# MASS FORMULAS AND EISENSTEIN CONGRUENCES IN HIGHER RANK

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Abstract. We use mass formulas to construct minimal parabolic Eisenstein congruences for algebraic modular forms on reductive groups compact at infinity. For unitary groups of prime degree, this construction yields Eisenstein congruences for non-endoscopic cuspidal automorphic forms on quasi-split unitary groups.

In supplementary sections, we also generalize previous weight 2 Eisenstein congruences for Hilbert modular forms, and prove some special congruence mod $p$ results between cusp forms on $\mathrm{U}(p)$.

## 1. Introduction

In [Mar17], we gave a construction for mod $p$ congruences of weight 2 cusp forms with Eisenstein series on $\mathrm{PGL}(2)$ using the Eichler mass formula for a definite quaternion algebra and the Jacquet–Langlands correspondence. This approach has certain advantages over previous approaches to Eisenstein congruences for elliptic modular forms (e.g., [Maz77], [Yoo19]): one can treat more general levels and primes $p$, as well as Hilbert modular forms, without much difficulty.

In this paper, we extend this approach to groups of higher rank. For $\mathrm{GL}(2)$, there is no difference between congruences of Hecke eigenvalues and congruences of Fourier coefficients. In higher rank, the relation between Fourier coefficients and Hecke eigenvalues is more mysterious, so these two types of congruences are not known to be equivalent.

We will only address congruences of Hecke eigenvalues for Eisenstein series attached to a *minimal* parabolic subgroup. As Hecke eigenvalues determine $L$-functions, it seems plausible that these congruences are related to $L$-value congruences with products of $\mathrm{GL}(1)$ $L$-functions. Relatedly, Bergström and Dummigan [BD16] relate Hecke eigenvalue congruences for Eisenstein series attached to maximal parabolic subgroups with the Bloch–Kato conjecture.

Suppose $\pi, \pi'$ are irreducible automorphic representations of a reductive group $G$ over a number field $F$, and that outside of a finite set of places $S$ there is a hyperspecial maximal compact subgroup $K_v \subset G(F_v)$ such that $\pi_v$ and $\pi'_v$ are both $K_v$-spherical. Then we say $\pi$ and $\pi'$ are Hecke congruent mod $p$ (away from $S$) if there exists a prime $\mathfrak{p}$ above $p$ in a sufficiently large number field such that, for

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$v \notin S$, the spherical Hecke eigenvalues for $\pi_v^{K_v}$ and those for $(\pi'_v)^{K_v}$ are congruent mod $\mathfrak{p}$.

1.1. **Main results.** Our first main result is a general congruence result for algebraic modular forms.

Let $F$ be a totally real number field and $G/F$ be a reductive group which is compact at infinity. Gross [Gro99] defined a notion of algebraic modular forms on $G$. Let $K = \prod K_v$ be a suitably nice compact open subgroup of $G(\mathbb{A})$. In particular, we assume $K_v$ is a hyperspecial maximal compact subgroup almost everywhere and $K_v = G_v$ for $v \mid \infty$. Let $\mathcal{A}(G, K)$ denote the space of algebraic modular forms with level $K$ and trivial weight. We may view $\mathcal{A}(G, K)$ as the space of $\mathbb{C}$-valued functions on the finite set $\mathrm{Cl}(K) = G(F)\backslash G(\mathbb{A})/K$. Let $x_1, \dots, x_h \in G(\mathbb{A})$ be a set of representatives for $\mathrm{Cl}(K)$ and put $w_i = |G(F) \cap x_i K x_i^{-1}|$. On $\mathcal{A}(G, K)$, we consider the inner product $(\phi, \phi') = \sum \frac{1}{w_i}\phi(x_i)\overline{\phi'(x_i)}$. This space has a basis of orthogonal eigenforms for the unramified Hecke algebra. The constant function $\mathbb{1}$ is an eigenform, which we think of as a compact analogue of an Eisenstein series associated to the minimal parabolic of the quasi-split form. Let $\mathcal{A}_0(G, K)$ be the orthogonal complement on $\mathbb{1}$ in $\mathcal{A}(G, K)$. The mass of $K$ is defined to be

$$m(K) = (\mathbb{1}, \mathbb{1}) = \frac{1}{w_1} + \cdots + \frac{1}{w_h}.$$

We say two eigenforms are Hecke congruent mod $p$ if their automorphic representations are.

**Theorem A.** *(Theorem 2.1) If $p \mid m(K)$, then there exists an eigenform $\phi \in \mathcal{A}_0(G, K)$ which is Hecke congruent to $\mathbb{1}$ mod $p$.*

Explicit mass formulas have been computed in a wide variety of settings— e.g., see [Shi06] and [GHY01]. We will explicate a mass formula for unitary groups below, but the point is this gives a simple numerical criterion for the existence of certain congruences.

Often one focuses on automorphic forms on quasi-split groups. Suppose $G$ is as above, and $G'$ is a quasi-split inner form of $G$. Then by Langlands' conjectures, automorphic representations of $G$ should transfer to $G'$, and thus Theorem A should imply a Hecke congruence on $G'$.

Such a congruence on $G'$ can be regarded as an Eisenstein congruence as follows. Suppose $G'/F$ is semisimple with a Borel subgroup $B$. For a character $\chi$ of the Levi of $B$, consider the principal series representation $I(\chi)$ induced from $\chi$. Choosing standard sections of $I(\chi)$ yields Eisenstein series, which are not in general $L^2$. In particular, if $\chi = \delta_G'^{-1/2}$ where $\delta_{G'}$ denotes the modulus character, then $I(\chi)$ contains the trivial representation $\mathbb{1}_{G'}$ as a subrepresentation, and $\mathbb{1}_{G'}$ contributes to the residual part of the discrete $L^2$ spectrum. Note that one can reformulate the weight 2 Eisenstein series congruence for elliptic modular forms from [Maz77], [Mar17] as congruences with $\mathbb{1}_{G'}$ for $G' = \mathrm{PGL}(2)$.

One would like to know when such congruence are "new" in the following sense. If one has an endoscopic lifting from $\mathrm{PGL}(2)$ to $G'$, then an Eisenstein congruence on $G'$ may just arise as a lift of an Eisenstein congruence on $\mathrm{PGL}(2)$. Indeed, in some but not all examples we computed with $G = \mathrm{SO}(5)$ and $G = \mathrm{U}(3)$, Eisenstein congruences coming from Theorem A numerically appear to be Saito–Kurokawa lifts or Kudla lifts of Eisenstein congruences on $\mathrm{PGL}(2)$.

In Theorem 3.8, we show that this construction yields non-endoscopic Eisenstein congruences on quasi-split unitary groups of prime degree over totally real fields. The point of using unitary groups (rather than, e.g., orthogonal groups) is because they possess inner forms which are compact mod center at a finite place. The restriction to prime degree is because there is a simple cuspidality criterion in this case, but potentially this could be removed with a concrete understanding of the non-cuspidal spectrum on inner forms.

For concreteness, and to minimize notation in the introduction, here we merely state this congruence result when the unitary group is attached to the quadratic extension $E/F = \mathbb{Q}(i)/\mathbb{Q}$ and the automorphic representation is spherical outside of a single prime $\ell$.

**Theorem B*.** *(Example 3.11) Let $n = 2m+1$ be an odd prime, $\chi$ the idele class character for $\mathbb{Q}$ associated to the quadratic extension $E = \mathbb{Q}(i)$, and $G' = \mathrm{U}(n)$ the quasi-split unitary group associated to $E/\mathbb{Q}$. Let $\mathbb{1}_{G'}$ denote the trivial representation of $G$. Fix a prime $\ell \equiv 1 \bmod 4$. Suppose $p > n$ is a prime such that either $p \mid (\ell^r - 1)$ for some $1 \le r \le n-1$ or that $p$ divides the numerator of the product $\prod_{r=1}^{m} B_{2r} \cdot \prod_{r=1}^{m} B_{2r+1,\chi}$ of generalized Bernoulli numbers.*

*Then there exists a holomorphic weight $n$ cuspidal representation $\pi$ of $G'(\mathbb{A})$ such that (i) $\pi_v$ is unramified at each finite odd $v \neq \ell$, (ii) $\pi_2$ is spherical, (iii) $\pi_\ell$ is an unramified twist of the Steinberg representation, (iv) the base change $\pi_E$ of $\pi$ to $\mathrm{GL}(n, \mathbb{A}_E)$ is cuspidal, and (v) $\pi$ is Hecke congruent to $\mathbb{1}_{G'} \bmod p$.*

The asterisk in the theorem refers to an underlying assumption of the endoscopic classification for unitary groups when $n > 3$, to be discussed below. This is long known for $n = 3$.

Condition (iv) implies that $\pi$ is not an endoscopic lift from lower rank groups. Conditions (i)—(iii) tell us that $\pi$ has "level $\ell$" with respect to Iwahori subgroups. The difference between conditions (i) and (ii) is due to the fact that our unitary group is ramified at 2. Moreover the condition that $\ell \equiv 1 \mod 4$, i.e., $\ell$ is split in $E/\mathbb{Q}$, is needed to use an inner form $G$ of $G'$ which is locally compact mod center at $\ell$.

The divisibility hypotheses on $p$ imply that $p$ divides the mass of a suitable compact open subgroup. The $p > n$ condition is not needed in general, but here it ensures $p$ does not divide the denominator of any Bernoulli numbers appearing in the mass. As a specific example of the Bernoulli number divisibility condition, for any $\ell \equiv 1 \bmod 4$, we may take $p = 61$ if $n = 7$ or $p \in \{19, 61, 277, 691\}$ if $n = 13$. For unitary groups $\mathrm{U}(n)$ attached to more general CM extensions $E/F$ (still with

$n$ prime), there are additional divisibility conditions in terms of $E$ to guarantee one gets a non-endoscopic congruence.

The Hecke eigenvalues for $\mathbb{1}_G$ are relatively simple to describe, being simply the degrees of the corresponding Hecke operators. For instance, if $G = \mathrm{U}(2)$ or $\mathrm{U}(3)$, the local spherical Hecke algebra is generated by a single Hecke operator $T_q$. For $G = \mathrm{U}(2)$, the spherical eigenvalue of $\mathbb{1}_G$ for $T_q$ is $q + 1$ if $q$ is split in $E/\mathbb{Q}$ and $q^2 + q$ if $q$ is inert in $E/\mathbb{Q}$. For $G = \mathrm{U}(3)$, the spherical eigenvalue of $\mathbb{1}_G$ for $T_q$ is $q^2 + q + 1$ if $q$ is split in $E/\mathbb{Q}$ and $q^4 + q$ if $q$ is inert in $E/\mathbb{Q}$. In general, there are many local Hecke operators at $q$.

To our knowledge, these are the first general non-endoscopic Eisenstein congruence results in higher rank for Eisenstein series attached to minimal parabolic subgroups.

We also use the ideas in the proofs to obtain two other congruence results. In Section 4, we refine our earlier results on weight 2 Eisenstein congruences for $\mathrm{GL}(2)$ from [Mar17]. In Section 5, we show that if $\pi$ is a cuspidal representation of $\mathrm{U}(p)$ with trivial central character such that $\pi_v$ is an unramified twist of Steinberg at some finite $v$, there exists a cuspidal $\pi'$ on $\mathrm{U}(p)$ with the same level structure as $\pi$ which is Hecke congruent to $\pi$ mod $p$ and $\pi'_v$ is Steinberg at $v$. This is not about Eisenstein congruences, but is a higher rank analogue of a mod 2 congruence result on $\mathrm{GL}(2)$ from [Mar17].

1.2. **Method of proof.** The proof of Theorem A is a straightforward generalization of the proof of Eisenstein congruences on definite quaternion algebras from [Mar17]. This essentially boils down to some linear algebra over rings.

