CONGRUENCES FOR MODULAR FORMS MOD 2 AND QUATERNIONIC S-IDEAL CLASSES

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ABSTRACT. We prove many simultaneous congruences mod 2 for elliptic and Hilbert modular forms among forms with different Atkin–Lehner eigenvalues. The proofs involve the notion of quaternionic S-ideal classes and the distribution of Atkin–Lehner signs among newforms.

1. INTRODUCTION

In this paper, we use the notion of quaternionic S-ideal classes and the Jacquet–Langlands correspondence to show certain behavior of Atkin–Lehner signs yields many simultaneous congruences of newforms mod 2. We begin by explaining our main results over $\mathbb{Q}$, and will discuss the extensions to Hilbert modular forms at the end of the introduction.

Let N be a squarefree product of an odd number of primes, $M|N$ and $k \in 2\mathbb{N}$. By a sign pattern ε for M we mean a collection of signs $\varepsilon_p \in \{\pm 1\}$ for each $p|M$. Denote the sign pattern with $\varepsilon_p = -1$ for all $p|M$ by $-_M$.

Let $S_k^{\mathrm{new}}(N)$ denote the span of newforms of weight k for $\Gamma_0(N)$. For a sign pattern ε for M, let $S_k^{\mathrm{new},\varepsilon}(N)$ be the subspace spanned by newforms f with p-th Atkin–Lehner eigenvalue $w_p(f) = \varepsilon_p$ for all $p|M$. The case $k = 2$ is a little different than $k \geq 4$, due to the interaction with the weight 2 Eisenstein series $E_{2,N}(z) := \sum_{d|N} \mu(d) d E_2(dz)$. To state our first result uniformly, we introduce the augmented space $S_k^{\mathrm{new}}(N)^*$, which is just $S_k^{\mathrm{new}}(N)$ if $k \geq 4$ but $S_2^{\mathrm{new}}(N)^* = S_2^{\mathrm{new}}(N) \oplus \mathbb{C}E_{2,N}$. Similarly, we set $S_k^{\mathrm{new},\varepsilon}(N)^* = S_k^{\mathrm{new},\varepsilon}(N)$ if $k \geq 4$ or $\varepsilon \neq -_M$ and $S_2^{\mathrm{new},-_M}(N)^* = S_2^{\mathrm{new},-_N}(N) \oplus \mathbb{C}E_{2,N}$.

Denote the n-th Fourier coefficient of a modular form f by $a_n(f)$. Our first main result is

Theorem 1.1. *Suppose M, N are as above such that, for each divisor $d|M$ with $d > 1$, there exists an odd prime $p|\frac{N}{M}$ such that $\left(\frac{-d}{p}\right) = 1$. If M is even, assume also $\frac{N}{M}$ is divisible by a prime which is 1 mod 4. Let $f \in S_k^{\mathrm{new}}(N)^*$ be a newform and ℓ a prime of $\bar{\mathbb{Q}}$ above 2.*

Then for any sign pattern ε for M, there exists an eigenform $g \in S_k^{\mathrm{new},\varepsilon}(N)^*$ such that $a_n(f) \equiv a_n(g) \bmod \ell$ for all $n \in \mathbb{N}$. Moreover, we may take $g \in S_k^{\mathrm{new}}(N)$ to be a cuspidal newform if $k \neq 2$, $\varepsilon \neq -_M$, or N is not an even product of 3 primes.

In particular this theorem applies if there exists a prime $p|\frac{N}{M}$ with $p \equiv 1$ mod 4 such that $\left(\frac{q}{p}\right) = 1$ for each prime $q|M$, e.g. $M = 26$ and $N = 17 \cdot M$. The theorem

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also holds with the alternative hypotheses that $M \equiv 7 \bmod 8$ is prime and N is even (see Remark 4.1).

Quadratic reciprocity implies that if $p_1 p_2 | N$ and $p_1 \equiv 3 \bmod 4$, then $M = p_1$ or $M = p_2$ satisfies the hypothesis of this theorem. This yields

Corollary 1.2. *Suppose N is composite and divisible by some $p \equiv 3 \bmod 4$. Let $k \in 2\mathbb{N}$ with $k \neq 2$ if $N = 2p_1 p_2$ for some primes p_1, p_2. Fix a prime ℓ of $\overline{\mathbb{Q}}$ above 2. Then for any newform $f \in S_k^{\mathrm{new}}(N)$, there exists a non-Galois-conjugate newform $g \in S_k^{\mathrm{new}}(N)$ such that $a_n(f) \equiv a_n(g) \bmod \ell$ for all $n \in \mathbb{N}$.*

Our second main result is

Theorem 1.3. *Let $f \in S_2^{\mathrm{new}}(N) \oplus \mathbb{C}E_{2,N}$ be an eigenform, and ℓ a prime of $\overline{\mathbb{Q}}$ above 2. Then there exists an eigenform $g \in S_2^{\mathrm{new}, -_N}(N) \oplus \mathbb{C}E_{2,N}$ such that $a_n(f) \equiv a_n(g) \bmod \ell$ for all $n \in \mathbb{N}$. Moreover, if N is not an even product of 3 primes, we may take $g \in S_2^{\mathrm{new}, -_N}(N)$.*

Note many existing congruence results exclude small primes or primes dividing the level, e.g. [Maz77], [Hid81] and [Yoo], whereas our method is specific to congruences mod 2 and does not require $2 \nmid N$. Moreover, these results indicate that congruences modulo (primes above) 2 are very common. Indeed, they seem much more common than congruences modulo odd primes appear to occur, since (at least large) congruence primes must divide the special value of an L-function (e.g., see [Hid81]). Further, while many congruence results are known, simultaneous congruence results seem harder to come by. However, our results exhibit many simultaneous congruences.

Namely, if $\omega(M)$ is the number of prime factors of M, Theorem 1.1 gives conditions for all newforms as well as $E_{2,N}$ to be congruent to at least $2^{\omega(M)}$ (non-Galois-conjugate) eigenforms. Further, Theorem 1.3 says that for any squarefree level N with $\omega(N)$ odd, there are at most $1 + \dim S_2^{\mathrm{new}, -_N}(N)$ congruence classes in $S_2^{\mathrm{new}}(N)$. An exact formula for $\dim S_2^{\mathrm{new}, -_N}(N)$ is given in [Mar2] and is approximately $2^{-\omega(N)} \dim S_2^{\mathrm{new}}(N)$, so there must be at least one congruence class containing many newforms when $\omega(N)$ is large.

In weight 2, we note there have been some recent results giving simultaneous congruences. Le Hung and Li [LHL16], in their investigations on level raising mod 2, have shown for certain forms in $S_2(N)$ one gets congruences with forms with prescribed Atkin–Lehner signs. Specifically, under the assumption that f is not congruent to an Eisenstein series mod 2, the methods in [LHL16] give both Theorem 1.3, and a version of Theorem 1.1 (at least in weight 2) where one can prescribe all but one Atkin–Lehner sign for g. We note that our methods seem quite different, though both make use of the Jacquet–Langlands correspondence.

On the other hand, Ribet and later Yoo (see [Yoo]) have investigated congruences of newforms in $S_2(N)$ with Eisenstein series modulo primes $p > 3$ with prescribed Atkin–Lehner signs, which gives simultaneous Eisenstein congruences under certain conditions. Taking $f = E_{2,N}$ in Theorem 1.1 gives an analogue of the sufficient conditions in [Yoo] for an Eisenstein congruence with prescribed Atkin–Lehner signs mod 2. However we cannot specify signs at all places (except when all signs are -1 by taking $f = E_{2,N}$ in Theorem 1.3).

Now let us discuss the proofs, which have a couple of features we find interesting, such as the connection with distributions of Atkin–Lehner signs and the connection

between certain eigenspaces of quaternionic modular forms and quaternionic S-ideal class.

First we discuss the distribution of Atkin–Lehner signs. Let us say the sign patterns ε for M are perfectly equidistributed in weight k and level N if $\dim S_k^{\mathrm{new},\varepsilon}(N)^*$ is independent of ε. We will find that perfect equidistribution in weight 2 implies perfect equidistribution in weight k. (This is also evident from [Mar2] under the hypotheses of Theorem 1.1.) Then we will prove that this perfect equidistribution implies the congruences in Theorem 1.1, and use [Mar2] to see the above hypotheses are sufficient for perfect equidistribution.

Theorem 1.3 is related to a different fact about distribution of sign patterns. In [Mar2], we showed that although the sign patterns are equidistributed asymptotically as the weight or level grows, there is a bias toward or against certain sign patterns in fixed spaces $S_k^{\mathrm{new}}(N)$. In particular, when $k = 2$ and $\omega(N)$ is odd, there is a bias towards $-_N$ in the sense that $\dim S_2^{\mathrm{new},\varepsilon}(N) \leq \dim S_2^{\mathrm{new},-_N}(N)^*$ for any sign pattern ε for N. Below we will give a simple proof of this using quaternion algebras, and the idea behind this proof is what allows us to construct the congruences in Theorem 1.3.

The overall strategy to get our theorems is the use of arithmetic of definite quaternion algebras to construct congruences between quaternionic modular forms, and then use the Jacquet–Langlands correspondence to deduce congruences for elliptic or Hilbert modular forms. This is why we restrict to $\omega(N)$ odd over $\mathbb{Q}$. We also used this idea in [Mar1] to get Eisenstein congruences in weight 2, generalizing results from [Maz77] and [Yoo]. Whereas in that paper we used mass formulas for quaternionic orders to get Eisenstein congruences, here we use the structure of quaternionic S-ideal classes to get our congruences.

