$. □

The lower bounds are not intended to be optimal, but merely sufficient to conclude the following, which implies part (2) of Theorem 1.1.

Corollary 3.3. *We have $B_0(p,d) \ge B'(p,d)$.*

This is an equality if (i) $d \le 2$; (ii) $p \ge 2d+1$; or (iii) if $2d = ap^m(p-1)$ for some $m \ge 0$ and $a < p$.

We have a strict inequality $B_0(p,d) < B'(p,d)$ if (i) $\max\{5,d\} \le p < 2d+1$; (ii) $5 \le p < d$ and $(p-1) \nmid d$; or (iii) $p \in \{2,3\}$, $d > p$, and $p \nmid d$.

Proof. This follows from the previous lemma. The cases $d \le 2$ are more easily seen from examining Table 1. □

3.2. **Sharpness of bounds.** Here we consider when the bounds $v_p(N) \le B_0(p,d)$ of Theorem 1.1 are sharp. Note that once there exists, say, a weight 2 newform f of degree d, one can replace f with a quadratic twist to ensure $v_p(N) \ge 2$. Thus we will look for examples where $v_p(N) = B_0(p,d) > 2$ is attained for $d \le 10$. This will justify the bolded entries in Table 1.

First we searched the LMFDB [LMFDB] for weight 2 newforms with trivial nebentypus which attain such bounds. The LMFDB contains data on all weight 2 newforms of level up to 10000. This data shows the bounds are sharp when (i) $d \le 4$; (ii) $d = 5$ and $p \ne 11$; (iii) $d = 6$ and $p \ne 13$; (iv) $d = 7$ and $p \ne 2$; (v) $d = 8$ and $p \ne 2, 17$; (vi) $d = 9$ and $p \ne 2, 3, 19$; and (vii) $d = 10$ and $p \ne 2, 5, 11$.

Note that $11^4 = 14641$ is already outside of the LMFDB range, so the LMFDB data is necessarily insufficient to search for $v_p(N) = B_0(p,d)$ when $p = 2d+1 \ge 11$. Similarly, 2^{14}, 3^9 and 5^6 are also outside of the LMFDB range.

For the other cases in Table 1, we used Magma [Magma] to compute newform decompositions (i.e., degrees of all newforms) for $S_2(N)$ for various $N > 10000$. Andrew Sutherland has also done some such calculations, and at least some of our data is contained in his. In particular, we found the following, which justifies the remaining bolded entries in Table 1. We indicate approximate runtimes and RAM usage for each newform decomposition calculation in parentheses.

- $N = 12032 = 2^8 \cdot 47$ has a degree 7 newform (32s, 245MB)
- $N = 14592 = 2^8 \cdot 57$ has a degree 9 newform (4min, 1.3GB)
- $N = 11264 = 2^{10} \cdot 11$ has a degree 10 newform (52s, 325MB)
- $N = 16384 = 2^{14}$ has a degree 8 newform (65s, 444MB)
- $N = 19683 = 3^9$ has a degree 9 newform (4min, 763MB)
- $N = 14641 = 11^4$ has a degree 5 newform (9min, 1.5GB)

The remaining cases to check are when $d = 10$ and $p = 5, 11$, or when $d = 6, 8, 9$ and $p = 2d+1$. In these cases, we searched for newforms where $v_p(N) = B_0(p,d)$ or $v_p(N) = B_0(p,d) - 1$ is attained, to the extent we could with moderate computational resources. We did not find any levels outside of the LMFDB range where either of these bounds is attained, but we summarize our attempts. We remark that these computations tend to be more accessible for levels of the form $2^m p^r$ than for $q \cdot p^r$ where q is an odd prime of comparable size to 2^m.

For $p = 5$, we computed newform decompositions for levels $N = 5^6 \cdot M$ where $M = 2, 4, 8$, as well as $N = 5^5 \cdot M$ for $M \le 12$ and $M = 14, 16, 18$. The decomposition for $N = 125000 = 8 \cdot 5^6$ took over 7 hours and used 46 GB of RAM. None of these levels have degree $d = 10$ forms, though there are degree 20 forms in levels $18750 = 6 \cdot 5^5$, $28125 = 9 \cdot 5^5$, $50000 = 16 \cdot 5^5$ and $56250 = 18 \cdot 5^5$. There are degree 10 newforms in the LMFDB with level $N = 8750 = 14 \cdot 5^4$, so at least $v_5(N) = B_0(5,10) - 2$ is attained.