Theorem A then yields Eisenstein congruences on definite unitary groups. To derive Theorem B*, we work with an inner form $G$ of $\mathrm{U}(n)$ which is compact at infinity and compact mod center at $\ell$, i.e., $G$ is unitary group over a division algebra. By comparing the endoscopic classification of discrete $L^2$ automorphic representations of $G$ with those of the quasi-split form $\mathrm{U}(n)$, one gets a transfer of automorphic representations of $G$ to those of $\mathrm{U}(n)$. Since $G$ is compact mod center at $\ell$ and $n$ is prime, if $\pi$ is a non-abelian (not 1-dimensional) automorphic representation of $G$, the transfer to $G'$ must be non-endoscopic and have cuspidal base change to $\mathrm{GL}(n, \mathbb{A}_E)$. When $E = \mathbb{Q}(i)$, there are no abelian automorphic representations occurring in $\mathcal{A}_0(G, K)$, which gives Theorem B*. For definite unitary groups associated to a general CM extensions $E/F$, one gets an Eisenstein congruence with a non-abelian $\phi$ provided that $p$ divides the numerator of $\frac{m(K)}{n|\mathrm{Cl}(\mathrm{U}_{E/F}(1)|}$. See Theorem 3.8 for a precise statement.

The endoscopic classification results that we use were obtained (conditional on stabilization of trace formulas) in [Mok15] for $\mathrm{U}(n)$ and were announced in [KMSW] for inner forms. However, the proof for the case of inner forms, while known in many situations, is still work in progress, and we assume this classification in Theorem B*. For $n = 3$, the endoscopic classification was completed for all inner forms in [Rog90], and thus our results are unconditional at least for $n = 3$.

**Notation.** Throughout, $F$ will denote a number field, $\mathfrak{o} = \mathfrak{o}_F$ its ring of integers, $\mathbb{A} = \mathbb{A}_F$ its adele ring, and $v$ a place of $F$. We also denote the finite adeles by $\mathbb{A}_f$ and put $\hat{\mathfrak{o}} = \prod_{v<\infty} \mathfrak{o}_v$. At a finite place $v$, we denote by $\mathfrak{p}_v$ the prime ideal and $q_v$ the size of the residue field.

For a group $G$, we denote its center by $Z(G)$, or just by $Z$ if $G$ is understood. For an algebraic group $G$ over $F$, we often write $G_v$ for $G(F_v)$. By an automorphic representation, by default we mean an irreducible $L^2$-discrete automorphic representation.

Finally $p$ will typically denote our congruence prime. To denote other primes, we generally use $v$ to denote other primes, or $\ell$ or $q$ when $F = \mathbb{Q}$. If $\alpha \in \mathbb{Q}$, by $p \mid \alpha$, we mean that $p$ divides the numerator of $\alpha$.

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## 2. Congruences from mass formulas

Let $F$ be a totally real number field. Let $G$ be a connected reductive linear algebraic group over $F$ such that $G_\infty$ is compact. Let $K = \prod K_v$ be an open compact subgroup of $G(\mathbb{A})$ such that $K_v = G_v$ for $v \mid \infty$. For later use of the theory of Hecke operators, we will also assume $K_v$ is a hyperspecial maximal compact subgroup for all $v$ outside of a finite set of places $S$, which contains all infinite places.

Fix a nonzero Haar measure $dg$ on $G(\mathbb{A})$ which is a product of local Haar measures $dg_v$. The mass of $K$ is defined to be

$$m(K) = \frac{\mathrm{vol}(G(F)\backslash G(\mathbb{A}), dg)}{\mathrm{vol}(K, dg)}. \tag{2.1}$$

(As usual, we give the discrete subgroup $G(F)$ the counting measure and the volume of the quotient $G(F)\backslash G(\mathbb{A})$ really means with respect to the quotient measure.) This is nonzero, finite, and independent of the choice of $dg$. Note that if $K' \subset K$ is also a compact open subgroup, then $m(K') = [K : K']m(K)$.

Consider the classes $\mathrm{Cl}(K) = G(F)\backslash G(\mathbb{A})/K$. We identify $\mathrm{Cl}(K)$ with a set of representatives $\{x_1, \ldots, x_h\}$, where $x_i \in G(\mathbb{A})$. Note $\mathrm{vol}(G(F)\backslash G(F)x_iK, dg) =$

$\frac{1}{w_i}\operatorname{vol}(K, dg)$ where $w_i = |G(F) \cap x_i K x_i^{-1}|$. Thus we can also express the mass as

$$m(K) = \frac{1}{w_1} + \cdots + \frac{1}{w_h}. \tag{2.2}$$

Consequently, $m(K) \in \mathbb{Q}$.

If $G$ is a unitary, symplectic or orthogonal group, and $K$ is the stabilizer of a lattice $\Lambda$, then this mass corresponds to the classical mass of $\Lambda$. Mass formulas have been calculated in a considerable amount of generality in many works, e.g. see [GHY01] or [Shi06]. We will explicate these in some cases below.

2.1. **Algebraic modular forms.** The basic theory of algebraic modular forms was developed in [Gro99]. Below, we will review aspects necessary for our applications.

We define the space of algebraic modular forms on $G(\mathbb{A})$ with level $K$ and trivial weight to be

$$\mathcal{A}(G, K) = \{\phi : \operatorname{Cl}(K) \to \mathbb{C}\}. \tag{2.3}$$

As $\mathcal{A}(G, K) \subset L^2(G(F)\backslash G(\mathbb{A}))$, we can decompose this space as

$$\mathcal{A}(G, K) = \bigoplus \pi^K, \tag{2.4}$$

where $\pi$ runs over irreducible automorphic representations of $G(\mathbb{A})$ with trivial infinity type. If $\pi^K \neq 0$, we will say $\pi$ occurs in $\mathcal{A}(G, K)$. Since $G(F)\backslash G(\mathbb{A})$ is compact, $L^2(G(F)\backslash G(\mathbb{A}))$ decomposes discretely and each $\pi$ above is finite dimensional. The usual inner product on $L^2(G(F)\backslash G(\mathbb{A}))$ restricts to an inner product on $\mathcal{A}(G, K)$, which after suitable normalization we can take to be

$$(\phi, \phi') = \sum \frac{1}{w_i}\phi(x_i)\overline{\phi'(x_i)}.$$

Let $Z$ denote the center of $G$ and $K_Z = K \cap Z(\mathbb{A})$. Note that $\operatorname{Cl}(K_Z) = Z(F)\backslash Z(\mathbb{A})/K_Z$ acts on elements of $\mathcal{A}(G, K)$ by (left or right) multiplication. Let $\omega : \operatorname{Cl}(K_Z) \to \mathbb{C}$ be a "class character".[1] Define the space of algebraic modular forms with central character $\omega$, level $K$ and trivial weight to be

$$\mathcal{A}(G, K; \omega) = \{\phi \in \mathcal{A}(G, K) : \phi(zg) = \omega(z)\phi(g) \text{ for } z \in Z(\mathbb{A}),\ g \in G(\mathbb{A})\}.$$

By decomposing $\mathcal{A}(G, K)$ with respect to the action of $\operatorname{Cl}(K_Z)$, we obtain a decomposition

$$\mathcal{A}(G, K) = \bigoplus_{\omega} \mathcal{A}(G, K; \omega),$$

where $\omega$ runs over characters of $\operatorname{Cl}(K_Z)$. We also have decompositions of the form (2.4) for each $\mathcal{A}(G, K; \omega)$, where now one runs over $\pi$ with central character $\omega$.

If $\chi : G(\mathbb{A}) \to \mathbb{C}$ is a 1-dimensional representation and $\ker\chi \supset G(F)K$, then we may view $\chi$ as an element of $\mathcal{A}(G, K)$. In particular, the space for trivial

[1] If we relax our compact at infinity condition to compact mod center at infinity, and suppose $Z = \mathrm{GL}(1)$ and $K_Z = \hat{\mathfrak{o}}_F^\times \times F_\infty^\times$, this is just an ideal class character of $F$.

representation is the span of the constant function $\mathbb{1} \in \mathcal{A}(G,K)$. Define the codimension 1 subspace

$$\mathcal{A}_0(G,K) = \{\phi \in \mathcal{A}(G,K) : (\phi, \mathbb{1}) = 0\},$$

and put $\mathcal{A}_0(G,K;\omega) = \mathcal{A}(G,K,\omega) \cap \mathcal{A}_0(G,K)$. We say $\phi \in \mathcal{A}(G,K)$ is non-abelian if it is not a linear combination of 1-dimensional representations of $G(\mathbb{A})$.

In the special case $G = B^\times$, where $B$ is a definite quaternion algebra over $F = \mathbb{Q}$ and $K$ is the multiplicative group of an Eichler order of level $N$, then $\mathbb{1} \in \mathcal{A}(G,K)$ corresponds to a weight 2 Eisenstein series on GL(2) and the Jacquet–Langlands correspondence gives a Hecke isomorphism of $\mathcal{A}_0(G,K)$ with the subspace of $S_2(N)$ which are $p$-new for $p$ ramified in $B$. However, in general $\mathcal{A}_0(G,K)$ may contain many abelian forms, as well as many non-abelian forms $\phi$ (even in $\pi^K$ for some $\pi$ occurring in $\mathcal{A}_0(G,K)$) which do not correspond to cusp forms via a generalized Jacquet–Langlands correspondence. For higher rank $G$, it is a difficult problem to describe the set of $\phi$ which correspond to cusp forms on the quasi-split form of $G$.

Finally, for a subring $\mathcal{O}$ of $\mathbb{C}$, and a space of algebraic modular forms (e.g., $\mathcal{A}(G,K)$) we denote with a superscript $\mathcal{O}$ (e.g., $\mathcal{A}^{\mathcal{O}}(G,K)$) the subring of $\mathcal{O}$-valued algebraic modular forms.

2.2. **Hecke operators.** For $g \in G(\mathbb{A})$ and $\phi \in \mathcal{A}(G,K)$, we define the Hecke operator

$$(T_g\phi)(x) = \sum \phi(xg_i), \quad KgK = \coprod g_iK. \tag{2.5}$$

By right $K$-invariance of $\phi$, this is independent of the choice of representatives $g_i$ in the coset decomposition $KgK = \coprod g_iK$, and $T_{g'} = T_g$ if $g' \in KgK$. Clearly each $\pi^K$ is stable under $T_g$ for each $\pi$ occurring in $\mathcal{A}(G,K)$. In particular, $T_g$ acts on the subspaces $\mathcal{A}_0(G,K)$ and $\mathcal{A}_0(G,K,\omega)$.

We also note that each $T_g$ is integral in the sense that, viewing each $\phi$ as a column vector $(\phi(x_i)) \in \mathbb{C}^h$, the action is given by left multiplication by an integral matrix in $M_h(\mathbb{Z})$. Consequently, for any subring $\mathcal{O} \subset \mathbb{C}$, $T_g$ restricts to an operator on $\mathcal{A}^{\mathcal{O}}(G,K)$ (and similarly, $\mathcal{A}_0^{\mathcal{O}}(G,K)$, etc.). Moreover, all eigenvalues for $T_g$ are algebraic integers.

Consider a representation $\pi = \bigotimes' \pi_v$ occurring in $\mathcal{A}(G,K)$. Take $v \notin S$. Then $\pi_v$ is $K_v$-spherical, and $\dim \pi_v^{K_v} = 1$. Viewing $g_v \in G(F_v)$ as an element of $G(\mathbb{A})$ which is $g_v$ at $v$ and 1 at all other places, we can consider the (global) Hecke operator $T_{g_v}$. Then $T_{g_v}$ acts by a scalar on $\pi_v^{K_v}$, and hence is diagonalizable on $\mathcal{A}(G,K)$.

In fact, the $T_{g_v}$'s are simultaneously diagonalizable for all $v \notin S$ and all $g_v \in G(F_v)$. Specifically, let us call any nonzero $\phi \in \mathcal{A}(G,K)$ such that $\phi \in \pi^K$ for some $\pi$ an eigenform. Such a $\phi$ is a simultaneous eigenform for all $T_{g_v}$'s with $v \notin S$. We denote the corresponding eigenvalue by $\lambda_{g_v}(\phi)$. Then any basis of $\mathcal{A}(G,K)$ of eigenforms simultaneously diagonalizes the $T_{g_v}$'s ($v \notin S$).

Note that $\mathbb{1}$ is always an eigenform, and $\lambda_{g_v}(\mathbb{1})$ is the degree of $T_{g_v}$, i.e., the number of $g_i$'s occurring in the decomposition $Kg_vK = \coprod g_iK$, which equals $\mathrm{vol}(K_vg_vK_v)/\mathrm{vol}(K_v)$.