In Section 2, we define the notion of S-ideal classes for quaternion algebras in an analogous way to the definition of S-ideal classes for number fields. The S-ideal class numbers interpolate between the usual class number and the type number of a quaternion algebra.

In Section 3 we review the theory of (definite) quaternionic modular forms. If B is a definite quaternion algebra of discriminant N and $\mathcal{O}$ is a maximal order of B, then the space $S_k^{\mathrm{new}}(N)^*$ corresponds to a space of $\mathcal{M}_{k-2}(\mathcal{O})$ of quaternionic modular forms. These can be viewed as certain vector-valued functions on the set of right $\mathcal{O}$-ideal classes $\mathrm{Cl}(\mathcal{O})$. In the case $k = 2$, $S_2^{\mathrm{new}}(N)^*$ simply corresponds to the space of all $\mathbb{C}$-valued functions on $\mathrm{Cl}(\mathcal{O})$.

In Section 4 we describe the action of ramified Hecke operators on quaternionic modular forms in terms of local involutions acting on $\mathrm{Cl}(\mathcal{O})$. This gives a realization of the space of quaternionic forms corresponding to $S_k^{\mathrm{new},\varepsilon}(N)^*$ as certain functions on the set of S-ideal classes $\mathrm{Cl}_S(\mathcal{O})$ for $\mathcal{O}$. However, the precise description of this space in general is somewhat complicated as it involves both the way the local involutions for different primes interact globally as well as the way they interact with the weight and the signs ε_p.

There are two situations where we can make this description simpler. One is if the local involutions act on $\mathrm{Cl}(\mathcal{O})$ both without fixed points as well as without fixing orbits of points under the other local involutions. This corresponds to the S-class numbers being as small as possible, which corresponds to perfect equidistribution of sign patterns in weight 2. From our description of quaternionic forms corresponding to $S_k^{\mathrm{new},\varepsilon}(N)$, we can deduce that perfect equidistribution in weight 2

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implies it in all weights. In this situation, this is enough construct the quaternionic congruences which imply Theorem 1.1, excluding the cuspidal condition in weight 2, upon applying our dimension formulas for $S_k^{\mathrm{new},\varepsilon}(N)$ in [Mar2] to determine when we have perfect equidistribution of signs.

The other situation where this description becomes simpler is in weight 2, so one only needs to understand how the local involutions interact. Namely, if $k = 2$, these forms are just the $\mathbb{C}$-valued functions on the set of S-ideal classes which are "admissible for $-\varepsilon$." Since all S-ideal classes are admissible when $\varepsilon = -_N$, this immediately gives bias towards the sign pattern $-_N$ in weight 2. This description also yields relations between type numbers or generally S-ideal class numbers and dimensions of of spaces of newforms, and allows us to construct the quaternionic congruences needed for the first part of Theorem 1.3.

To show one can take g to be a cusp form in Theorem 1.3 (and thus also Theorem 1.1) when N is not an even product of 3 primes, we prove two auxiliary results. By a variant of our argument in [Mar1], we show in Proposition 5.4 that $E_{2,N}$ is congruent to a newform in $S_2^{\mathrm{new},-_N}(N)$ under certain conditions; in particular if $\omega(N) > 3$ or N is a product of 3 odd primes. We treat the N prime case by showing that lack of perfect equidistribution of Atkin–Lehner signs means the congruent quaternionic modular form we construct must be cuspidal (Proposition 5.5). Using dimension formulas from [Mar2], we see that lack of perfect equidistribution is automatic for N prime. These auxiliary results in fact give other conditions when $N = 2p_1p_2$ where we can still take g cuspidal in Theorem 1.3—for instance, if p_1 or p_2 is 1 mod 4 and $N > 258$. See Section 5.3 for details. We note that some exceptions to taking g cuspidal when $N = 2p_1p_2$ are in fact necessary, e.g., $S_2^{\mathrm{new}}(42)$ and $S_2^{\mathrm{new}}(70)$ are 1-dimensional but not all Atkin–Lehner operators act by -1.

Now we summarize what we can say in the case of Hilbert modular forms. For simplicity, we only work over totally real fields F of narrow class number $h_F^+ = 1$, however we expect that our arguments can be suitably modified to remove this restriction. (See Section 3 for comments on what needs to be modified.) The proofs we have described above then go through for Hilbert modular forms over F with the exception of the explicit determination of when we have perfect equidistribution of signs, as we have not worked out an analogue of [Mar2] over totally real fields. In other words, one does not have the explicit criteria in terms of quadratic residue symbols for the Hilbert analogue of Theorem 1.1 (see Corollary 5.2), nor does one have exactly analogous conditions on the level for when one can take g cuspidal in the analogue of Theorem 1.3 (see Corollary 5.3). However, we can still give some conditions on when we can take g cuspidal in Corollary 5.3 by Proposition 5.4 which gives Eisenstein congruences under certain hypotheses.

Lastly, we remark in [Mar1] we worked with quaternionic orders which were not necessarily maximal (or even Eichler), which allowed us to get Eisenstein congruences for any level N which is not a perfect square, though we could not always say the congruent cusp form is new. We expect that our basic strategy here should be generalizable to non-maximal orders, so we would not need to assume $\omega(N)$ is odd (when $F = \mathbb{Q}$) or N is squarefree. However, our dimension formulas from [Mar2] are only for squarefree level because the trace formula we used is significantly more complicated for non-squarefree level, though the method should apply to arbitrary level. Potentially, this could make the hypotheses for a non-squarefree analogue of Theorem 1.1 considerably more complicated.

2. Quaternionic S-ideal classes

Let F be a totally real number field with narrow class number $h_F^+ = 1$, and B/F be a totally definite quaternion algebra of discriminant $\mathfrak{N}$. Fix a maximal order $\mathcal{O}$ of B. For any (finite) prime $\mathfrak{p}$ of F, we have the local completions $B_{\mathfrak{p}} = B \otimes_F F_{\mathfrak{p}}$ and $\mathcal{O}_{\mathfrak{p}} = \mathcal{O} \otimes_{\mathbb{Z}} \mathfrak{o}_{F,\mathfrak{p}}$. Then $B_{\mathfrak{p}}/F_{\mathfrak{p}}$ is a division algebra if and only if $\mathfrak{p}|\mathfrak{N}$. Let $\hat{\mathcal{O}}^{\times} = \prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^{\times}$ and $\hat{B}^{\times} = \prod_{\mathfrak{p}}' B_{\mathfrak{p}}^{\times}$ denote the finite ideles of B, i.e., the restricted direct product of the $B_{\mathfrak{p}}^{\times}$'s with respect to the $\mathcal{O}_{\mathfrak{p}}^{\times}$'s.

When we restrict to $F = \mathbb{Q}$, we write N for $\mathfrak{N}$, p for $\mathfrak{p}$, and so on.

Recall there is a canonical bijection of

$$\text{(2.1)} \qquad \text{Cl}(\mathcal{O}) := B^{\times} \backslash \hat{B}^{\times} / \hat{\mathcal{O}}^{\times}$$

with the set (not a group) of right (locally principal) ideal classes of $\mathcal{O}$. The class number $h_B = |\text{Cl}(\mathcal{O})|$ is independent of the choice of $\mathcal{O}$.

The number of maximal orders in B up to $B^{\times}$-conjugacy is called the type number t_B of B. The conjugacy classes of maximal orders are in bijection with

$$\text{(2.2)} \qquad B^{\times} \backslash \hat{B}^{\times} / \hat{G}(\mathcal{O}),$$

where $\hat{G}(\mathcal{O}) = \prod' G(\mathcal{O}_{\mathfrak{p}})$ is the stabilizer subgroup with local components $G(\mathcal{O}_{\mathfrak{p}}) = \{x \in B_{\mathfrak{p}}^{\times} : x\mathcal{O}_{\mathfrak{p}}x^{-1} = \mathcal{O}_{\mathfrak{p}}\}$. Here $G(\mathcal{O}_{\mathfrak{p}}) = F_{\mathfrak{p}}^{\times} \mathcal{O}_{\mathfrak{p}}^{\times}$ if $\mathfrak{p} \nmid \mathfrak{N}$ and $G(\mathcal{O}_{\mathfrak{p}}) = B_{\mathfrak{p}}^{\times}$ if $\mathfrak{p}|\mathfrak{N}$. The latter part follows as any finite-dimensional $\mathfrak{p}$-adic division algebra has a unique maximal order. Hence $\hat{G}(\mathcal{O}) = \hat{F}^{\times} \hat{\mathcal{O}}^{\times} \cdot \prod_{\mathfrak{p}|\mathfrak{N}} B_{\mathfrak{p}}^{\times}$. Since t_B is the cardinality of (2.2) and $[B_{\mathfrak{p}}^{\times} : F_{\mathfrak{p}}^{\times} \mathcal{O}_{\mathfrak{p}}^{\times}] = 2$ at ramified places (and $h_F = 1$), one deduces that $\frac{h_B}{2^{\omega(\mathfrak{N})}} \leq t_B \leq h_B$, where $\omega(\mathfrak{N})$ is the number of prime ideals dividing $\mathfrak{N}$.