For $p = 11$, we computed newform decompositions for levels $N = 11^4 \cdot M$ for $M = 1, 2, 4, 8$. None of these levels have degree $d = 10$ forms. There are however many degree 10 forms in the LMFDB with $v_{11}(N) = 3$, e.g., there is one in level $1331 = 11^3$, so at least $v_{11}(N) = B_0(11,10) - 1$ is attained.

For $p = 13$, we computed newform decompositions for levels $N = 13^4 \cdot M$ with $M = 1, 4$, as we well as levels $N = 13^3 \cdot M$ for $M \leq 16$ and $M = 18, 20, 24, 32$. The decomposition for $N = 114244 = 4 \cdot 13^4$ took 11 hours and used 61 GB of RAM. None of these levels have degree 6 newforms, though there are degree 12 newforms in levels $35152 = 16 \cdot 13^3$, $39546 = 18 \cdot 13^3$ and $70304 = 32 \cdot 13^3$.

For $p = 17$, we computed newform decompositions for levels $N = 17^3 \cdot M$ for $M \leq 9$ and $M = 12, 16$. (For level $7 \cdot 17^3$, we had to use an alternate method of factoring Hecke polynomials.) None of these levels have degree 8 newforms, though there are degree 16 newforms in level $44217 = 9 \cdot 17^3$.

For $p = 19$, the LMFDB lists degree 9 newforms in level $6859 = 19^3$.

3.3. **Restrictions on RM fields.** Let A be simple abelian variety with maximal RM of conductor N^d, where $d = \dim A$. A further application of Proposition 2.1 (combined with Lemma 3.1) is that large powers of p dividing N force restrictions on the endomorphism algebra $K = \mathrm{End}^0(A)$, sometimes determining it completely. Moreover, different primes can interact in this phenomenon, creating "global" restrictions on local conductor exponents.

We can think of this as follows. Say $v_p(N)$ is large, and let r be as in Proposition 2.1. This forces $K = \mathrm{End}^0(A)$ to contain $\mathbb{Q}(\zeta_{p^r})^+$, and leaves only $d' = d/(\frac{1}{2}p^{r-1}(p-1))$ "degrees of freedom" inside K for other cyclotomic subfields. So if q is another prime dividing N, we have the stronger conductor bound $v_q(N) \leq B_0(q, d')$, since the cyclotomic fields $\mathbb{Q}(\zeta_{p^r})$ and $\mathbb{Q}(\zeta_{q^s})$ are disjoint for $q \neq p$.

Applying this reasoning case-by-case for $2 \leq d \leq 6$ proves Proposition 1.4.

The $d = 2$ case of this proposition immediately implies Corollary 1.5 once one knowns $A = \mathrm{Jac}(C)$ is simple. Indeed, if it were not, it would be a product of elliptic curves E_1 and E_2, and then N is the product of the conductors of E_1 and E_2. But by local conductor bounds for elliptic curves, this is impossible if $v_2(N) > 16$ or $v_5(N) > 4$.

3.4. **Abelian varieties of GL(2)-type.** Now suppose A is a d-dimensional simple abelian variety of GL(2)-type, not necessarily with maximal RM. Then A is isogenous to a newform $f \in S_2(N, \varepsilon)$. Suppose p, r are as in Proposition 2.1. When p is odd or $p = 2$ and $v_2(N)$ is even, [Bru95, Theorem 5.5] tells us that $\mathbb{Q}(\zeta_p^r)^+\mathbb{Q}(\varepsilon) \subset K_f$. If $p = 2$ and $v_2(N)$ is odd, Brumer's theorem says $\mathbb{Q}(\zeta_p^{r-1})^+\mathbb{Q}(\varepsilon) \subset K_f$. Thus using [Bru95, Theorem 5.5] in place of Proposition 2.1 in the proof of Theorem 1.1(1) yields the statement in Remark 1.2.

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