2.3. **Congruences.** Let $\phi, \phi' \in \mathcal{A}(G, K)$ be eigenforms. We say $\phi$ and $\phi'$ are Hecke congruent mod $p$ (away from $S$) if, for all $v \notin S$ and all $g_v \in G(F_v)$, $\lambda_{g_v}(\phi) \equiv \lambda_{g_v}(\phi') \bmod \mathfrak{p}$, where $\mathfrak{p} \mid p$ is a prime of some finite extension of $\mathbb{Q}$.

For a subring $\mathcal{O} \subset \mathbb{C}$, ideal $\mathfrak{n}$ in $\mathcal{O}$ and $\phi_1, \phi_2 \in \mathcal{A}^{\mathcal{O}}(G, K)$, we write $\phi_1 \equiv \phi_2 \bmod \mathfrak{n}$ if $\phi_1(x_i) \equiv \phi_2(x_i) \bmod \mathfrak{p}$ for all $x_i \in \mathrm{Cl}(K)$. Note if $\phi_1$ and $\phi_2$ are eigenforms which are nonzero mod $\mathfrak{p}$, then $\phi_1 \equiv \phi_2 \bmod \mathfrak{p}$ implies $\phi_1$ and $\phi_2$ are Hecke congruent mod $\mathfrak{p}$.

**Theorem 2.1.** *Suppose $p \mid m(K)$. Then there exists an eigenform $\phi \in \mathcal{A}_0(G, K)$ which is Hecke congruent to $\mathbb{1} \bmod p$.*

*Proof.* One can use the same arguments as those given for GL(2) in [Mar17] and [Mar18b]. In fact we give a slightly more refined argument than what we need for this proposition in order to use it later in Section 3.3.

Let $r = v_p(m(K)) \geq 1$. The first step is to note that there exists a $\mathbb{Z}$-valued $\phi' \in \mathcal{A}_0^{\mathbb{Z}}(G, K)$ such that $\phi' \equiv \mathbb{1} \bmod p^r$, i.e., $\phi'(x_i) \equiv 1 \bmod p^r$ for $i = 1, \dots, h$. To see this, consider $\phi' \in \mathcal{A}^{\mathbb{Z}}(G, K)$ such that each $\phi'(x_i) = 1 + p^r a_i$ for some $a_i \in \mathbb{Z}$. We claim we can choose the $a_i$'s so that $(\phi', \mathbb{1}) = 0$, i.e., $p^r \sum \frac{a_i}{w_i} = -\sum \frac{1}{w_i} = -m(K)$. Let $w = \prod w_i$ and $w_i^* = \frac{w}{w_i}$. Then we want $a_i \in \mathbb{Z}$ such that $\sum a_i w_i^* = -w\frac{m(K)}{p^r}$. Note that $p^j \mid w_i^*$ for some $i$ implies $p^j \mid w$ and thus $p^{j+r} \mid wm(K)$. Thus $\gcd(w_1^*, \dots, w_h^*) \mid w\frac{m(K)}{p^r}$, and we may choose the $a_i$'s as claimed.

Take such a $\phi'$, which is a mod $p$ eigenform—i.e., for each $v \notin S$ and $g_v \in G(F_v)$ there exists a $\lambda$ such that $T_{g_v}\phi' \equiv \lambda\phi' \bmod p$. Now we want to pass from $\phi'$ to an eigenform $\phi$ which is Hecke congruent to $\phi'$ mod $p$. For this, one can either use the Deligne–Serre lifting lemma as in the proof of [Mar18b, Theorem 5.1] or the reduction argument as in proof of [Mar17, Theorem 2.1]. Specifically, the subsequent Lemma 2.2 is a slight refinement of the latter, and applying it with $\mathcal{O} = \mathbb{Z}$, $\phi_1 = \mathbb{1}$, $\phi_2 = \phi'$ and $W = \mathcal{A}_0(G, K)$ gives the desired $\phi$. ☐

**Lemma 2.2.** *Let $\mathcal{O}$ be the ring of integers of a number field $L$, and $\mathfrak{p}$ a prime of $\mathcal{O}$ above a rational prime $p$. Let $\phi_1 \in \mathcal{A}^{\mathcal{O}}(G, K)$ be an eigenform. Let $W$ be a Hecke-stable subspace of $\mathcal{A}(G, K)$. Suppose there exists $\phi_2 \in \mathcal{A}^{\mathcal{O}}(G, K)$ such that $\phi_2 \equiv \phi_1 \bmod \mathfrak{p}$ and $\phi_2$ has nonzero orthogonal projection to $W$. Then there exists an eigenform $\phi \in W$ such that $\phi$ is Hecke congruent to $\phi_1 \bmod p$ for all Hecke operators $T_g$.*

*Proof.* Enlarge $L$ if necessary to assume that $\mathcal{A}^{\mathcal{O}}(G, K)$ contains a basis of eigenforms $\psi_1, \dots, \psi_h$. Let $\Phi$ denote the collection of $\phi \in \mathcal{A}^{\mathcal{O}}(G, K)$ such that $\phi$ is congruent to a nonzero multiple of $\phi_1 \bmod \mathfrak{p}$ and $\phi$ has nonzero orthogonal projection to $W$. The hypothesis on $\phi_2$ means $\Phi \neq \emptyset$. Let $m$ be minimal such that, after a possible reordering of $\psi_1, \dots, \psi_h$, there exists $\phi = c_1\psi_1 + \cdots + c_m\psi_m \in \Phi$ with each $c_i \in L^\times$ and $\psi_1 \in W$. Take such a $\phi$.

Fix any Hecke operator $T_g$, and put $\phi' = [T_g - \lambda_g(\psi_1)]\phi$. Then note that

$$\phi' \equiv (\lambda_g(\phi_1) - \lambda_g(\psi_1))\phi \mod \mathfrak{p}.$$

Hence $\phi' \in \Phi$ unless $\lambda_g(\psi_1) \equiv \lambda_g(\phi_1) \bmod \mathfrak{p}$. But $\phi'$ is of the form $c_2'\psi_2 + \cdots + c_m'\psi_m$ for some $c_i' \in L^\times$. Thus $\phi' \notin \Phi$ by minimality of $m$. Consequently, $\lambda_g(\psi_1) \equiv \lambda_g(\phi_1) \bmod \mathfrak{p}$ for all $g$, and we may take $\psi_1$ for our desired $\phi$. $\square$

**Remark 2.3.** Let $\mathfrak{p}$ be the prime above $p$ in a sufficiently large extension of $\mathbb{Q}_p$, with ramification index $e$. The work [BKK14] considers the notion of *depth* of congruences, which is $\frac{1}{e}$ times the number of Hecke eigensystems satisfying a congruence mod $\mathfrak{p}$ counted with multiplicity (a congruence mod $\mathfrak{p}^r$ means multiplicity $r$). Combining this theorem with Proposition 4.3 of *op. cit.* gives a lower bound on the depth of congruences of $v_p(m(K))$.

We can also guarantee the existence of such a $\phi$ with trivial central character.

**Corollary 2.4.** *Set $\bar{G} = G/Z$. Suppose that $\bar{G}(k) = G(k)/Z(k)$ holds for any field $k$ of characteristic zero, and $p \mid \frac{m(K)}{m(K_Z)}$. Then there exists an eigenform $\phi \in \mathcal{A}_0(G, K; 1)$ which is Hecke congruent to $\mathbb{1} \bmod p$.*

*Proof.* Let $\bar{K} = Z(\mathbb{A})K/Z(\mathbb{A})$. Then $\mathcal{A}_0(G, K; 1)$ may be identified with $\mathcal{A}_0(\bar{G}, \bar{K})$. Now note that $m(\bar{K}) = \frac{m(K)}{m(K_Z)}$, and apply the proposition to $\mathcal{A}(\bar{G}, \bar{K})$. $\square$

The assumption for $\bar{G}$ in Corollary 2.4 is satisfied when $G$ is a unitary group of odd degree.

## 3. Eisenstein congruences for unitary groups

Let $E/F$ be a CM extension of number fields, and $G' = \mathrm{U}(n)$ be the associated quasi-split unitary group over $F$ in $n$ variables. Explicitly, if $\Phi$ is an $n \times n$ matrix with alternating $\pm 1$'s on the anti-diagonal and zeros elsewhere, then we may represent $G' = \{g \in \mathrm{GL}(n, E) : {}^t\bar{g}\Phi g = \Phi\}$. Here bar denotes the Galois automorphism of $E/F$ (in this case applied coordinate-wise to $g$.

Let $G$ be an inner form of $G'$. We can realize $G$ as follows. There exist (i) a central simple algebra $A/E$ of degree $n$, i.e., $\dim_E A = n^2$, and (ii) an involution $\alpha \mapsto \alpha^*$ of $A$ of the second kind with $\alpha^* = \bar{\alpha}$ for $\alpha \in E$, such that

$$G = \{g \in A^\times : g^*g = 1\}.$$

We remark that $G$ is the automorphism group of the Hermitian form $\langle \alpha, \beta \rangle = \alpha^*\beta$ on $A$. The center of $G$ is $E^\times \cap G$ (viewing $E^\times$ as the algebraic group $\mathrm{Res}_{E/F}\mathbb{G}_m$), which we may identify with $\mathrm{U}(1) = E^1 = \{a \in E^\times : a\bar{a} = 1\}$.

To specify $A$ and/or $*$ below, we will also denote $G = \mathrm{U}_A(n) = \mathrm{U}_{A,*}(n)$. (The isomorphism class depends on both $A$ and $*$, but as we will typically only be concerned about specifying $A$ we often just write $\mathrm{U}_A(n)$.) Landherr's theorem on the classification of involutions of the second kind tells us that if $v$ is inert or ramified in $E/F$, then $A_v$ is split. Moreover, if $v$ splits in $E/F$ as $v = ww'$, then $*$ interchanges the factors of $A_w$ and $A_{w'}$, giving an isomorphism $A_w \simeq A_{w'}^{\mathrm{opp}}$ and $G_v = \mathrm{U}_A(n, F_v) \simeq A_w^\times \simeq A_{w'}^\times$.

We will now assume $G = \mathrm{U}_{A,*}(n)$ is a definite unitary group, i.e., the associated Hermitian form is totally definite. This means $G_v$ is compact for all $v \mid \infty$. Note that one can make a definite involution on $A$ from any involution by conjugation (see [Sch85, Remark 10.6.11]).

Let det denote the reduced norm on $A$. By restriction to $G = \mathrm{U}_A(n)$, we may view det as a homomorphism of algebraic groups $\det : G \to \mathrm{U}(1)$. The derived subgroup $\mathrm{SU}_A(n)$ of $G$ is the kernel of det, so any 1-dimensional automorphic representation of $G(\mathbb{A})$ factors through det.

**Lemma 3.1.** *The map* $\det : G(k) \to \mathrm{U}(1, k)$ *is a surjective map of rational points for any localization* $k = F_v$ *as well as for* $k = F$.

*Proof.* By the Hasse principle for the norm map of unitary groups ([PR94, Theorem 6.28]), the result for $F$ follows from the result for each $F_v$. If $v$ is split in $E_v/F_v$, the local result follows from surjectivity of reduced norm for central simple algebras over $p$-adic fields. Otherwise, $G(F_v)$ is an honest unitary group, and it is clear det restricted to the diagonal torus surjects onto $\mathrm{U}(1, k)$. ☐

3.1. **Endoscopic classification.** Here we briefly explain certain aspects of the endoscopic classification for unitary groups as asserted in [KMSW, Theorem* 1.7.1], and refer the reader to *op. cit.* and [Mok15] for more precise details.

The endoscopic classification was treated by Rogawski [Rog90] for U(3) and its inner forms (as well as quasi-split U(2)), by Mok [Mok15] for quasi-split $\mathrm{U}(n)$, and by Kaletha–Minguez–Shin–White [KMSW] for inner forms of $\mathrm{U}(n)$ under some hypotheses. (See [Mok15, Section 2.6] for a summary of some intermediary results.) These latter results rely on the stabilization of the twisted trace formula which was established in [MW17], and also require the general weighted fundamental lemma which is expected to be finished by Chaudouard and Laumon. Work in progress of Kaletha–Minguez–Shin is expected to complete the proof of [KMSW, Theorem* 1.7.1], and we will assume this in our subsequent congruence results.