Let S be a set of primes dividing $\mathfrak{N}$. We define the (right) S-ideal classes of $\mathcal{O}$ to be

$$\text{Cl}_S(\mathcal{O}) := B^{\times} \backslash \hat{B}^{\times} / \hat{\mathcal{O}}_S^{\times},$$

where

$$\mathcal{O}_S^{\times} = \prod_{\mathfrak{p} \in S} B_{\mathfrak{p}}^{\times} \times \prod_{\mathfrak{p} \notin S} \mathcal{O}_{\mathfrak{p}}^{\times}.$$

This interpolates (2.1) and (2.2), and is analogous to the definition of the S-ideal class group for number fields: if $S = \emptyset$ one gets (2.1), and if $S = \{\mathfrak{p} : \mathfrak{p}|\mathfrak{N}\}$ one gets (2.2). (The factor $\mathbb{A}_F^{\times}$ in the quotient (2.2) makes no difference since $h_F = 1$.) The set $\text{Cl}_S(\mathcal{O})$ is always finite. Denote the S-ideal class number $|\text{Cl}_S(\mathcal{O})|$ by $h_{B,S}$. If $\mathfrak{M} = \prod_{\mathfrak{p} \in S} \mathfrak{p}$, we sometimes also write $\text{Cl}_S(\mathcal{O}) =: \text{Cl}_{\mathfrak{M}}(\mathcal{O})$ and $h_{B,S} =: h_{B,\mathfrak{M}}$.

3. Quaternionic modular forms

Let F, B, and $\mathcal{O}$ be as above. Let $\mathbf{k} = (k_1, \ldots, k_d) \in (2\mathbb{Z}_{\geq 0})^d$, where $d = [F : \mathbb{Q}]$. Let $\tau_1, \ldots, \tau_d$ denote the embeddings of F into $\mathbb{R}$, and put $B_{\infty}^{\times} = \prod B_{\tau_i}^{\times}$. View each

$B_{\tau_i}^\times$ as a subgroup of $\mathrm{GL}_2(\mathbb{C})$. Let (ρ_{k_i}, V_{k_i}) be the twist $\det^{-k_i/2} \otimes \mathrm{Sym}^{k_i}$ of the k_i-th symmetric power representation Sym^{k_i} of $\mathrm{GL}_2(\mathbb{C})$ into $\mathrm{GL}_{k_i+1}(\mathbb{C})$ restricted to $B_{\tau_i}^\times$. The twist here gives ρ_{k_i} trivial central character. Put $(\rho_{\mathbf{k}}, V_{\mathbf{k}}) = \bigotimes(\rho_{k_i}, V_{k_i})$.

We define the space $\mathcal{M}_{\mathbf{k}}(\mathcal{O})$ of weight $\mathbf{k}$ quaternionic modular forms of level $\mathcal{O}$ to be the space of functions $\varphi : \hat{B}^\times \times B_\infty^\times \to V_{\mathbf{k}}$ satisfying

$$\varphi(\gamma x u, \gamma y g) = \rho_{\mathbf{k}}(g^{-1})\varphi(x, y) \quad \text{for } x \in \hat{B}^\times, y \in B_\infty^\times, \gamma \in B^\times, u \in \hat{\mathcal{O}}^\times, g \in B_\infty^\times.$$

Alternatively, $\mathcal{M}_{\mathbf{k}}(\mathcal{O})$ is the space of functions on $B^\times \backslash B^\times(\mathbb{A})/\hat{\mathcal{O}}^\times$ on which $B_\infty^\times$ acts on the right by $\rho_{\mathbf{k}}$. We note that a consequence of our assumption $h_F^+ = 1$ is that all forms in $\mathcal{M}_{\mathbf{k}}(\mathcal{O})$ are invariant under translation by the center $\mathbb{A}_F^\times$ of $B^\times(\mathbb{A})$. Without this assumption, we could restrict to the subspace of forms with trivial central character as in [Mar1].

For the invariance conditions on φ to be compatible with the transformation condition on $B_\infty^\times$, it is necessary and sufficient that $\varphi(x, 1) \in V_{\mathbf{k}}^{\Gamma(x)}$, where $\Gamma(x) = x\hat{\mathcal{O}}^\times x^{-1} \cap B^\times$. Write

$$\mathrm{Cl}(\mathcal{O}) = \{x_1, \ldots, x_h\}$$

for some fixed choice of $x_1, \ldots, x_h$ in $B^\times(\mathbb{A})$. Put $\Gamma_i = \Gamma(x_i)$. Then we can and will view the elements $\varphi \in \mathcal{M}_{\mathbf{k}}(\mathcal{O})$ as precisely the set of functions

$$(3.1) \qquad \varphi : \mathrm{Cl}(\mathcal{O}) \to \bigsqcup V_{\mathbf{k}}^{\Gamma_i}, \quad \varphi(x_i) \in V_{\mathbf{k}}^{\Gamma_i} \text{ for } 1 \le i \le h.$$

Namely, we can view φ as a function of $\hat{B}^\times$ by $\varphi(x) := \varphi(x, 1)$. Since $\mathrm{Cl}(\mathcal{O})$ is precisely the set of orbits of $B^\times \backslash B^\times(\mathbb{A})/\hat{\mathcal{O}}^\times$ under $B_\infty^\times$, any $\varphi \in \mathcal{M}_{\mathbf{k}}(\mathcal{O})$ is completely determined by its values on $x_1, \ldots, x_h$. Consequently $\mathcal{M}_{\mathbf{k}}(\mathcal{O}) \simeq \bigoplus V_{\mathbf{k}}^{\Gamma_i}$.

Note that viewing φ as a function of $\hat{B}^\times$ (which we do from now on except where explicated), φ is invariant under $\hat{F}^\times = Z(\hat{B}^\times)$, right $\hat{\mathcal{O}}^\times$-invariant and transforms on the left by $\rho_{\mathbf{k}}$ under $B^\times$ since

$$(3.2) \qquad \varphi(\gamma x) = \varphi(\gamma x, 1) = \varphi(x, \gamma^{-1}) = \rho_{\mathbf{k}}(\gamma)\varphi(x, 1) = \rho_{\mathbf{k}}(\gamma)\varphi(x), \quad \gamma \in B^\times.$$

If $\mathbf{k} = \mathbf{0} := (0, 0, \ldots, 0)$, then $\rho_{\mathbf{k}}$ is the trivial representation so $\mathcal{M}_{\mathbf{k}}(\mathcal{O})$ is simply the set of functions $\varphi : \mathrm{Cl}(\mathcal{O}) \to \mathbb{C}$. Here we define the Eisenstein subspace $\mathcal{E}_{\mathbf{0}}(\mathcal{O})$ to be the space of $\varphi \in \mathcal{M}_{\mathbf{0}}(\mathcal{O})$ which factors through the reduced norm $N_{B/F}$. By the assumption that $h_F^+ = 1$, $\mathcal{E}_{\mathbf{0}}(\mathcal{O}) = \mathbb{C}\mathbb{1}$, where $\mathbb{1}$ denotes the constant function on $\mathrm{Cl}(\mathcal{O})$. (For general F, $\mathcal{E}_{\mathbf{0}}(\mathcal{O})$ is h_F^+-dimensional.)

In this case, we can define a normalized inner product on $\mathcal{M}_{\mathbf{0}}(\mathcal{O})$ to be

$$(3.3) \qquad (\varphi, \varphi') = \sum \frac{1}{[\Gamma_i : \mathfrak{o}_F^\times]} \varphi(x_i)\overline{\varphi'(x_i)}.$$

Then we define the cuspidal subspace $\mathcal{S}_{\mathbf{0}}(\mathcal{O})$ of $\mathcal{M}_{\mathbf{0}}(\mathcal{O})$ to be the orthogonal complement of the Eisenstein subspace: $\mathcal{M}_{\mathbf{0}}(\mathcal{O}) = \mathcal{E}_{\mathbf{0}}(\mathcal{O}) \oplus \mathcal{S}_{\mathbf{0}}(\mathcal{O})$.

If $\mathbf{k} \ne \mathbf{0}$, then nothing nonzero in $\mathcal{M}_{\mathbf{k}}(\mathcal{O})$ can factor through $\mathbb{A}_F^\times$, so we put $\mathcal{E}_{\mathbf{k}}(\mathcal{O}) = 0$ and $\mathcal{S}_{\mathbf{k}}(\mathcal{O}) = \mathcal{M}_{\mathbf{k}}(\mathcal{O})$.

3.1. **Hecke operators.** In this section, $g \in \hat{B}^\times$ and we view elements of $\mathcal{M}_{\mathbf{k}}(\mathcal{O})$ as functions on $\hat{B}^\times$ by (3.1).

Fix a Haar measure dg on $\hat{B}^\times$ which gives $\hat{\mathcal{O}}^\times$ volume 1. For $\alpha \in \hat{B}^\times$, we associate the Hecke operator $T_\alpha : \mathcal{M}_{\mathbf{k}}(\mathcal{O}) \to \mathcal{M}_{\mathbf{k}}(\mathcal{O})$ given by

$$(3.4) \qquad (T_\alpha \varphi)(x) = \int_{\hat{\mathcal{O}}^\times \alpha \hat{\mathcal{O}}^\times} \varphi(xg)\, dg.$$

Writing $\hat{\mathcal{O}}^\times \alpha \hat{\mathcal{O}}^\times = \bigsqcup \beta_j \hat{\mathcal{O}}^\times$ for some finite collection of $\beta_j \in \hat{\mathcal{O}}^\times$, we can rewrite (3.4) as the finite sum

$$(T_\alpha \varphi)(x) = \sum \varphi(x\beta_j).$$

For $\mathfrak{p}$ a prime of F, let $\varpi_\mathfrak{p}$ denote a uniformizer in $F_\mathfrak{p}$. Then for $\mathfrak{p} \nmid \mathfrak{N}$, identify $B_\mathfrak{p}^\times$ with $\mathrm{GL}_2(F_\mathfrak{p})$ and set $\alpha_\mathfrak{p} = \begin{pmatrix} \varpi_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix} \in B_\mathfrak{p}^\times$. For $\mathfrak{p}|\mathfrak{N}$, let $E_\mathfrak{p}$ be the unramified quadratic extension of $F_\mathfrak{p}$, write $B_\mathfrak{p} = \{\begin{pmatrix} x & \varpi_\mathfrak{p} y \\ \bar{x} & \bar{y} \end{pmatrix} : x, y \in E_\mathfrak{p}\}$ and set $\alpha_\mathfrak{p} = \varpi_{B_\mathfrak{p}} = \begin{pmatrix} 0 & \varpi_\mathfrak{p} \\ 1 & 0 \end{pmatrix} \in B_p^\times$. Here we used the notation $\varpi_{B_\mathfrak{p}}$ to indicate that this element is a uniformizer for $B_\mathfrak{p}$.