In fact the cases that we need are in some sense easier than cases already established in the literature (e.g., [HT01], [Lab11], [Shi11], [Mok15]), as the only non-quasi-split forms we consider are certain compact forms, where the trace formula analysis is simpler and one does not have endoscopic contributions. However, to our knowledge the cases we use (definite unitary groups over division algebras) have not been explicitly dealt with in the literature.

To describe the classification, in this section we let $G$ be an arbitrary inner form of $G' = \mathrm{U}(n)$. In particular, we allow $G = G'$.

As in [Mok15], the set of formal global parameters for $G'$ is the set $\Psi(G')$ consisting of formal sums (up to equivalence) $\psi = \psi_1 \boxplus \cdots \boxplus \psi_m$ of formal tensors

$\psi_i = \mu_i \boxtimes \nu_i$, where $\mu_i$ is a cuspidal automorphic representation of $\mathrm{GL}(n_i, \mathbb{A}_E)$ and $\nu_i$ is the $r_i$-dimensional irreducible representation of $\mathrm{SU}(2)$, such that $\sum n_i r_i = n$ and the parameter $\psi$ is conjugate self-dual. If $m = 1$, we call $\psi$ simple. If each $\nu_i = 1$, we call $\psi$ generic. Set $\dim \psi_i = n_i r_i$.

According to the Moeglin–Waldspurger classification, $\mu_i \boxtimes \nu_i$ corresponds to a discrete automorphic representation $\sigma_{\psi_i}$ of $\mathrm{GL}(n_i r_i, \mathbb{A}_E)$, which is cuspidal if $r_i = 1$. Thus by Langlands theory of Eisenstein series, $\psi$ corresponds to an automorphic representation $\sigma_\psi$ of $\mathrm{GL}_n(\mathbb{A}_E)$. Let $\Psi_2(G')$ denote the subset of square-integrable parameters, which are of the form $\psi = \psi_1 \boxplus \cdots \boxplus \psi_m$ where the $\psi_i$'s are all distinct and each $\psi_i$ is conjugate self-dual. Let $\Psi_2(G', \mathrm{std})$ be the subset of $\Psi_2(G')$ which "factor through" the standard $L$-embedding $\mathrm{std} : {}^L G' \to {}^L \mathrm{Res}_{E/F}(G')$ (this set is denoted $\Psi_2(G', \xi_1)$ in [Mok15, Definition 2.4.5]).

Let $\psi = \psi_1 \boxplus \cdots \boxplus \psi_m \in \Psi_2(G')$. One associates to $\psi$ a component group $\mathcal{S}_\psi \simeq (\mathbb{Z}/2\mathbb{Z})^{m'}$ (denoted $\bar{\mathcal{S}}_\psi$ in [KMSW]), and a canonical sign character $\epsilon_\psi$ of $\mathcal{S}_\psi$. Here $0 \leq m' \leq m$—see [Mok15, (2.4.14)] for a precise description of $m'$. We note $\epsilon_\psi = 1$ if $\psi$ is generic. Then there is a global packet $\Pi_\psi(G) = \Pi_\psi(G, \xi, \epsilon_\psi)$ of representations attached to an inner twist $(G, \xi)$ that is a certain subset of a restricted product of local packets consisting of elements which are globally compatible with $\epsilon_\psi$. (Here $\xi$ is an $\bar{F}$-isomorphism from $G$ to $G'$ exhibiting $G$ as an inner form of $G'$.) The role of $\epsilon_\psi$ is to give a parity condition for a product of members of local packets to lie in the global packet.

The packet $\Pi_\psi(G)$ is necessarily empty if $\psi$ is not locally relevant everywhere for $G$. Specifically, if $v$ is split in $E/F$, and $G(F_v) \simeq \mathrm{GL}(r_v, D_v)$ where $D_v$ is a central $F_v$-division algebra of degree $d_v$, then for $\psi = \psi_1 \boxplus \cdots \boxplus \psi_m$ to be relevant it is necessary that $d_v \mid \dim \psi_i$ for each $i$.

For $\psi \in \Psi_2(G', \mathrm{std})$ and $\pi \in \Pi_\psi(G)$, we call the associated automorphic representation $\pi_E := \sigma_\psi$ of $\mathrm{GL}(n, \mathbb{A}_E)$ the (standard) base change of $\pi$. Note that $\pi_E$ is cuspidal if and only if $\psi = \pi_E \boxtimes 1$, i.e., if and only if $\psi$ is simple generic. If $\pi_E$ is cuspidal and $v = ww'$ is a split place for $E/F$, then $\pi_v \simeq \pi_{E,w}$ when $A_v$ is split, and more generally $\pi_v$ corresponds to $\pi_{E,w}$ via the Jacquet–Langlands correspondence for $\mathrm{GL}(n)/E$.

Then the $\kappa = 1$ and $\chi_\kappa = 1$ case of [KMSW, Theorem* 1.7.1] states that we have a $G(\mathbb{A})$-module isomorphism:

$$L^2_{\mathrm{disc}}(G(F) \backslash G(\mathbb{A})) \simeq \bigoplus_{\psi \in \Psi_2(G', \mathrm{std})} \bigoplus_{\pi \in \Pi_\psi(G)} \pi. \tag{EC-U}$$

A consequence of this is a generalized Jacquet–Langlands correspondence for unitary groups. Namely, fix an inner form $G$ of $G'$, so $G(F_v) \simeq G'(F_v)$ for almost all $v$. For simplicity, assume $\psi \in \Psi_2(G', \mathrm{std})$ is simple generic, so we may view $\psi$ as a conjugate self-dual cuspidal representation of $\mathrm{GL}_n(\mathbb{A}_E)$. Then the packet $\Pi_\psi(G')$ is non-empty—in fact it contains a cuspidal generic representation of $G'$

[Mok15, Corollary 9.2.4]. If $\pi \in \Pi_\psi(G)$ we write $\mathrm{JL}(\pi) = \Pi_\psi(G')$ for the Jacquet–Langlands correspondent to the packet $\Pi_\psi(G)$. For $v$ split in $E/F$ and $\pi' \in \mathrm{JL}(\pi)$, $\pi_v$ and $\pi'_v$ correspond via the local Jacquet–Langlands correspondence for $\mathrm{GL}_n(F_v)$, and necessarily $\pi'_v \simeq \pi_v$ if $G(F_v) \simeq G'(F_v) \simeq \mathrm{GL}_n(F_v)$.

It is expected that generic packets are tempered. If $\psi$ is cohomological, then Shin [Shi11] (together with [CH13] when $n$ is even and $\psi_\infty$ is not Shin-regular) guarantees that $\psi_v$ is tempered at all finite $v$. Now let us also assume $\psi$ is cohomological.

For $\pi' \in \Pi_\psi(G')$, the local packets $\Pi_{\psi_v}$ for $\pi$ and $\pi'$ are the same at almost all places. But, by definition, elements of the global packets correspond locally to the trivial character of the component group (and thus unramified local parameters $\psi_v$) almost everywhere. Since $\psi_v$ is generic and bounded (tempered), the local packet $\Pi_{\psi_v}(G'(F_v))$ is in bijection with the dual of the component group at nonarchimedean $v$ ([Mok15, Theorem 2.5.1(b)]). Consequently, $\pi_v \simeq \pi'_v$ for almost all $v$.

In fact we can say more. Since $\psi$ is simple generic, we have $|\mathcal{S}_\psi| = 1$ ([Mok15, (2.4.14)]). This means there is no parity condition associated to $\epsilon_\psi$ required for a product $\pi' = \otimes \pi'_v$ of local components of packets to lie in the global packet $\Pi_\psi(G')$. Hence, given $\pi$, we may always choose $\pi' \in \mathrm{JL}(\pi)$ such that $\pi'_v \simeq \pi_v$ whenever $G(F_v) \simeq G'(F_v)$. Moreover, at all other $v$, we can choose $\pi'_v$ freely within the local packet $\Pi_{\psi_v}(G'(F_v))$.

3.2. **A cuspidality criterion for base change.** For the remainder of this section, we return to our assumption that $G = \mathrm{U}_{A,*}(n)$ is a definite unitary group.

**Proposition 3.2.** *Assume* (EC-U). *Suppose $n$ is prime and $A_w$ is a division algebra for some finite prime $w$ of $E$. If $\pi$ occurs in $\mathcal{A}(G,K)$, and $\pi$ is not 1-dimensional, then $\pi_E$ is cuspidal.*

*Proof.* Necessarily, there is a finite prime $v$ of $F$ which splits as $v = ww'$ for some $w'$. Then $G(F_v) \simeq A_w^\times$ is the multiplicative group of a degree $n$ division algebra. Let $\psi \in \Psi_2(G', \mathrm{std})$ be the parameter associated to $\pi$. Then for $\psi$ to be relevant, we need $\psi$ to be simple, i.e., $\psi = \mu \boxtimes \nu$ for some cuspidal automorphic representation $\mu$ of $\mathrm{GL}_m(\mathbb{A}_E)$ and $\nu$ of dimension $r = \frac{n}{m}$.

Since $n$ is prime, either $m = 1$ or $m = n$. If $m = n$, we are done. Otherwise, the proposition follows from the following lemma, which was kindly explained to us by Sug Woo Shin. □

**Lemma 3.3.** *Assume* (EC-U). *Suppose $\pi$ is an automorphic representation of $G$ associated to a simple parameter $\psi = \mu \boxtimes \nu$ where $\mu$ is a representation of $\mathrm{GL}(1, \mathbb{A}_E)$. Then $\pi$ is 1-dimensional.*

*Proof.* Suppose $v = ww'$ is split in $E$. The local base change $\pi_{E,w}$ is a 1-dimensional representation of $\mathrm{GL}_n(E_w)$. Then $\pi_v \simeq \pi_{E,w}$, so $\pi_v$ is 1-dimensional. Since the strong approximation property with respect to $v$ is satisfied by $G^1 = \{g \in G :$

$\det g = 1\}$ (see [PR94, Theorem 7.12]), $\pi_v$ trivial on $G^1(F_v)$ implies $\pi$ is trivial on $G^1(\mathbb{A})$. Thus $\pi$ is 1-dimensional. □

3.3. **Eisenstein congruences.** Let $K = \prod K_v \subset G(\mathbb{A})$ be a compact open subgroup which is hyperspecial and maximal at almost all $v$. We assume that $K_v = G_v$ for $v \mid \infty$, and place the following assumptions on $K_v$ for $v < \infty$.

First suppose $v$ splits in $E/F$. Then we can write $G_v = \mathrm{GL}_{r_v}(D_v)$ for some division algebra $D_v$ of degree $d_v$ with $d_v r_v = n$. Let $\mathcal{O}_v$ be an order of $D_v$ containing the unramified field extension of $F_v$ of degree $d_v$ (e.g., $\mathcal{O}_v$ is the maximal order in $D_v$). We assume the diagonal subgroup $(\mathcal{O}_v^\times)^{r_v} \subset K_v$. This holds, for instance, when $K_v$ is the stabilizer of a lattice of the form $\mathcal{I}_1 \oplus \cdots \oplus \mathcal{I}_{r_v} \subset D_v^{r_v}$ where each $\mathcal{I}_i$ is left $\mathcal{O}_v$-ideal on $D_v$.

Next suppose $v$ is ramified or inert in $E/F$, so $A_v$ is split. Assume $G_v$ has a maximal torus $T_v \simeq (E_v^\times)^r \times (E_v^1)^s$ for some $r, s$ with $2r + s = n$, such that the integral points of $T_v$ are contained in $K_v$, i.e., $(\mathfrak{o}_{E_v}^\times)^r \times (E_v^1)^s \subset K_v$. This holds for instance if $K_v$ is the stabilizer of a lattice of the form $\mathcal{I}_1 \oplus \cdots \oplus \mathcal{I}_n \subset E_v^n$ where each $\mathcal{I}_i$ is a $\mathfrak{o}_{E_v}$-ideal (in which case $s = n$).

The above assumptions guarantee that for all $v < \infty$, (i) $K_v \cap Z(G_v) = \mathrm{U}(1, \mathfrak{o}_v) = \{a \in \mathfrak{o}_{E,v}^\times : a\bar{a} = 1\}$, and (ii) $\det K_v = \mathrm{U}(1, \mathfrak{o}_v)$. Note for $v < \infty$, if $E_v/F_v$ is a field then $\mathrm{U}(1, \mathfrak{o}_v) = \mathrm{U}(1, F_v) = E_v^1$, whereas if $E_v/F_v$ is split then $\mathrm{U}(1, \mathfrak{o}_v) \simeq \mathfrak{o}_v^\times$.

Consequently, if $\pi$ occurs in $\mathcal{A}(G, K, \omega)$, then $\omega$ is a character of $\mathrm{U}(1, \mathbb{A})$ which is invariant under $\mathrm{U}(1, F)$ and $K \cap Z(\mathbb{A}) = \mathrm{U}(1, \hat{\mathfrak{o}})\,\mathrm{U}(1, F_\infty)$. Thus the relevant central characters for us will be characters $\omega$ of the class group $\mathrm{Cl}(\mathrm{U}(1)) = \mathrm{U}(1, F) \backslash \mathrm{U}(1, \mathbb{A}_f) / \mathrm{U}(1, \hat{\mathfrak{o}})$.

Any 1-dimensional representation $\pi$ occurring in $\mathcal{A}(G, K)$ is of the form $\pi = \chi \circ \det$, where $\chi$ is a character of $\mathrm{U}(1, \mathbb{A})$. From Lemma 3.1 and our assumptions on $K$, we in fact see that $\chi$ must be a character of $\mathrm{Cl}(\mathrm{U}(1))$.

We can apply Theorem 2.1 or Corollary 2.4 to construct congruences on $\mathcal{A}(G, K)$. However, since $\mathcal{A}(G, K)$ admits many 1-dimensional representations in general, even with trivial central character, we need more to guarantee we get congruences with non-abelian forms.

3.3.1. *Congruence modules.* Fix a finite abelian group $H$ and let $L$ be a number field which contains all character values for $H$. Let $X(R)$ be the set of $R$-valued class functions for $R = \mathbb{Z}$ or $R = L$. Endow $X(L)$ with the usual inner product $(\cdot, \cdot)$. Decompose $X(L) = X_{\mathbb{1}}(L) \oplus X_0(L)$ where $\mathbb{1}$ is the trivial character of $H$ and $X_{\mathbb{1}}(L) = L\mathbb{1}$. Let $X_{\mathbb{1}}(\mathbb{Z}) = X_{\mathbb{1}}(L) \cap X(\mathbb{Z}) = \mathbb{Z}\mathbb{1}$ and $X_0(\mathbb{Z}) = X_0(L) \cap X(\mathbb{Z})$. Also, let $X^{\mathbb{1}}(\mathbb{Z})$ (resp. $X^0(\mathbb{Z})$) be the image of the orthogonal projection $X(\mathbb{Z}) \to X_{\mathbb{1}}(L)$ (resp. $X(\mathbb{Z}) \to X_0(L)$). Then $X_{\mathbb{1}}(\mathbb{Z}) \oplus X_0(\mathbb{Z}) \subset X(\mathbb{Z}) \subset X^{\mathbb{1}}(\mathbb{Z}) \oplus X^0(\mathbb{Z})$. We consider the congruence module $C_0(H) = X(\mathbb{Z})/(X_{\mathbb{1}}(\mathbb{Z}) \oplus X_0(\mathbb{Z}))$. (See [Gha02] for an introduction to congruence modules.) One readily sees that the projection $X(L) \to X_{\mathbb{1}}(L)$ induces an isomorphism $C_0(H) \simeq X(\mathbb{Z})/(X_{\mathbb{1}}(\mathbb{Z}) \oplus X_0(\mathbb{Z})) \simeq X^{\mathbb{1}}(\mathbb{Z})/X_{\mathbb{1}}(\mathbb{Z})$. One similarly has an isomorphism with $X^0(\mathbb{Z})/X_0(\mathbb{Z})$.

**Lemma 3.4.** *For a positive integer $n$, there exists $\phi \in X_0(\mathbb{Z})$ such that $\phi \equiv \mathbb{1} \mod n$ if and only if $C_0(H)$ contains an element of order $n$.*

*Proof.* First note if $\phi \in X_0(\mathbb{Z})$ such that $\phi \equiv \mathbb{1} \mod n$, then the projection of $\frac{1}{n}(\phi - \mathbb{1})$ to $X_{\mathbb{1}}(L)$ is the element $-\frac{1}{n}\mathbb{1} \in X^{\mathbb{1}}(\mathbb{Z})$, and thus gives an element of order $n$ in $C_0(H)$. Conversely, suppose $\psi \in X(\mathbb{Z})$ is an element of order $n$ in $C_0(H)$. Then we can write $\psi = \frac{a}{n}\mathbb{1} - \frac{1}{n}\phi$ where $a \in \mathbb{Z}$ and $\phi \in X_0(\mathbb{Z})$. Since projection gives the isomorphism $C_0(H) \simeq X^{\mathbb{1}}(\mathbb{Z})/\mathbb{Z}\mathbb{1}$, $\frac{a}{n}$ has order $n$ mod $\mathbb{Z}$. Thus after scaling $\psi$ (and correspondingly $\phi$) we may assume $a \equiv 1 \mod n$. Then $\phi \equiv \mathbb{1} \mod n$. $\square$

**Lemma 3.5.** *As $\mathbb{Z}$-modules, $C_0(H) \simeq H$.*

*Proof.* First suppose that $H = H_1 \times H_2$. For $i = 1, 2$, write $X(R; H_i)$, $X_0(R; H_i)$, etc. for the corresponding objects for the group $H_i$. It is not hard to see that $X(\mathbb{Z}) = X(\mathbb{Z}; H) = \{\phi_1 \otimes \phi_2 : \phi_i \in X(\mathbb{Z}; H_i)\}$. Thus we may identify $X(\mathbb{Z}; H) = X(\mathbb{Z}; H_1) \oplus X(\mathbb{Z}; H_2)$. This identifies the $\mathbb{Z}$-submodule $X_{\mathbb{1}}(\mathbb{Z}; H) \oplus X_0(\mathbb{Z}; H)$ with $X_{\mathbb{1}}(\mathbb{Z}; H_1) \oplus X_0(\mathbb{Z}; H_1) \oplus X_{\mathbb{1}}(\mathbb{Z}; H_2) \oplus X_0(\mathbb{Z}; H_2)$. Hence $C_0(H) \simeq C_0(H_1) \oplus C_0(H_2)$. This reduces the proof to the case that $H = \langle g \rangle$ is cyclic of order $n$, which we assume now.

If $\chi_1, \dots, \chi_n$ are the irreducible characters of $H$, then $\frac{1}{n}(\chi_1 + \cdots + \chi_n) \in X(\mathbb{Z})$. Hence $\frac{1}{n}\mathbb{1} \in X^{\mathbb{1}}(\mathbb{Z})$. Conversely, suppose $\frac{1}{m}\mathbb{1} \in X^{\mathbb{1}}(\mathbb{Z})$. Then there exists $\phi \in X_0(\mathbb{Z})$ such that $\phi \equiv \mathbb{1} \mod m$. Let $a_j = \phi(g^j)$ for $1 \leq j \leq n$. Then $n \cdot (\chi, \mathbb{1}) = \sum a_j = 0$ but also $\sum a_j \equiv n \mod m$, hence $m \mid n$. Therefore $C_0(H) \simeq X^1(\mathbb{Z})/\mathbb{Z}\mathbb{1} \simeq H$. $\square$

The relevant consequence for us is the following. Let $e(H)$ denote the exponent of a finite group $H$: if $p^r \nmid e(H)$ then there is no congruence mod $p^r$ between the trivial character of $H$ and any $\mathbb{Z}$-valued linear combination of the non-trivial characters of $H$.

**Proposition 3.6.** *Let $h_E^1 = |\mathrm{Cl}(\mathrm{U}(1))|$ and $e_E^1$ be the exponent of $\mathrm{Cl}(\mathrm{U}(1))$. Suppose $p \mid \frac{m(K)}{\gcd(n, e_E^1) h_E^1}$ and $n$ is odd. Then there is a non-abelian eigenform $\phi \in \mathcal{A}(G, K, 1)$ such that $\phi$ is Hecke congruent to $\mathbb{1}$ mod $p$.*

**Remark 3.7.** When $F = \mathbb{Q}$, $h_E^1 = 2^{-t} h_E$, where $h_E$ is the class number of $E$ and $t$ is the number of primes of $\mathbb{Q}$ ramified in $E$. See [Shi97, Section 24.5] for the general case.

*Proof.* By our assumptions on $K$, we have $K_Z = \mathrm{U}(1, \hat{\mathfrak{o}}) U(1, F_\infty)$, so $m(K_Z) = |\mathrm{Cl}(\mathrm{U}(1))|$. Thus Corollary 2.4 says there exists an eigenform $\phi \in \mathcal{A}_0(G, K, 1)$ which is Hecke congruent to $\mathbb{1}$ mod $p$. We want to show we can take $\phi$ to be non-abelian.

Let $\bar{G} = G/Z$ and $\bar{K} = Z(\mathbb{A})K/Z(\mathbb{A})$. Note the abelian elements of $\mathcal{A}(G, K, 1) = \mathcal{A}(\bar{G}, \bar{K})$ are generated by the characters $\chi \circ \det$ where $\chi$ is a character of $\mathrm{Cl}(\mathrm{U}(1))$ of order dividing $n$. We may view such $\chi$ as factoring through the largest quotient $H$ of $\mathrm{Cl}(\mathrm{U}(1))$ of exponent dividing $n$.

Recall that the existence of such a $\phi$ arose from an integral element $\phi' \in \mathcal{A}_0^{\mathbb{Z}}(\bar{G}, \bar{K})$ such that $\phi' \equiv \mathbb{1} \bmod p^r$ where $r = v_p(m(\bar{K})) = v_p(m(K)) - v_p(h_E^1)$. For a suitable rationality field $L$, decompose $\mathcal{A}_0^L(\bar{G}, \bar{K}) = X_1(L) \oplus X_2(L)$ where $X_1(L)$ consists of the abelian forms orthogonal to $\mathbb{1}$ and $X_2(L)$ is spanned by the non-abelian eigenforms.

We claim $\phi' \notin X_1(L)$. Since $\det : G(\mathbb{A}) \to \mathrm{U}(1, \mathbb{A})$ is surjective, our assumptions on $K$ imply that det induces a surjective map $\Delta : \mathrm{Cl}(\bar{K}) \to H$. Thus if we had $\phi' \in X_1(L)$, composing it with $\Delta$ gives $\mathbb{Z}$-valued class function $\psi$ on $H$ such that $\psi \equiv \mathbb{1} \bmod p^r$. But this is impossible by the above lemmas as $v_p(\frac{p^r}{\gcd(n, e_E^1)}) > 0$ implies $p^r$ does not divide the exponent of $H$.

Hence $\phi'$ has nonzero projection to $X_2(L)$. Therefore applying the lifting lemma, Lemma 2.2, with $W = X_2(L)$, we obtain an eigenform $\phi \in X_2(L)$ which is Hecke congruent to $\mathbb{1} \bmod p$. $\square$

3.3.2. *Non-endoscopic congruences.* We now define the notion of congruences on the quasi-split form $G'$. For convenience, we talk about congruences of representations. Suppose $K' = \prod K'_v$ is an open compact subgroup of $G'$ which is hyperspecial at all $v \notin S$, and $\pi$ and $\pi'$ are automorphic representations of $G'(\mathbb{A})$ which are $K'_v$-unramified at all $v \notin S$. For $\alpha_v \in G'_v$, we let $\lambda_{\alpha_v}(\pi)$ be the eigenvalue of the local Hecke operator $K'_v \alpha_v K'_v$ on $\pi_v^{K'_v}$. We say $\pi$ and $\pi'$ are Hecke congruent (away from $S$) mod $p$ if $\lambda_{\alpha_v}(\pi) \equiv \lambda_{\alpha_v}(\pi') \bmod \mathfrak{p}$ for some prime $\mathfrak{p}$ of $\bar{\mathbb{Q}}$ above $p$ and all $v \notin S$, $\alpha_v \in G'_v$.