For any prime $\mathfrak{p}$, let $T_\mathfrak{p} = T_{\alpha_\mathfrak{p}}$, where we view $\alpha_\mathfrak{p} \in B_\mathfrak{p}^\times$ as the element $\beta = (\beta_v)_v \in \hat{B}^\times$ satisfying $\beta_v = \alpha_\mathfrak{p}$ when $v = \mathfrak{p}$ and $\beta_v = 1$ otherwise. When $\mathfrak{p} \nmid \mathfrak{N}$, this definition agrees with the (suitably normalized) definition of unramified Hecke operators for holomorphic Hilbert modular forms.

Suppose $\mathfrak{p}|\mathfrak{N}$. Since $\mathcal{O}_\mathfrak{p}$ is the unique maximal order of $B_\mathfrak{p}$, it is fixed under conjugation by $\alpha_\mathfrak{p} = \varpi_{B_\mathfrak{p}}$. (In fact explicit calculation shows that conjugation by $\alpha_\mathfrak{p}$ in $B_\mathfrak{p}$ acts as the canonical involution of $B_\mathfrak{p}$.) Consequently $\hat{\mathcal{O}}^\times \alpha_\mathfrak{p} \hat{\mathcal{O}}^\times = \varpi_{B_\mathfrak{p}} \hat{\mathcal{O}}^\times$, and the definition of the Hecke operator means

(3.5) $$(T_\mathfrak{p} \varphi)(x) = \varphi(x\varpi_{B_\mathfrak{p}}), \quad \mathfrak{p}|\mathfrak{N}.$$

Hence, for ramified primes, since $\varpi_{B_\mathfrak{p}}^2 = \varpi_\mathfrak{p} \in Z(\hat{B}^\times)$, we have $(T_\mathfrak{p}^2 \varphi)(x) = \varphi(x\varpi_\mathfrak{p}) = \varphi(x)$, i.e., $T_\mathfrak{p}$ acts on $\mathcal{M}_\mathbf{k}(\mathcal{O})$ with order 2.

In this paper, we say $\varphi \in \mathcal{M}_\mathbf{k}(\mathcal{O})$ is an eigenform if it is an eigenfunction of *all* $T_\mathfrak{p}$'s. Then $\mathcal{M}_\mathbf{k}(\mathcal{O})$ has a basis of eigenforms as $(T_\mathfrak{p})_\mathfrak{p}$ is a commuting family of diagonalizable operators. Recall that this is not quite true for Hilbert (or elliptic) modular forms—rather, one either needs to restrict the definition of eigenforms to be eigenfunctions of the unramified Hecke operators or restrict to a subspace of newforms. In our quaternionic situation, all eigenforms are "new" because we are working with a maximal order. The diagonalizability of the ramified Hecke operators $T_\mathfrak{p}$, $\mathfrak{p}|\mathfrak{N}$, follows from the fact that they are involutions.

Any eigenform $\varphi \in \mathcal{M}_\mathbf{k}(\mathcal{O})$ lies in an irreducible cuspidal automorphic representation π of $B^\times(\mathbb{A})$ with trivial central character. (Our definition of cusp forms does not exactly match up with the usual notion of cuspidal automorphic representations—the eigenform $\mathbb{1} \in \mathcal{M}_\mathbf{0}(\mathcal{O})$ is not a cusp form, and it generates the trivial automorphic representation, which is a cuspidal representation of $B^\times(\mathbb{A})$ using the standard definition. However, it will not correspond to a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$, which is why we do not call the form $\mathbb{1}$ a cusp form.) At a ramified prime $\mathfrak{p}$, the local representation $\pi_\mathfrak{p}$ is 1-dimensional and factors through the reduced norm map $N_{B_\mathfrak{p}/F_\mathfrak{p}}$. Because we are working with trivial central character, either $\pi_\mathfrak{p}$ is the trivial representation $\mathbb{1}_\mathfrak{p}$ or the reduced norm map composed with the unramified quadratic character $\eta_\mathfrak{p}$ of $F_\mathfrak{p}^\times$. Since $T_\mathfrak{p} \varphi = \pi(\varpi_{B_\mathfrak{p}})\varphi$, we see that $T_\mathfrak{p}$ acts on φ by $+1$ (resp. -1) if $\pi_\mathfrak{p} = \mathbb{1}_\mathfrak{p}$ (resp. $\eta_\mathfrak{p} \circ N_{B_\mathfrak{p}/F_\mathfrak{p}}$).

3.2. **The Jacquet–Langlands correspondence.** The Jacquet–Langlands correspondence, proved in the setting of automorphic representations, gives an isomorphism:

$$\mathcal{S}_\mathbf{k}(\mathcal{O}) \simeq S_{\mathbf{k+2}}^{\mathrm{new}}(\mathfrak{N}),$$

where $\mathbf{2} := (2, \ldots, 2) \in \mathbb{N}^d$. This isomorphism respects the action of $T_{\mathfrak{p}}$ for $\mathfrak{p} \nmid \mathfrak{N}$, i.e., it is an isomorphism of modules for the unramified Hecke algebra. (To get the right normalization of Hecke operators, we take the convention of viewing the space of Hilbert modular forms $M_{\mathbf{k}}(\mathfrak{N})$ adelically and defining the Hecke operators analogously to (3.4).)

Let $\mathrm{St}_{\mathfrak{p}}$ denote the Steinberg representation of $\mathrm{GL}_2(F_{\mathfrak{p}})$. For $\mathfrak{p}|\mathfrak{N}$, the Atkin–Lehner operator $W_{\mathfrak{p}}$ acts on an eigenform $f \in \mathcal{S}_{\mathbf{k+2}}^{\mathrm{new}}(\mathfrak{N})$ with eigenvalue -1 (resp. $+1$) if the associated local representation $\pi_{f,\mathfrak{p}}$ is $\mathrm{St}_{\mathfrak{p}}$ (resp. $\mathrm{St}_{\mathfrak{p}} \otimes \eta_{\mathfrak{p}}$). In fact, we can take this to be the definition of the Atkin–Lehner operator on the space of Hilbert modular newforms of squarefree level. (See [SW93] for a more classical approach to Atkin–Lehner operators for Hilbert modular forms.) A standard computation shows that the (normalized) ramified Hecke eigenvalue $a_{\mathfrak{p}}(f) = -w_{\mathfrak{p}}(f)$, i.e., $T_{\mathfrak{p}} = -W_{\mathfrak{p}}$ for $\mathfrak{p}|\mathfrak{N}$.

Since the local Jacquet–Langlands correspondence associates $\mathbb{1}_{\mathfrak{p}}$ with $\mathrm{St}_{\mathfrak{p}}$ and $\eta_{\mathfrak{p}} \circ N_{B_{\mathfrak{p}}/F_{\mathfrak{p}}}$ with $\mathrm{St}_{\mathfrak{p}} \otimes \eta_{\mathfrak{p}}$, we see that the action of the ramified Hecke operators $T_{\mathfrak{p}}$ on $\mathcal{S}_{\mathbf{k}}(\mathcal{O})$ corresponds to the action of $T_{\mathfrak{p}} = -W_{\mathfrak{p}}$ on $\mathcal{S}_{\mathbf{k+2}}^{\mathrm{new}}(\mathfrak{N})$ under the Jacquet–Langlands correspondence. This can be viewed as a representation-theoretic generalization of the relationship between the Fricke involution on the space of weight 2 elliptic cusp forms and quaternionic theta series given by Pizer [Piz77].

While the Jacquet–Langlands correspondence is technically only a correspondence of cusp forms (or rather, cuspidal representations which are not 1-dimensional), we can extend the above Hecke module isomorphism to include all of $\mathcal{M}_{\mathbf{k}}(\mathcal{O})$.

Namely, it suffices to assume $\mathbf{k} = \mathbf{0}$, so $\mathcal{M}_{\mathbf{k}}(\mathcal{O})$ is just the space of $\mathbb{C}$-valued functions on $\mathrm{Cl}(\mathcal{O})$. Then $\mathcal{E}_{\mathbf{0}}(\mathcal{O}) = \mathbb{C}\mathbb{1}$, and the $\mathfrak{p}$-th eigenvalue of $\mathbb{1} \in \mathcal{E}_{\mathbf{0}}(\mathcal{O})$ is simply the degree of $T_{\mathfrak{p}}$, i.e., $1 + N(\mathfrak{p})$ if $\mathfrak{p} \nmid \mathfrak{N}$ or 1 if $\mathfrak{p}|\mathfrak{N}$. There is an Eisenstein series $E_{\mathbf{2},\mathfrak{N}} \in M_{\mathbf{2}}(\mathfrak{N})$ with these same Hecke eigenvalues for all $\mathfrak{p}$. When $F = \mathbb{Q}$, we may take $E_{2,N} := \sum_{d|N} \mu(d) d E_2(dz)$ where E_2 is the quasimodular weight 2 Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$ and μ is the Möbius function. Thus when $\mathbf{k} = \mathbf{0}$, we can extend the above Hecke module isomorphism of cuspidal spaces to a Hecke module isomorphism:
$$\mathcal{M}_{\mathbf{0}}(\mathcal{O}) \simeq \mathbb{C}E_{\mathbf{2},\mathfrak{N}} \oplus \mathcal{S}_{\mathbf{2}}^{\mathrm{new}}(\mathfrak{N}).$$
We take $w_{\mathfrak{p}}(E_{\mathbf{2},\mathfrak{N}}) = -a_{\mathfrak{p}}(E_{\mathbf{2},\mathfrak{N}}) = -1$ for all $\mathfrak{p}|\mathfrak{N}$.

We remark that for general h_F^+, the reduced norm map from B to F induces a surjective map $N_{B/F} : \mathrm{Cl}(\mathcal{O}) \to \mathrm{Cl}^+(\mathfrak{o}_F)$, and a basis of eigenforms for $\mathcal{E}_{\mathbf{0}}(\mathcal{O})$ is just the collection of maps $\lambda \circ N_{B/F}$ where λ ranges over characters of $\mathrm{Cl}^+(\mathfrak{o}_F)$. We can still extend the Jacquet–Langlands correspondence to all of $\mathcal{M}_{\mathbf{0}}(\mathcal{O})$ by associating $\lambda \circ N_{B/F}$ to $E_{\mathbf{2},\mathfrak{N}} \otimes \lambda$.

3.3. **Relation with quaternionic S-ideal classes.** Let $\mathfrak{M}$ be an integral ideal dividing $\mathfrak{N}$, which we just write as M when $F = \mathbb{Q}$. By a sign pattern $\chi = \chi_{\mathfrak{M}}$ for $\mathfrak{M}$, we mean a collection of signs $\chi_{\mathfrak{p}} \in \{\pm 1\}$ for all prime ideals $\mathfrak{p}|\mathfrak{M}$. If $\chi_{\mathfrak{p}} = +1$ (resp. -1) for all $\mathfrak{p}|\mathfrak{M}$, we denote the sign pattern by $+_{\mathfrak{M}}$ (resp. $-_{\mathfrak{M}}$). Also, if χ is a sign pattern for $\mathfrak{M}$, denote by $-\chi$ the sign pattern given by signs $-\chi_{\mathfrak{p}}$ for all $\mathfrak{p}|\mathfrak{M}$.

Consider the subspace of $\mathcal{M}_{\mathbf{k}}(\mathcal{O})$ with this collection of Hecke signs:
$$\mathcal{M}_{\mathbf{k}}^{\chi}(\mathcal{O}) = \langle \varphi \in \mathcal{M}_{\mathbf{k}}(\mathcal{O}) \text{ is an eigenform} : T_{\mathfrak{p}}\varphi = \chi_{\mathfrak{p}}\varphi \text{ for all } \mathfrak{p}|\mathfrak{M} \rangle.$$
Similarly we define $\mathcal{S}_{\mathbf{k}}^{\chi}(\mathcal{O}) = \mathcal{M}_{\mathbf{k}}^{\chi}(\mathcal{O}) \cap \mathcal{S}_{\mathbf{k}}(\mathcal{O})$. Note that $\mathcal{M}_{\mathbf{k}}^{\chi}(\mathcal{O}) = \mathcal{S}_{\mathbf{k}}^{\chi}(\mathcal{O}) \oplus \mathbb{C}\mathbb{1}$ if $\mathbf{k} = \mathbf{0}$ and $\chi = +_{\mathfrak{M}}$; otherwise $\mathcal{M}_{\mathbf{k}}^{\chi}(\mathcal{O}) = \mathcal{S}_{\mathbf{k}}^{\chi}(\mathcal{O})$.

To keep notation consistent with [Mar2] when $F = \mathbb{Q}$, we denote the space of Hilbert newforms with fixed Atkin–Lehner (rather than Hecke) signs by

$$S_{\mathbf{k}}^{\mathrm{new},\varepsilon}(\mathfrak{N}) = \langle f \in S_{\mathbf{k}}(\mathfrak{N}) \text{ is a newform} : W_{\mathfrak{p}}f = \varepsilon_{\mathfrak{p}}f \text{ for all } \mathfrak{p}|\mathfrak{M}\rangle,$$

for a sign pattern ε for $\mathfrak{M}$. The description of the Jacquet–Langlands correspondence above tells us we have Hecke module isomorphisms:

$$\text{(3.6)} \qquad \mathcal{S}_{\mathbf{k}}^{\chi}(\mathcal{O}) \simeq S_{\mathbf{k+2}}^{\mathrm{new},-\chi}(\mathfrak{N}),$$

and

$$\text{(3.7)} \qquad \mathcal{M}_{\mathbf{k}}^{\chi}(\mathcal{O}) \simeq \begin{cases} \mathbb{C}E_{2,\mathfrak{N}} \oplus S_{\mathbf{2}}^{\mathrm{new},-\chi}(\mathfrak{N}) & \text{if } \mathbf{k} = \mathbf{0} \text{ and } \chi = +_{\mathfrak{M}}, \\ S_{\mathbf{k+2}}^{\mathrm{new},-\chi}(\mathfrak{N}) & \text{else.} \end{cases}$$

If $\varphi \in \mathcal{M}_{\mathbf{k}}^{\chi}(\mathcal{O})$, then it is right $B_{\mathfrak{p}}^{\times}$-invariant (i.e., $\varphi(x\alpha_{\mathfrak{p}}) = \varphi(x)$ for all $\alpha_{\mathfrak{p}} \in B_{\mathfrak{p}}^{\times}$) if and only if $\chi_{\mathfrak{p}} = +1$. This implies we can view forms in $\mathcal{M}_{\mathbf{k}}^{+\mathfrak{M}}(\mathcal{O})$ as certain functions on $\mathrm{Cl}_S(\mathcal{O})$. In particular, for weight zero we see that

$$\text{(3.8)} \qquad \mathcal{M}_{\mathbf{0}}^{+\mathfrak{M}}(\mathcal{O}) \simeq \{\varphi : \mathrm{Cl}_{\mathfrak{M}}(\mathcal{O}) \to \mathbb{C}\}.$$

Hence

$$\text{(3.9)} \qquad h_{B,\mathfrak{M}} = \dim \mathcal{M}_{\mathbf{0}}^{+\mathfrak{M}}(\mathcal{O}) = 1 + \dim S_{\mathbf{2}}^{\mathrm{new},-\mathfrak{M}}(\mathfrak{N}).$$

We remark that when $F = \mathbb{Q}$ and $N = p$, we have $h_{B,p} = t_B$ so (3.9) yields $t_B = 1 + S_2^{\mathrm{new},-p}(p)$, which was already known to Deuring. More generally, but still with $F = \mathbb{Q}$, a relation between type numbers and the full (not new) space of cusp forms with given Atkin–Lehner eigenvalues was given by Hasegawa and Hashimoto [HH95], which is similar to, but slightly different than, (3.9). Note they do not restrict to squarefree level, and their approach is essentially to use explicit formulas for type numbers and dimensions, rather than looking through the lens of the Jacquet–Langlands correspondence as we do here.

When $F = \mathbb{Q}$, a formula for $\dim S_k^{\mathrm{new},\varepsilon}(N)$ was given in [Mar2], This translates into an explicit formula for the S-ideal class numbers $h_{B,S}$ by (3.9). The general case is somewhat complicated, so here we just explain the formula in a simple case which will arise for us later: when $S = \{p\}$, we have $h_{B,p} = \frac{1}{2}h_B = \frac{1}{2}(1 + \dim S_2^{\mathrm{new}}(N))$ if (and only if) p satisfies condition (a), (b), or (c) of Proposition 5.5 below.

In the next section, we will generalize (3.8) to treat spaces $\mathcal{M}_{\mathbf{k}}^{\chi}(\mathcal{O})$ of higher weight and other sign patterns χ.

4. Action of local involutions

Keep the notation of the previous section. Here, for a prime $\mathfrak{p}$ at which B is ramified, we will study the action of $\varpi_{B_{\mathfrak{p}}}$ on $\mathrm{Cl}(\mathcal{O})$. This will give a "local involution" $\sigma_{\mathfrak{p}}$ on the global space $\mathrm{Cl}(\mathcal{O})$, which by (3.5) will tell us about the action of ramified Hecke operators on $\mathcal{M}_{\mathbf{k}}(\mathcal{O})$. This will result in an algebro-combinatorial description of the spaces $\mathcal{M}_{\mathbf{k}}^{\chi}(\mathcal{O})$ for prescribed sign patterns χ.