Consider the simple parameter $\psi_0 = 1 \boxtimes \nu(n) \in \Psi_2(G', \mathrm{std})$, where $\nu(n)$ is the irreducible $n$-dimensional representation of $\mathrm{SU}(2)$. This is the parameter of the trivial representations $\mathbb{1}_G$ and $\mathbb{1}_{G'}$ of $G$ and $G'$. The base change of $\mathbb{1}_{G'}$ to $\mathrm{GL}_n(\mathbb{A}_E)$ is the residual contribution of the Eisenstein series induced from the $\delta_{\mathrm{GL}(n)}^{-1/2}$ of the Borel.

**Theorem 3.8.** *Suppose $n$ is an odd prime and assume (EC-U) for $n$. Let $A/E$ be a degree $n$ central simple algebra which is division at a non-empty set $\mathrm{Ram}_0(A)$ of finite places of $F$ which split in $E/F$, and let $S_0 \subset \mathrm{Ram}_0(A)$. Consider a definite unitary group $G = \mathrm{U}_A(n)$ over $A$ as above. Let $K = \prod K_v \subset G(\mathbb{A})$ be a compact open subgroup satisfying the assumptions at the beginning of this section, and also assume that $K_v = G^1(F_v)$ for $v \in S_0$.*

*Suppose that $p \mid \frac{m(K)}{\gcd(n, e_E^1) h_E^1}$. Then there exists a cuspidal automorphic representation $\pi$ of $G'(\mathbb{A})$ with trivial central character such that: (i) the base change $\pi_E$ is cuspidal, (ii) $\pi_{v_0}$ is an unramified twist of Steinberg for $v_0 \in S_0$; (iii) $\pi_v$ has a nonzero $K_v$-fixed vector when $G(F_v) \simeq G'(F_v)$; (iv) $\pi_v$ is a holomorphic weight $n$ discrete series for $v \mid \infty$; and (v) $\pi$ is Hecke congruent to $\mathbb{1}_{G'}$ mod $p$.*

Note that by the classification of central simple algebras over number fields and Landherr's theorem, given $E/F$ and any non-empty finite set $\Sigma$ of finite places of $F$ split in $E/F$, there exists $G = \mathrm{U}_A(n)$ as in Theorem 3.8 with $\mathrm{Ram}_0(A) = \Sigma$.

We make a few remarks on such a $\pi$ as in the theorem. First, it cannot arise as an endoscopic lift from smaller unitary groups, so this congruence is "native" to $\mathrm{U}(n)$. Second, by the central character condition, (ii) means $\pi_{v_0}$ is a twist of Steinberg by an unramified character of order dividing $n$. Also, (iii) implies $\pi_v$ will be unramified whenever $K_v$ is hyperspecial. Moreover, if every finite place $v \notin S_0$ satisfies $G(F_v) \simeq G(F'_v)$, and if $K_v$ is good special maximal compact subgroup at all of these places, then we have strong control over $\pi$ at all places: (iv) describes $\pi_\infty$ completely; (ii) says $\pi_v$ is an unramified twist of Steinberg for $v \in S_0$, and (iii) says $\pi_v$ is $K_v$-spherical at all remaining $v$.

*Proof.* First Proposition 3.6 tells us there exists a non-abelian eigenform $\phi \in \mathcal{A}(G, K, 1)$ which is Hecke congruent to $\mathbb{1} \bmod p$. Let $\sigma$ be the associated automorphic representation of $G(\mathbb{A})$. By Proposition 3.2, we know $\sigma_E$ is cuspidal. We may take $\pi \in \mathrm{JL}(\sigma)$ such that (iv) holds and $\pi_v \simeq \sigma_v$ when $G(F_v) \simeq G'(F_v)$. For $v \in S_0$, since $K_{v_0} = G^1_{v_0}$ we must have that $\sigma_{v_0} = \chi_{v_0} \circ \det$, where $\chi_{v_0}$ is an unramified character of $F^\times_{v_0}$, so the local Jacquet–Langlands correspondent $\pi_{v_0}$ is Steinberg twisted by $\chi_{v_0}$. Finally, $\pi$ satisfies (v) because $\mathbb{1}_{G'}$ has the same Hecke eigenvalues as $\mathbb{1} \in \mathcal{A}(G, K, 1)$ at almost all places. ☐

We now describe $m(K)$ for nice maximal compact subgroups $K$ using [GHY01]. For simplicity we restrict to odd $n$. If desired, one can obtain masses for smaller compact subgroups $K' \subset K$ by recalling that $m(K') = [K : K']m(K)$. Let $\chi_{E/F}$ be the quadratic idele class character of $F$ associated to $E/F$.

**Proposition 3.9.** *Let $G = \mathrm{U}_A(n)$ be a definite unitary group over $A$ where $n$ is odd. Let $\mathrm{Ram}_f(E)$ (resp. $\mathrm{Ram}_f(A)$) denote the set of finite primes of $F$ above which $E$ (resp. $A$) is ramified. Assume $A_w$ is division for each $w$ above $v \in \mathrm{Ram}_f(A)$. Let $S = \mathrm{Ram}_f(E) \cup \mathrm{Ram}_f(A)$. Take $K = \prod K_v$ such that $K_v$ is maximal hyperspecial for finite $v \notin S$, $K_v = G^1(F_v)$ for $v \in \mathrm{Ram}_f(A)$, $K_v$ is the stabilizer of a maximal lattice for $v \in \mathrm{Ram}_f(E)$, and $K_v = G(F_v)$ for $v \mid \infty$. Then*

$$m(K) = 2^{1-nd-|\mathrm{Ram}_f(E)|} \times \prod_{r=1}^{n} L(1-r, \chi^r_{E/F}) \times \prod_{v \in \mathrm{Ram}_f(A)} \left( \prod_{r=1}^{n-1} (q_v^r - 1) \right), \tag{3.1}$$

*where $d = [F : \mathbb{Q}]$.*

*Proof.* A general mass formula is given in [GHY01, Proposition 2.13], which is explicated for definite odd unitary groups over fields in Proposition 4.4 of *op. cit.* From those calculations, it follows that

$$m(K) = 2^{1-nd} \times \prod_{r=1}^{n} L(1-r, \chi^r_{E/F}) \times \prod_{v \in S} \lambda_v,$$

where $\lambda_v$ is as follows. For a finite place $v$, let $\underline{H}'_v$ be Gross's canonical integral model of $H_v := G'_v$. Let $\underline{G}_v$ be the smooth integral model associated to a parahoric such that $K_v = \underline{G}_v(\mathfrak{o}_v)$. By our hypotheses, $S$ is the set of finite places such that

$\underline{G}_v \not\simeq \underline{H}_v^0$. Let $\bar{G}_v$ and $\bar{H}_v^0$ be the maximal reductive quotients of the special fibers of $\underline{G}_v$ and $\underline{H}_v^0$, which are reductive groups over $k_v = \mathfrak{o}_v/\mathfrak{p}_v$, with $\bar{G}_v$ possibly being disconnected. Then for $v \in S$,

$$\lambda_v = \frac{q_v^{-N(\bar{H}_v^0)}|\bar{H}_v^0(k_v)|}{q_v^{-N(\bar{G}_v)}|\bar{G}_v(k_v)|},$$

where $N(\cdot)$ denotes the number of positive roots over $\bar{k}_v$. When $G_v$ is quasi-split, *loc. cit.* tells us $\lambda_v = \frac{1}{2}$ if $E_v/F_v$ is ramified.

So we need only to compute $\lambda_v$ for $v \in \mathrm{Ram}_f(A)$. In this case $v$ splits in $E/F$ so $\bar{H}_v^0 \simeq \mathrm{GL}(n, k_v)$. Let $\mathcal{O}_v = A_v$ and $\mathfrak{P}_v$ the prime ideal of $\mathcal{O}_v$. Then $G_v \simeq \mathcal{O}_v^\times/(1+\mathfrak{P}_v) \simeq \mathbb{F}_{q_v^n}^\times$, which gives $\lambda_v = q_v^{-n(n-1)/2}\prod_{r=1}^{n-1}(q_v^n - q_v^r) = \prod_{r=1}^{n-1}(q_v^r - 1)$. $\square$

**Remark 3.10.** By [GHY01], we can extend the formula (3.1) to include finite places $v$ such that $G_v$ is quasi-split and $K_v$ is a special but not hyperspecial maximal compact. Each such place will contribute a factor of $\lambda_v = \frac{q^n+1}{q+1}$ to $m(K)$.

Consequently, Theorem 3.8 gives non-endoscopic Eisenstein congruences mod $p$ which are Steinberg at $v$ whenever $p$ is a sufficiently large (depending on $n$ and $E/F$) prime dividing some $q_v^r - 1$ (for $1 \le r \le n-1$).

**Example 3.11.** *Suppose $F = \mathbb{Q}$, $E = \mathbb{Q}(i)$. Then $|\mathrm{Cl}(\mathrm{U}(1))| = 1$. Let $A/E$ be a central division algebra of odd prime degree $n = 2m+1$ which is ramified only at the primes of $E$ above a fixed rational prime $\ell \equiv 1 \bmod 4$ (so necessarily division at $w \mid \ell$). Write $\chi = \chi_{E/F}$. It is well known that $L(1-r, \chi^r) = -\frac{1}{r}B_{r,\chi^r}$ (generalized Bernoulli number). Thus taking $G$ and $K$ as in Proposition 3.9, we get*

$$m(K) = \frac{1}{2^n n!}\prod_{r=1}^{m} B_{2r} \times \prod_{r=1}^{m} B_{2r+1,\chi} \times \prod_{r=1}^{n-1}(\ell^r - 1).$$

*Suppose $p > n$ is a prime dividing some $\ell^r - 1$ where $1 \le r \le n-1$. Since $p > n$, the von Staudt–Clausen theorem tells us that $p$ does not divide the denominators of any of the Bernoulli numbers $B_2, B_4, \ldots, B_{2m}$. Also $B_{1,\chi}, B_{3,\chi}, \ldots, B_{n,\chi}$ all have denominator 2. Hence Theorem 3.8 yields a non-endoscopic holomorphic weight $n$ cuspidal representation $\pi$ of $G'(\mathbb{A}) = \mathrm{U}(n, \mathbb{A})$ Hecke congruent to $\mathbb{1}_{G'} \bmod p$ such that $\pi$ is (i) unramified at each odd finite $v \ne \ell$, (ii) spherical at $v = 2$, and (iii) an unramified twist of Steinberg at $v = \ell$. (By working with smaller compact subgroups $K$, one can remove the condition $p > n$.)*

*The same result is true for some additional values of $p$, independent of $\ell$, coming from numerators of Bernoulli numbers. For instance, we can always take $p = 61$ for $7 \le n \le 59$ as $61 \mid B_{7,\chi}$; we can take $p \in \{277, 2659\}$ if $11 \le n < p$ as $277 \cdot 2659 \mid B_{9,\chi}$; we can take $p = 19$ if $n = 13, 17$ as $19 \mid B_{11,\chi}$; or we can take $p \in \{43, 691, 967\}$ if $13 \le n < p$ as $691 \mid B_{12}$ and $43 \cdot 97 \mid B_{13,\chi}$.*

## 4. Eisenstein congruences for GL(2)

In this section, we discuss weight 2 Eisenstein congruences in the case of GL(2) (or rather PGL(2)). This was treated in [Mar17] over totally real number fields $F$ originally under the assumption that $h_F = h_F^+$. However, as pointed out to us by Jack Shotton, the published argument only gives cuspidal congruences mod $p$ when $p \nmid h_F$ and $h_F$ is odd.[2]

Here we explain how to remove this class number condition by working with PGL(2) rather than GL(2) and using congruence modules as in Section 3.3.1. Moreover, even in the case that $p \nmid h_F$ and $h_F$ is odd, we slightly refine our earlier result by making use of [Mar20] together with congruence modules.