4.1. **Action on ideal classes.** Let $\mathfrak{p}$ be a prime at which B ramifies. For $S = \{\mathfrak{p}\}$, we also write $\mathrm{Cl}_S(\mathcal{O})$ as $\mathrm{Cl}_{\mathfrak{p}}(\mathcal{O})$. Now we have a surjective map

$$\text{(4.1)} \qquad \mathrm{Cl}(\mathcal{O}) \to \mathrm{Cl}_{\mathfrak{p}}(\mathcal{O})$$

given by quotienting out on the right by $B_{\mathfrak{p}}^\times$. Since $B_{\mathfrak{p}}^\times = F_{\mathfrak{p}}^\times(\mathcal{O}_{\mathfrak{p}}^\times \sqcup \varpi_{B_{\mathfrak{p}}}\mathcal{O}_{\mathfrak{p}}^\times)$, given any $x \in \hat{B}^\times$ the associated $\{\mathfrak{p}\}$-ideal class $[x]_{\mathfrak{p}} := B^\times x\hat{\mathcal{O}}^\times B_{\mathfrak{p}}^\times$ is either $[x]$ or $[x] \sqcup [x\varpi_{B_p}]$, where $[x] := B^\times x\hat{\mathcal{O}}^\times$. Thus the map (4.1) has fibers of size 1 or 2.

Put another way, right multiplication by $\varpi_{B_{\mathfrak{p}}}$ induces an involution, i.e. a permutation of order 2, on $\mathrm{Cl}(\mathcal{O})$, and the orbits of this involution are precisely the fibers of (4.1). Denote this involution by $\sigma_{\mathfrak{p}}$, so $\sigma_{\mathfrak{p}}([x]) = [x\varpi_{B_{\mathfrak{p}}}]$ for any $x \in \hat{B}^\times$.

It will be useful to know certain objects associated to ideal classes are invariant under $\sigma_{\mathfrak{p}}$.

For a right ideal $\mathcal{I}$ of $\mathcal{O}$, let $\mathcal{O}_l(\mathcal{I}) = \{\alpha \in B : \alpha\mathcal{I} \subset \mathcal{I}\}$ denote the left order of $\mathcal{I}$. If $\mathcal{I}$ corresponds to x, we also write the left order as $\mathcal{O}_l(x)$. Note $x\hat{\mathcal{O}}x^{-1} \cap B$ is a maximal order of B since it locally is. Since it preserves $x\hat{\mathcal{O}}$ by left multiplication, we have $\mathcal{O}_l(x) = x\hat{\mathcal{O}}x^{-1} \cap B$. From this it is easy to see that $\mathcal{O}_l(x) = \mathcal{O}_l(x')$ for $x' \in [x]$, so we may unambiguously call this the left order $\mathcal{O}_l([x])$ of the ideal class $[x]$. Similarly, since $\Gamma(x) = \mathcal{O}_l(x)^\times$, this group only depends on $[x]$ and we may also write it as $\Gamma([x])$.

Lemma 4.1. *For $x \in \hat{B}^\times$, $\mathcal{O}_l([x]) = \mathcal{O}_l(\sigma_{\mathfrak{p}}([x]))$ and $\Gamma([x]) = \Gamma(\sigma_{\mathfrak{p}}([x]))$.*

Proof. It suffices to prove the statement about left orders. By the above adelic description of left orders, it suffices to show $\hat{\mathcal{O}}^\times = \varpi_{B_{\mathfrak{p}}}\hat{\mathcal{O}}^\times\varpi_{B_{\mathfrak{p}}}^{-1}$. Clearly these groups are the same away from $\mathfrak{p}$, and they are the same at $\mathfrak{p}$ since $B_{\mathfrak{p}}$ has a unique maximal order. $\square$

In this subsection, we needed to distinguish between x, $[x]$ and $[x]_{\mathfrak{p}}$ for $x \in \hat{B}^\times$, but below this is less crucial so we will use x_i for an both element of $\mathrm{Cl}(\mathcal{O})$ and a representative in $\hat{B}^\times$ as in Section 3.

4.2. **Action on quaternionic modular forms.** Fix a set of representatives $x_1, \ldots, x_h$ for $\mathrm{Cl}(\mathcal{O})$ and let $\mathfrak{p}|\mathfrak{N}$. Then we may view $\sigma_{\mathfrak{p}}$ as a permutation on $\{x_1, \ldots, x_h\}$. Writing $\sigma_{\mathfrak{p}}(x_i) = \gamma x_i\varpi_{B_{\mathfrak{p}}}u$ for some $\gamma \in B^\times$, $u \in \hat{\mathcal{O}}^\times$, then by (3.2) we see

$$\varphi(\sigma_{\mathfrak{p}}(x_i)) = \varphi(\gamma x_i\varpi_{B_{\mathfrak{p}}}) = \rho_{\mathbf{k}}(\gamma)\varphi(x_i\varpi_{B_{\mathfrak{p}}}).$$

Note that $\gamma^{-1} \in \Gamma^{\sigma_{\mathfrak{p}}}(x_i) := x_i\varpi_{B_{\mathfrak{p}}}\hat{\mathcal{O}}^\times\sigma_{\mathfrak{p}}(x_i)^{-1} \cap B^\times$. Thus the ramified Hecke action in (3.5) can be rewritten as

$$(4.2) \qquad (T_{\mathfrak{p}}\varphi)(x_i) = \rho_{\mathbf{k}}(\gamma)\varphi(\sigma_{\mathfrak{p}}(x_i)), \quad \text{for some } \gamma \in \Gamma^{\sigma_{\mathfrak{p}}}(x_i), \text{ for all } 1 \leq i \leq h.$$

We remark that for any fixed $\gamma_0 \in \Gamma^{\sigma_{\mathfrak{p}}}(x_i)$, we can write any $\gamma \in \Gamma^{\sigma_{\mathfrak{p}}}(x_i)$ as $\gamma = \gamma_0\gamma'$ where $\gamma' \in \Gamma(\sigma_{\mathfrak{p}}(x_i))$. Hence if the equation in (4.2) holds for a fixed i and some $\gamma \in \Gamma^{\sigma_{\mathfrak{p}}}(x_i)$, it holds for all such γ for that i by (3.1).

Now let χ be a sign pattern for some $\mathfrak{M}|\mathfrak{N}$, and let $\gamma_{i,\mathfrak{p}} \in \Gamma^{\sigma_{\mathfrak{p}}}(x_i)$ for each $1 \leq i \leq h$, $\mathfrak{p}|\mathfrak{M}$. Then for $\varphi \in \mathcal{M}_{\mathbf{k}}(\mathcal{O})$, we see that $\varphi \in \mathcal{M}_{\mathbf{k}}^\chi(\mathcal{O})$ if and only if

$$(4.3) \qquad \varphi(x_i) = \chi_{\mathfrak{p}}\rho_{\mathbf{k}}(\gamma_{i,\mathfrak{p}})\varphi(\sigma_{\mathfrak{p}}(x_i)), \quad \text{for } 1 \leq i \leq h, \mathfrak{p}|\mathfrak{M}.$$

In the case $\mathbf{k} = \mathbf{0}$ so $\rho_{\mathbf{k}}$ is trivial, (4.3) simply becomes

$$(4.4) \qquad \varphi(x_i) = \chi_{\mathfrak{p}}\varphi(\sigma_{\mathfrak{p}}(x_i)), \quad \text{for } 1 \leq i \leq h, \mathfrak{p}|\mathfrak{M}.$$

If $\sigma_{\mathfrak{p}}(x_i) = x_i$, put $V_{\mathbf{k}}^{\Gamma_i, \chi_{\mathfrak{p}}} = \{v \in V_{\mathbf{k}}^{\Gamma_i} : \rho_{\mathbf{k}}(\gamma_{i,\mathfrak{p}})v = \chi_{\mathfrak{p}}v\}$. Note that in this case $\gamma_{i,\mathfrak{p}}^2 \in Z(B^\times)$, so $\gamma_{i,\mathfrak{p}}$ acts as an involution and we have $V_{\mathbf{k}}^{\Gamma_i} \simeq V_{\mathbf{k}}^{\Gamma_i, +_{\mathfrak{p}}} \oplus V_{\mathbf{k}}^{\Gamma_i, -_{\mathfrak{p}}}$. If x_i is not fixed by $\sigma_{\mathfrak{p}}$, put $V_{\mathbf{k}}^{\Gamma_i, \chi_{\mathfrak{p}}} = V_{\mathbf{k}}^{\Gamma_i}$.

Lemma 4.2. *Fix $\chi_{\mathfrak{p}}$ a sign for some $\mathfrak{p}|\mathfrak{N}$. Order $x_1, \ldots, x_h$ so that $x_1, \ldots, x_t$ is a set of representatives for $\mathrm{Cl}_{\mathfrak{p}}(\mathcal{O})$, where $t = h_{B,\mathfrak{p}}$. Then we have an isomorphism*

$$\mathcal{M}_{\mathbf{k}}^{\chi_{\mathfrak{p}}}(\mathcal{O}) \simeq \{\varphi : \mathrm{Cl}_{\mathfrak{p}}(\mathcal{O}) \to \bigsqcup V_{\mathbf{k}}^{\Gamma_i, \chi_{\mathfrak{p}}} \mid \varphi(x_i) \in V_{\mathbf{k}}^{\Gamma_i, \chi_{\mathfrak{p}}} \text{ for } 1 \leq i \leq t\}.$$

Proof. Let φ be an element of the set on the right, which we temporarily denote by $A(\chi_{\mathfrak{p}})$. Then we extend φ to $\mathrm{Cl}(\mathcal{O})$ as follows: for $t < j \leq h$, write $x_j = \sigma_{\mathfrak{p}}(x_i)$ for some $1 \leq i \leq t$, and put $\varphi(x_j) = \chi_{\mathfrak{p}} \rho_{\mathbf{k}}(\gamma_{j,\mathfrak{p}}) \varphi(x_i)$. Note that $\varphi(x_j) \in V_{\mathbf{k}}^{\Gamma_j}$ by Lemma 4.1. This defines an embedding of $A(\chi_{\mathfrak{p}})$ into $\mathcal{M}_{\mathbf{k}}^{\chi_{\mathfrak{p}}}(\mathcal{O})$. We will show surjectivity by a dimension argument.

For $1 \leq i \leq t$, let $A_i(\chi_{\mathfrak{p}})$ be the subspace of $A(\chi_{\mathfrak{p}})$ consisting of elements φ such that $\varphi(x_j) = 0$ if $i \neq j$, $1 \leq j \leq t$. If $\sigma_{\mathfrak{p}}$ fixes x_i, then $V_{\mathbf{k}}^{\Gamma_i} \simeq V_{\mathbf{k}}^{\Gamma_i, +\mathfrak{p}} \oplus V_{\mathbf{k}}^{\Gamma_i, -\mathfrak{p}}$ implies $\dim A_i(+\mathfrak{p}) + \dim A_i(-\mathfrak{p}) = \dim V_{\mathbf{k}}^{\Gamma_i}$. Otherwise $\sigma_{\mathfrak{p}}(x_i) = x_j$ for some $j > t$, and $\dim A_i(+\mathfrak{p}) = \dim A_i(-\mathfrak{p}) = \dim V_{\mathbf{k}}^{\Gamma_i} = \dim V_{\mathbf{k}}^{\Gamma_j}$. Hence

$$\dim A(+\mathfrak{p}) + \dim A(-\mathfrak{p}) = \sum_{i=1}^{h} \dim V_{\mathbf{k}}^{\Gamma_i} = \dim \mathcal{M}_{\mathbf{k}}(\mathcal{O}),$$

and thus our embedding of $A(\chi_{\mathfrak{p}})$ into $\mathcal{M}_{\mathbf{k}}^{\chi_{\mathfrak{p}}}(\mathcal{O})$ must be surjective. $\qquad\square$

There are two situations where the above description of $\mathcal{M}_{\mathbf{k}}^{\chi_{\mathfrak{p}}}(\mathcal{O})$ becomes simpler. First, if $\sigma_{\mathfrak{p}}$ has no fixed points, then we can identify this space of forms with the functions φ on $\mathrm{Cl}_{\mathfrak{p}}(\mathcal{O})$ such that $\varphi(x_i) \in V_{\mathbf{k}}^{\Gamma_i}$ for each $1 \leq i \leq t$. Second, if $\mathbf{k} = \mathbf{0}$ then we can identify this space with functions $\varphi : \mathrm{Cl}_{\mathfrak{p}}(\mathcal{O}) \to \mathbb{C}$ such that $\varphi(x_i) = 0$ if $\sigma_{\mathfrak{p}}(x_i) = x_i$ and $\chi_{\mathfrak{p}} = -1$.

4.3. **Actions without fixed points.** Let $s_{\mathfrak{p}}$ denote the number of orbits of size 2 for $\sigma_{\mathfrak{p}}$, so $h - 2s_{\mathfrak{p}}$ is the number of fixed points of $\sigma_{\mathfrak{p}}$. For $\varphi \in \mathcal{M}_{\mathbf{0}}(\mathcal{O})$, note the equation $T_{\mathfrak{p}}\varphi = \varphi$ imposes $s_{\mathfrak{p}}$ linear constraints on φ: $\varphi(x_i) = \varphi(\sigma_{\mathfrak{p}}(x_i))$ for x_i in any orbit of size 2. On the other hand, $T_{\mathfrak{p}}\varphi = -\varphi$ forces $\varphi(x_i) = 0$ for any x_i fixed by $\sigma_{\mathfrak{p}}$ and $\varphi(x_i) = -\varphi(\sigma_{\mathfrak{p}}(x_i))$ for x_i in an orbit of size 2. Hence for a sign pattern $\chi_{\mathfrak{p}}$ for $\mathfrak{p}$, we have

$$(4.5) \qquad \dim \mathcal{M}_{\mathbf{0}}^{\chi_{\mathfrak{p}}}(\mathcal{O}) = \begin{cases} h - s_{\mathfrak{p}} & \chi_{\mathfrak{p}} = +1 \\ s_{\mathfrak{p}} & \chi_{\mathfrak{p}} = -1. \end{cases}$$

Consequently, we can compute $s_{\mathfrak{p}}$ from (3.7) and a dimension formula for $S_2^{\mathrm{new}, -\chi_{\mathfrak{p}}}(\mathfrak{N})$. In particular, $\sigma_{\mathfrak{p}}$ acts without fixed points if and only if

$$\dim S_2^{\mathrm{new}, +\mathfrak{p}}(\mathfrak{N}) = \dim S_2^{\mathrm{new}, -\mathfrak{p}}(\mathfrak{N}) + 1.$$

Now we assume $F = \mathbb{Q}$, and will use a trace formula for the Atkin–Lehner operator W_p on $S_2^{\mathrm{new}}(N)$ from [Mar2] to give necessary and sufficient criteria for σ_p to act on $\mathrm{Cl}(\mathcal{O})$ without fixed points, which is equivalent to $s_p = \frac{h}{2}$.

Lemma 4.3. *Let $p|N$.*

(a) For $p > 2$, $s_p = \frac{h}{2}$ if and only if $\left(\frac{-p}{q}\right) = 1$ for some odd prime $q|N$ or if N is even and $p \equiv 7 \bmod 8$.

(b) For $p = 2$, $s_p = \frac{h}{2}$ if and only if N is divisible by a prime which is 1 mod 4 and $\left(\frac{-2}{q}\right) = 1$ for some prime $q|N$.

Proof. By (4.5), $s_p = \frac{h}{2}$ if and only if $\dim S_2^{\mathrm{new},+_p} = 1 + \dim S_2^{\mathrm{new},-_p}$, i.e., if and only if $\mathrm{tr}_{S_2^{\mathrm{new}}(N)} W_p = 1$. This trace is computed in [Mar2, Prop 1.4].

Let $N' = N/p$. For $m \in \mathbb{N}$, let $m_{\mathrm{odd}} = 2^{-v_2(m)} m$ be the odd part of m. We define a constant $b(p, N')$ by the following table:

$p \bmod 8$	$b(p, N')$ for N' odd	$b(p, N')$ for N' even
1, 2, 5, 6	1	-1
3	4	-2
7	2	0

If $p > 3$, the trace of interest is

$$(4.6) \qquad \mathrm{tr}_{S_2^{\mathrm{new}}(N)} W_p = 1 - \frac{1}{2}|\mathrm{Cl}(\mathbb{Q}(\sqrt{-p}))|b(p, N') \prod_{q | N'_{\mathrm{odd}}} \left(\left(\frac{-p}{q} \right) - 1 \right).$$

This is 1 if and only if the second term on the right is 0, which gives part (a) when $p > 3$. If $p = 3$, this trace is

$$(4.7) \qquad \mathrm{tr}_{S_2^{\mathrm{new}}(N)} W_3 = 1 - (-1)^{v_2(N')} \prod_{q | N'_{\mathrm{odd}}} \left(\left(\frac{-3}{q} \right) - 1 \right).$$

This finishes (a).

If $p = 2$, this trace is

$$(4.8) \qquad \mathrm{tr}_{S_2^{\mathrm{new}}(N)} W_2 = 1 - \frac{1}{2} \left(\prod_{q | N'} \left(\left(\frac{-2}{q} \right) - 1 \right) + \prod_{q | N'} \left(\left(\frac{-1}{q} \right) - 1 \right) \right).$$

This gives (b). $\qquad\qquad\square$

We remark that knowing the traces of the Atkin–Lehner operator W_p on $S_2^{\mathrm{new}}(N)$ is the same as knowing the S-ideal class numbers $h_{B,p}$ together with h (see (3.9) and [Mar2]), so one may view the above result as an application of formulas for S-ideal class numbers, i.e., an application of the refined dimension formulas for $S_2^{\mathrm{new},\varepsilon}(N)$.

4.4. **Weight zero spaces.** To study the spaces $\mathcal{M}_{\mathbf{k}}^\chi(\mathcal{O})$ in more detail, we need to understand how the involutions $\sigma_{\mathfrak{p}}$ interact for the various primes $\mathfrak{p}|\mathfrak{M}$. It will be convenient to describe this in terms of a graph. The general case is somewhat complicated, so here we treat weight zero before discussing higher weights.

Fix an integral ideal $\mathfrak{M}|\mathfrak{N}$ and a sign pattern χ for $\mathfrak{M}$. We associate to χ a (signed multi)graph Σ_χ as follows. Let the vertex set of Σ_χ be $\mathrm{Cl}(\mathcal{O}) = \{x_1, \ldots, x_h\}$. For $\mathfrak{p}|\mathfrak{M}$, let $E(\chi_{\mathfrak{p}})$ denote the set of signed edges $\{\chi_{\mathfrak{p}} \cdot (x_i, \sigma_{\mathfrak{p}}(x_i))\}$ where x_i runs over a complete set of representatives for the orbits of $\sigma_{\mathfrak{p}}$. (By signed edges, we mean weighted edges, where the weights are ± 1 according to whether $\chi_{\mathfrak{p}} = \pm 1$.) Then we let the edge set of Σ_χ be the disjoint union of the $E(\chi_{\mathfrak{p}})$'s. In other words, to construct our graph Σ_χ on $\mathrm{Cl}(\mathcal{O})$, for all $1 \le i \le j \le h$ and $\mathfrak{p}|\mathfrak{M}$, we add an (undirected) edge between x_i and x_j with sign $\chi_{\mathfrak{p}}$ if and only if $x_j = \sigma_{\mathfrak{p}}(x_i)$. Note that Σ_χ may have loops as well as multiple edges with the same or opposite signs.