Let $F$ be a totally real number field of degree $d$, and $B/F$ be a definite quaternion algebra. Let $\mathcal{O}$ be a special order of $B$ (in the sense of Hijikata–Pizer–Shemanske) of the following type. For a prime $v$ split in $B$, assume $\mathcal{O}_v$ is an Eichler order of level $\mathfrak{p}_v^{r_v}$ (with $r_v = 0$ for almost all $v$). For $v$ a finite prime at which $B$ ramifies, assume $\mathcal{O}_v$ is of the form $\mathfrak{o}_{E,v} + \mathfrak{P}_v^{2m}$ where $m$ is a non-negative integer, $\mathfrak{o}_{E,v}$ is the ring of integers of the unramified quadratic extension $E_v/F_v$ and $\mathfrak{P}_v$ is the unique prime ideal for $B_v$. In the latter case we say $\mathcal{O}_v$ is a special order of level $\mathfrak{p}_v^{2m+1}$ (of unramified quadratic type). Let $\mathfrak{N}_1$ (resp. $\mathfrak{N}_2$) be $\prod_v \mathfrak{p}_v^{r_v}$ where $v$ ranges over the finite primes such that $B/F$ splits (resp. ramifies) and $\mathfrak{p}_v^{r_v}$ is the level of $\mathcal{O}_v$. Let $\mathfrak{N} = \mathfrak{N}_1\mathfrak{N}_2$. Let $E_{2,\mathfrak{N}}$ be a parallel weight 2 Eisenstein eigenform over $F$ of level $\mathfrak{N}$ which has Hecke eigenvalue $q_v$ (resp. 1) for $v \mid \mathfrak{N}_1$ (resp. $v \mid \mathfrak{N}_2$), and Hecke eigenvalue $q_v + 1$ for finite $v \nmid \mathfrak{N}$.

**Theorem 4.1.** *Suppose $p$ is a rational prime which divides*

$$2^{1-d-e-|\{v|\mathfrak{N}_1\}|}|\zeta_F(-1)| \prod_{v|\mathfrak{N}_1} q_v^{r_v-1}(q_v-1) \prod_{v|\mathfrak{N}_2} q_v^{r_v-1}(q_v+1), \tag{4.1}$$

*where $e$ is the 2-exponent of the narrow class group $\mathrm{Cl}^+(F)$. Then there exists a parallel weight 2 cuspidal Hilbert eigenform $f$ of level $\mathfrak{N}$ and trivial nebentypus such that $f$ is Hecke congruent to $E_{2,\mathfrak{N}}$ mod $p$ at all finite $v$ such that $r_v \leq 1$. Moreover, for $v \mid \mathfrak{N}_1$ we may take $f$ such that the $v$-part of the exact level of $f$ is $\mathfrak{p}_v^{s_v}$, where (i) $s_v$ is odd; (ii) $s_v = 1$ if $p \nmid q_v$; and (iii) $s_v = r_v$ for any single chosen $v \mid \mathfrak{N}_1$ lying above $p$ (if such a $v$ exists).*

*Proof.* Let $G = PB^\times$ and $K = \prod K_v$, where $K_v$ the image of $\mathcal{O}_v^\times$ in $PB^\times$ for $v < \infty$ and $K_v = G_v$ for $v \mid \infty$. From the SO(3) case of the mass formula in [GHY01], one deduces that (4.1) is $2^{-e}m(K)$ (compare with the mass formula in [Mar17]). As explained in [Mar17], the constant function $\mathbb{1}$ on $\mathrm{Cl}(K)$ is a Hecke eigenfunction of all Hecke operators $T_v$ ($v$ finite), with the same Hecke eigenvalues as the modular form $E_{2,\mathfrak{N}}$ for any $v$ with $r_v \leq 1$. Then by Theorem 2.1, there exists an eigenform $\phi \in \mathcal{A}_0(G,K)$ such that $\phi$ is Hecke congruent to $\mathbb{1}$ mod $p$.

[2] See `arXiv:1601.03284v4` for a corrected version of [Mar17].

This congruence is also valid for ramified Hecke eigenvalues when $r_v = 1$ (again, see *op. cit.*).

Now we want to show we can take $\phi$ to be non-abelian. The abelian forms in $\mathcal{A}_0(G, K)$, viewed as functions on $\mathbb{A}^\times \backslash B^\times(\mathbb{A})/B^\times(F_\infty)$, are generated by the forms $\psi \circ N$, where $N : B^\times \to F^\times$ is the reduced norm and $\psi$ is a quadratic character of $\mathrm{Cl}^+(F)$. Necessarily, such a form can only be congruent to $\mathbb{1}$ mod $p$ if $p = 2$. Using the same argument as in Proposition 3.6 (the relevant congruence module for the space of abelian forms orthogonal to $\mathbb{1}$ has 2-exponent $e$, whereas the congruence module for $\mathcal{A}_0(G, K)$ has 2-exponent $v_2(m(K))$), gives such a non-abelian $\phi$.

Let $\mathcal{S}(G, K)$ be the orthogonal complement of the abelian subspace of $\mathcal{A}(G, K)$. By the Jacquet–Langlands correspondence for modular forms from [Mar20], we have an isomorphism of Hecke modules, for the Hecke algebras away from the set of $v \mid \mathfrak{N}_1$ with $r_v > 1$,

$$\mathcal{S}(G, K) \simeq \bigoplus S_2^{\mathfrak{M}\text{-new}}(\mathfrak{M}\mathfrak{N}_2), \quad \mathfrak{M} = \prod_{v \mid \mathfrak{N}_1} \mathfrak{p}_v^{2m_v+1},\ 1 \le 2m_v + 1 \le r_v.$$

The spaces on the right are the spaces of parallel weight 2 Hilbert cusp forms of level $\mathfrak{M}\mathfrak{N}_2$ which are locally new at each $v \mid \mathfrak{M}$ (the associated local representation of $\mathrm{PGL}_2(F_v)$ has conductor $\mathfrak{p}_v^{2m_v+1}$), and $\mathfrak{M}$ runs over divisors of $\mathfrak{N}_1$ which have odd exponent at every $v \mid \mathfrak{N}_1$. This shows (i).

Let $f$ be a Hilbert modular form corresponding to $\phi$. If $v \mid \mathfrak{N}_1$ such that $p \nmid q_v$, then if necessary we may enlarge $K$ by taking $K_v = \mathcal{O}_{B,v}^\times$ at $v$ so that $r_v = 1$. This forces $f$ to have exact level $\mathfrak{p}_v$ at $v$, i.e., we may assume (ii).

For (iii), suppose there exists $v \mid \mathfrak{N}_1$ such that $p \mid q_v$. If $r_v = 1$, there is nothing to show, so assume $r_v \ge 3$. Then we may use the above decomposition of $\mathcal{S}(G, K)$ together with the argument from Proposition 3.6. Namely, for a sufficiently large rationality field $L$, we may decompose $\mathcal{A}_0^L(G, K) = X_1(L) \oplus X_2(L)$, where $X_1(L)$ is generated by abelian forms together with cuspidal eigenforms which have level at most $\mathfrak{p}_v^{r_v-2}$ at $v$, and $X_2(L)$ is generated by cuspidal eigenforms which have exact level $\mathfrak{p}_v^{r_v}$ at $v$. Now $X_1(L) = A_0^L(G, K')$ where $K'$ is defined in the same way as $K$ except replacing $r_v$ with $r_v - 2$. Then the $p$-exponent of the congruence module for $X_1$ is simply $v_p(m(K'))$. But this is strictly less than $v_p(m(K))$, so the argument of Proposition 3.6 gives an eigenform in $X_2(L)$ which is Hecke congruent to $\mathbb{1}$ mod $p$. $\square$

**Remark 4.2.** If $v \mid \mathfrak{N}_2$ such that $p \mid (q_v + 1)$ if $r_v = 1$ (resp. $p \mid q_v$ if $r_v > 1$), we expect that we can also assume the $f$ in the theorem is locally new at $v$. Similarly, we expect we can impose (iii) for all $v$ such that $p \mid q_v$. This is because then the local factor at $v$ contributes to the $v_p(m(K))$, i.e., contributes to the $p$-exponent of the relevant congruence module. Alternatively, this factor contributes to the depth of the congruence mentioned in Remark 2.3. In order to prove this along the lines of our argument for (iii), we would need to know the $p$-exponent of the congruence module for the $v$-old forms. We do not attempt to study this here.

**Remark 4.3.** Ribet and Yoo (see [Yoo19]) have studied weight 2 Eisenstein congruences with fixed Atkin–Lehner signs for elliptic modular forms of squarefree level under some conditions. If $p > 2$ and $\mathfrak{N}$ is squarefree, then $f$ as in the theorem necessarily has Atkin–Lehner sign $-1$ at each $v \mid \mathfrak{N}_1$, and Atkin–Lehner sign $+1$ at each $v \mid \mathfrak{N}_2$ such that the $v$-part of the exact level of $f$ is $\mathfrak{p}_v$.

**Corollary 4.4.** *Let $F = \mathbb{Q}$ and $p$ be prime. Then for any $m \geq 1$ (resp. $m \geq 3$) if $p$ is odd (resp. $p = 2$), there exists a newform $f \in S_2(p^{2m+1})$ which is Hecke congruent to $E_{2,p} \bmod p$ away from $p$.*

## 5. Special mod $p$ congruences for $\mathrm{U}(p)$

Given a weight 2 cuspidal newform $f$ on $\mathrm{PGL}(2)$ whose $p$-th Fourier coefficient is $-1$ for a $p$ dividing the level (i.e., locally is the unramified quadratic twist of Steinberg at $p$), one can use quaternionic modular forms to construct a newform $g$ of the same weight and level which is congruent to $f$ mod 2 and has Fourier coefficient $+1$ at $p$ (i.e., locally is the untwisted Steinberg at $p$), at least in the case that the level is a squarefree product of an odd number of primes [Mar18b]. In general $g$ may be Eisenstein, but under some simple explicit conditions it can be chosen to be cuspidal. Here we extend this to higher rank in the setting of unitary groups.

Let $E/F$ be a CM extension of number fields. Let $S$ be a non-empty finite set of finite places of $F$ which split in $E$. Consider a definite unitary group $G = \mathrm{U}_A(n)$, where $A/E$ is a degree $n$ central division algebra such that, for each finite $v \in S$, $G(F_v) \simeq D_v^\times$ for some division algebra $D_v/F_v$. Let $K \subset G(\mathbb{A})$ be as in the beginning of Section 3.3 such that $K_v \simeq \mathcal{O}_{D_v}^\times$ for $v \in S$.

If $\pi$ occurs in $\mathcal{A}(G, K; 1)$, then, for $v \in S$, $\pi_v$ is 1-dimensional, and thus of the form $\mu_v \circ \det$ for some unramified character $\mu_v : F_v^\times \to \mathbb{C}^\times$ such that $\mu_v^n = 1$. (Here det denotes the reduced norm from $D_v$ to $F_v$.) Consider a collection $\mu_S = (\mu_v)_{v \in S}$ of such $\mu_v$. We denote by $\mathcal{A}(G, K; 1)^{\mu_S}$ the subspace of $\mathcal{A}(G, K; 1)$ generated by $\pi^K$ where $\pi$ runs over all $\pi$ contributing to $\mathcal{A}(G, K; 1)$ such that $\pi_v \simeq \mu_v \circ \det$ for all $v \in S$. When $\mu_v = 1$ for all $v \in S$, we write this as $\mathcal{A}(G, K; 1)^{1_S}$. Let $\zeta_m = e^{2\pi i/m}$.

**Lemma 5.1.** *Fix $p \mid n$. Suppose $\mu_v$ has prime power order $p^{r_v} \mid n$ for all $v \in S$. Let $\mathcal{O}$ be the ring of integers of some number field containing $\zeta_{p^r}$, and $\mathfrak{p}$ a prime of $\mathcal{O}$ above $p$. Then for any nonzero $\phi \in \mathcal{A}^{\mathcal{O}}(G, K; 1)^{\mu_S}$, there exists a nonzero $\phi' \in \mathcal{A}^{\mathcal{O}}(G, K; 1)^{1_S}$ such that $\phi' \equiv \phi \bmod \mathfrak{p}$.*

*Proof.* Let $\bar{G} = G/Z$ and $\bar{K} = Z(\mathbb{A})K/Z(\mathbb{A})$. Then we may view $\phi$ as a function on $\mathrm{Cl}(\bar{K})$. For $v \in S$, fix a uniformizer $\varpi_{D,v}$ of $D_v$ such that $\det \varpi_{D,v} = \varpi_v$. Then $\varpi_{D,v}$ acts on $\mathrm{Cl}(\bar{K})$ via right multiplication with order dividing $n$. Denote this action by $\sigma_v$. Let $Y_1, \ldots, Y_t$ be the orbits of the ensuing action of $\Gamma = \prod_{v \in S} \langle \varpi_{D,v} \rangle$ on $\mathrm{Cl}(\bar{K})$.

Note that for $\phi \in \mathcal{A}(G, K; 1)$, we have $\phi \in \mathcal{A}(G, K; 1)^{\mu_S}$ if and only if $\phi(\sigma_v(y)) = \mu_v(\varpi_v)\phi(y)$ for all $y \in \mathrm{Cl}(\bar{K})$, $v \in S$. Fix some orbit $Y_i$ and write $Y_i = \{y_1, \ldots, y_s\}$. Then for any $y_j \in Y_i$, there is some sequence of $\sigma_v$'s (with $v \in S$) whose composition sends $y_1$ to $y_j$. Hence $\phi(y_j) = \zeta\phi(y_1)$ for some $p$-power root of unity $\zeta$. Since $\zeta \equiv 1 \bmod \mathfrak{p}$, defining $\phi'(y_j) = \phi(y_1)$ for $1 \leq j \leq s$ gives a function on $Y_i$ which is congruent to $\phi$ mod $\mathfrak{p}$. Defining $\phi'$ this way on each orbit completes the proof. □

The following is a partial analogue of [Mar18b, Theorem 1.3] in higher rank, and the proof is similar in spirit.

**Theorem 5.2.** *Let $n = p$ be an odd prime, and assume (EC-U) for $n$. Let $S$ be a finite set of finite places of $F$ which are split in $E/F$. Suppose $p$ does not divide $|\mathrm{Cl}(\mathrm{U}(1))|$ nor*

$$\prod_{r=1}^{p} L(1-r, \chi_{E/F}^r) \times \prod_{v \in S} \left( \prod_{r=1}^{p-1} (q_v^r - 1) \right).$$

*For each $v \in S$, let $\mu_v$ be an unramified character of $F_v^\times$ of order $1$ or $p$. For finite $v \notin S$, assume $K_v$ is a hyperspecial maximal compact open subgroup of $\mathrm{U}_p(F_v)$.*

*Let $\pi$ be an automorphic representation of $G'(\mathbb{A}) = \mathrm{U}(p, \mathbb{A})$ holomorphic of parallel weight $p$ with trivial central character such that $\pi_E$ is cuspidal, $K_v$-spherical for all finite $v \notin S$, and $\pi_v \simeq \mathrm{St}_v \otimes \mu_v$ for all $v \in S$. Then there exists an automorphic representation $\pi'$ of $G'(\mathbb{A})$, also holomorphic of parallel weight $p$ with trival central character and $\pi'_E$ cuspidal, such that $\pi'_v$ is $K_v$-spherical for all finite $v \notin S$, $\pi'_v \simeq \mathrm{St}_v$ for all $v \in S$ and $\pi$ is Hecke congruent to $\pi'$ mod $p$.*

*Proof.* Let $G = \mathrm{U}_A(p)$ be a totally definite inner form of $G'$ which is locally isomorphic to $G'$ at all finite places outside of $S$ and compact at each $v \in S$. Now $\pi$ corresponds to a simple generic formal parameter $\psi$, which we may think of as the cuspidal representation $\pi_E$ of $\mathrm{GL}_p(\mathbb{A}_E)$. Then there exists an automorphic representation $\sigma \in \Pi_\psi(G)$ such that $\sigma_v \simeq \pi_v$ for all finite $v \notin S$, $\sigma_v \simeq \mu_v \circ \det$ for $v \in S$, and $\sigma_v$ is trivial for $v \mid \infty$.

For $v \in S$, let $D_v/F_v$ be a division algebra isomorphic to $A_w/E_w$ for some $w \mid v$ and put $K_v = \mathcal{O}_{D_v}^\times$. For $v \mid \infty$, put $K_v = G_v$. Set $K = \prod K_v$. Then $\sigma$ occurs in $\mathcal{A}_0(G, K; 1)$ and we may take a nonzero $\phi \in \sigma^K$ to have values in the ring of integers $\mathcal{O}$ of some number field $L$. Let $\mathfrak{p}$ be a prime of $\mathcal{O}$ above $p$.

If $\phi \equiv 0 \bmod \mathfrak{p}$, we may consider the Hilbert class field $H_L$ of $L$ so that $\mathfrak{p}$ is unramified and principal in $H_L$. Thus we may scale $\phi$ by an element of $H_L$ to assume that $\phi \not\equiv 0 \bmod \mathfrak{p}$, and moreover $\phi \not\equiv 0 \bmod \mathfrak{P}$ for some prime $\mathfrak{P}$ of $H_L$ above $p$. Hence by replacing $L$ with $H_L$ and $\mathfrak{p}$ with $\mathfrak{P}$ if necessary, we may and will assume $\phi \not\equiv 0 \bmod \mathfrak{p}$.

By Lemma 5.1, there exists a nonzero $\phi' \in \mathcal{A}^{\mathcal{O}}(G, K; 1)^{1_S}$ such that $\phi' \equiv \phi \bmod \mathfrak{p}$. We claim $\phi'$ is non-abelian. First note that, since $p \nmid |\mathrm{Cl}(\mathrm{U}(1))|$, the only non-abelian forms in $\mathcal{A}(G, K; 1)$ are constant functions. However, if $\phi' = c\mathbb{1}$ for some $c \in \mathcal{O}$, then $\phi \in \mathcal{A}_0(G, K; 1)$ implies $0 = (\phi, \mathbb{1}) \equiv c(\mathbb{1}, \mathbb{1}) \equiv cm(K) \bmod \mathfrak{p}$. This would mean $\mathfrak{p} \mid m(K)$, since $\phi' \equiv \phi \not\equiv 0 \bmod \mathfrak{p}$ implies $c \not\equiv 0 \bmod \mathfrak{p}$. But this is impossible by our indivisibility assumption together with Proposition 3.9.

Then, as in the proofs of Theorem 2.1 and Theorem 3.8, we can transfer this to a mod $\mathfrak{p}$ Hecke congruence with a non-abelian eigenform $\phi''$ on $G$, and obtain a congruent $\pi'$ on $G'$ as asserted. □

**Remark 5.3.** It is clear from the proof that one can allow $K_v$ to be a finite index subgroup of a hyperspecial maximal compact $K_v^0$ at a finite number of $v \notin S$ by also imposing the conditions $p \nmid [K_v^0 : K_v]$. At such $v$, then the appropriate statement is that both $\pi_v$ and $\pi_v'$ have nonzero $K_v$-fixed vectors.

**Remark 5.4.** In the case of weight 2 elliptic modular forms of squarefree level, we showed in [Mar18a] that there is a strict (though small) bias towards local ramified factors being Steinberg as opposed to the unramifed quadratic twist of Steinberg. In [Mar18b], this bias was shown to be related to the existence of mod 2 congruences of forms which are twisted Steinberg at certain places to untwisted Steinberg at these places. Similarly, the above congruence result suggests a bias towards local untwisted Steinberg representations on $\mathrm{U}_p(\mathbb{A})$. Specifically, in the notation of the proof, we expect that the number of representations occurring in $\mathcal{A}(G, K; 1)^{1_S}$ is always at least the number of representations occurring in $\mathcal{A}(G, K; 1)^{\mu_S}$. The above result implies the analogous statement is true for mod $p$ Hecke congruence classes of representations.

## References


[BKK14] Tobias Berger, Krzysztof Klosin, and Kenneth Kramer, *On higher congruences between automorphic forms*, Math. Res. Lett. **21** (2014), no. 1, 71–82. ↑2.3

[BD16] Jonas Bergström and Neil Dummigan, *Eisenstein congruences for split reductive groups*, Selecta Math. (N.S.) **22** (2016), no. 3, 1073–1115. ↑1

[CH13] Gaëtan Chenevier and Michael Harris, *Construction of automorphic Galois representations, II*, Camb. J. Math. **1** (2013), no. 1, 53–73. ↑3.1

[GHY01] Wee Teck Gan, Jonathan P. Hanke, and Jiu-Kang Yu, *On an exact mass formula of Shimura*, Duke Math. J. **107** (2001), no. 1, 103–133. ↑1.1, 2, 3.3.2, 3.3.2, 3.10, 4

[Gha02] Eknath Ghate, *An introduction to congruences between modular forms*, Currents trends in number theory (Allahabad, 2000), Hindustan Book Agency, New Delhi, 2002, pp. 39–58. ↑3.3.1

[Gro99] Benedict H. Gross, *Algebraic modular forms*, Israel J. Math. **113** (1999), 61–93. ↑1.1, 2.1

[HT01] Michael Harris and Richard Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001. With an appendix by Vladimir G. Berkovich. ↑3.1

[KMSW] Tasho Kaletha, Alberto Minguez, Sug Woo Shin, and Paul-James White, *Endoscopic classification of representations: inner forms of unitary groups* (2014), preprint, available at https://arxiv.org/abs/1409.3731v3. ↑1.2, 3.1

[Lab11] J.-P. Labesse, *Changement de base CM et séries discrètes*, On the stabilization of the trace formula, Stab. Trace Formula Shimura Var. Arith. Appl., vol. 1, Int. Press, Somerville, MA, 2011, pp. 429–470 (French). ↑3.1

[Mar17] Kimball Martin, *The Jacquet-Langlands correspondence, Eisenstein congruences, and integral L-values in weight 2*, Math. Res. Lett. **24** (2017), no. 6, 1775–1795. Corrected version available at https://arxiv.org/abs/1601.03284v4. ↑1, 1.1, 1.2, 2.3, 4, 4, 2

[Mar18a] ———, *Refined dimensions of cusp forms, and equidistribution and bias of signs*, J. Number Theory **188** (2018), 1–17. ↑5.4

[Mar18b] ———, *Congruences for modular forms mod 2 and quaternionic S-ideal classes*, Canad. J. Math. **70** (2018), no. 5, 1076–1095. ↑2.3, 5, 5, 5.4

[Mar20] ———, *The basis problem revisited*, Trans. Amer. Math. Soc. **373** (2020), no. 7, 4523–4559. ↑4, 4

[Maz77] B. Mazur, *Modular curves and the Eisenstein ideal*, Inst. Hautes Études Sci. Publ. Math. **47** (1977), 33–186 (1978). ↑1, 1.1

[MW17] Colette Moeglin and Jean-Loup Waldspurger, *Stabilisation de la formule des traces tordue. Vol.s 1 and 2*, Progress in Mathematics, vol. 316 and 317, Basel: Birkhäuser/Springer, 2016 and 2017. ↑3.1

[Mok15] Chung Pang Mok, *Endoscopic classification of representations of quasi-split unitary groups*, Mem. Amer. Math. Soc. **235** (2015), no. 1108, vi+248. ↑1.2, 3.1, 3.1

[PR94] Vladimir Platonov and Andrei Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics, vol. 139, Academic Press, Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen. ↑3, 3.2

[Rog90] Jonathan D. Rogawski, *Automorphic representations of unitary groups in three variables*, Annals of Mathematics Studies, vol. 123, Princeton University Press, Princeton, NJ, 1990. ↑1.2, 3.1

[Sch85] Winfried Scharlau, *Quadratic and Hermitian forms*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 270, Springer-Verlag, Berlin, 1985. ↑3

[Shi97] Goro Shimura, *Euler products and Eisenstein series*, CBMS Regional Conference Series in Mathematics, vol. 93, Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1997. MR1450866 ↑3.7

[Shi06] ———, *Quadratic Diophantine equations, the class number, and the mass formula*, Bull. Amer. Math. Soc. (N.S.) **43** (2006), no. 3, 285–304. ↑1.1, 2

[Shi11] Sug Woo Shin, *Galois representations arising from some compact Shimura varieties*, Ann. of Math. (2) **173** (2011), no. 3, 1645–1741. ↑3.1, 3.1

[Yoo19] Hwajong Yoo, *Non-optimal levels of a reducible mod $\ell$ modular representation*, Trans. Amer. Math. Soc. **371** (2019), no. 6, 3805–3830. ↑1, 4.3



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