Let $X_1, \ldots, X_t$ denote the (vertex sets of the) connected components of Σ_χ. We note that $X_1, \ldots, X_t$ do not depend upon χ—the sign pattern only affects the signs of the edges in Σ_χ. Moreover, x_j lies in the connected component of x_i if and only

if it lies in the orbit of x_i under the permutation group generated by $\{\sigma_{\mathfrak{p}} : \mathfrak{p} | \mathfrak{M}\}$. By the description of $\sigma_{\mathfrak{p}}$ in terms of (4.1), this is equivalent to x_j lying in the same S-ideal class as x_i, where $S = \{\mathfrak{p} : \mathfrak{p} | \mathfrak{M}\}$. Hence, viewing the S-ideal classes as subsets of $\mathrm{Cl}(\mathcal{O})$, we may write $\mathrm{Cl}_S(\mathcal{O}) = \{X_1, \ldots, X_t\}$, and we see $t = h_{B,S}$.

Let E_i be the edge set for X_i in Σ_χ and partition $E_i = E_i^+ \sqcup E_i^-$, where $E_i^{\pm}$ denotes the subset of edges with sign ± 1. We say X_i is χ-admissible if there is a partition $X_i = X_i^+ \sqcup X_i^-$ such that the set of edges in E_i which connect a vertex in X_i^+ with a vertex in X_i^- is precisely E_i^-. In this case, we call the partition $X_i^+ \sqcup X_i^-$ χ-admissible. Note that if $\chi = +_{\mathfrak{M}}$, then $X_i^+ = X_i$ and $X_i^- = \emptyset$ is always a χ-admissible partition of X_i.

Denote the set of χ-admissible $X_i \in \mathrm{Cl}_S(\mathcal{O})$ by $\mathrm{Cl}_S(\mathcal{O})^{\chi-\mathrm{adm}}$.

Proposition 4.4. *Let χ be a sign pattern for $\mathfrak{M}|\mathfrak{N}$, $S = \{\mathfrak{p} : \mathfrak{p}|\mathfrak{M}\}$, and write $\mathrm{Cl}_S(\mathcal{O}) = \{X_1, \ldots, X_t\}$. Then we have an isomorphism*

$$\mathcal{M}_{\mathbf{0}}^\chi(\mathcal{O}) \simeq \{\varphi : \mathrm{Cl}_S(\mathcal{O})^{\chi-\mathrm{adm}} \to \mathbb{C}\}.$$

Note that when $\chi = +_{\mathfrak{M}}$, every class in $\mathrm{Cl}_S(\mathcal{O})$ is χ-admissible so this gives (3.8).

Proof. Order $x_1, \ldots, x_h$ so that $x_i \in X_i$ for $1 \le i \le t$. Let $\varphi \in \mathcal{M}_{\mathbf{0}}^\chi(\mathcal{O})$. By (4.4), if x_{j_1} are x_{j_2} are vertices in X_i connected by an edge with sign ± 1, then $\varphi \in \mathcal{M}_{\mathbf{0}}^\chi(\mathcal{O})$ means $\varphi(x_{j_1}) = \pm\varphi(x_{j_2})$. Hence the value of $\varphi(x_j)$ is determined by $\varphi(x_i)$ (namely, is $\pm\varphi(x_i)$) whenever $x_j \in X_i$. This gives a map from $\mathcal{M}_{\mathbf{0}}^\chi(\mathcal{O})$ into the space of functions on $\mathrm{Cl}_S(\mathcal{O})^{\chi-\mathrm{adm}}$ by restricting φ to be a function on the elements x_i, $1 \le i \le t$, such that X_i is χ-admissible.

To show this map is a bijection, it suffices to show that for $1 \le i \le t$ there exists $\varphi \in \mathcal{M}_{\mathbf{0}}^\chi(\mathcal{O})$ such that $\varphi(x_i) \ne 0$ if and only if X_i is χ-admissible. If $\varphi \in \mathcal{M}_{\mathbf{0}}^\chi(\mathcal{O})$ with $\varphi(x_i) \ne 0$, then the partition of X_i into the two sets $X_i^+ = \{x_j \in X_i : \varphi(x_j) = \varphi(x_i)\}$ and $X_i^- = \{x_j \in X_i : \varphi(x_j) = -\varphi(x_i)\}$ is a χ-admissible partition of X_i. Conversely, if $X_i^+ \sqcup X_i^-$ is a χ-admissible partition of X_i, then we can define an element of $\varphi \in \mathcal{M}_{\mathbf{0}}^\chi(\mathcal{O})$ by setting $\varphi(x_j) = \pm 1$ if $x_j \in X_i^{\pm}$ and $\varphi(x_j) = 0$ if $x_j \notin X_i$. $\square$

Thus $\dim \mathcal{M}_{\mathbf{0}}^\chi(\mathcal{O})$ is the number of χ-admissible classes in $\mathrm{Cl}_S(\mathcal{O})$, which generalizes (3.9). For congruences applications, we want to know more about which X_i are admissible. Clearly we have

Corollary 4.5. *All $X_i \in \mathrm{Cl}_S(\mathcal{O})$ are χ-admissible if and only if $\dim \mathcal{M}_{\mathbf{0}}^\chi(\mathcal{O}) = \dim \mathcal{M}_{\mathbf{0}}^{+_{\mathfrak{M}}}(\mathcal{O})$.*

It does not seem easy to say exactly what Σ_χ looks like in general, however we can get some information from considering how the edge sets $E(\chi_{\mathfrak{p}})$ can interact for various $\mathfrak{p}$.

Lemma 4.6. *If $\mathfrak{M} = \mathfrak{p}\mathfrak{M}_0$ and $X \in \mathrm{Cl}_{\mathfrak{M}_0}(\mathcal{O})$, then there exists $X' \in \mathrm{Cl}_{\mathfrak{M}_0}(\mathcal{O})$ such that $x_i \in X$ implies $\sigma_{\mathfrak{p}}(x_i) \in X'$.*

Proof. The projection $\mathrm{Cl}_{\mathfrak{M}_0}(\mathcal{O}) \to \mathrm{Cl}_{\mathfrak{M}}(\mathcal{O})$ has fibers of size 1 or 2. If the fiber containing X has size 1, the lemma is true with $X' = X$. Otherwise, let X' be the other element in the fiber containing X. Then there exists $x_i \in X$ such that $\sigma_{\mathfrak{p}}(x_i) \in X'$, i.e., $x_i \varpi_{B_{\mathfrak{p}}} \in X'$. One easily sees that this implies $x_j \varpi_{B_{\mathfrak{p}}} \in X'$ for all $x_j \in X = B^\times x_i \hat{\mathcal{O}}^\times \prod_{\mathfrak{q}|\mathfrak{M}_0} B_{\mathfrak{q}}^\times$. $\square$

Thus if we think of building Σ_χ in stages by adding the edge sets $E(\chi_\mathfrak{p})$ one prime at a time, we see that at each stage each connected component comprises exactly one or two connected components from the previous stage. Furthermore, if a connected component is obtained by linking two connected components X and X' from the previous stage, then involution $\sigma_\mathfrak{p}$ linking X and X' must be a bijection between the set of right $\mathcal{O}$-ideal classes in X and those in X'.

Consequently, each connected component $X_i \in \Sigma_\chi$ has cardinality 2^m for some $0 \leq m \leq 2^{\omega(\mathfrak{M})}$.

4.5. Admissibility in higher weight.
Now we return to arbitrary weight $\mathbf{k} \in (2\mathbb{Z}_{\geq 0})^d$.

As before, let $\mathfrak{M}|\mathfrak{N}$ and put $S = \{\mathfrak{p}|\mathfrak{M}\}$. Write $\mathrm{Cl}_S(\mathcal{O}) = \{X_1, \ldots, X_t\}$ and $\mathrm{Cl}(\mathcal{O}) = \{x_1, \ldots, x_h\}$ with $x_i \in X_i$ for $1 \leq i \leq t$. For a sign pattern χ for $\mathfrak{M}$, we say X_i is χ-admissible in weight $\mathbf{k}$ if for any $v \in V_\mathbf{k}^{\Gamma_i}$ there exists $\varphi \in \mathcal{M}_\mathbf{k}^\chi(\mathcal{O})$ such that $\varphi(x_i) = v$. By the proof of Proposition 4.4, being χ-admissible in weight $\mathbf{0}$ is just the notion of χ-admissible from the previous section.

If every X_i is χ-admissible in weight $\mathbf{k}$, then similar to previous sections to we get an isomorphism

$$(4.9) \qquad \mathcal{M}_\mathbf{k}^\chi(\mathcal{O}) \simeq \{\varphi : \mathrm{Cl}_S(\mathcal{O}) \to \bigsqcup V_\mathbf{k}^{\Gamma_i} \mid \varphi(x_i) \in V_\mathbf{k}^{\Gamma_i} \text{ for } 1 \leq i \leq t\}$$

by simply restricting $\varphi \in \mathcal{M}_\mathbf{k}^\chi(\mathcal{O})$ to $x_1, \ldots, x_t$. Without the admissibility condition, there is always an injection from the set on the left to the set on the right, and we see that $\dim \mathcal{M}_\mathbf{k}^\chi(\mathcal{O}) = \sum_{i=1}^t \dim V_\mathbf{k}^{\Gamma_i}$ if and only if each X_i is χ-admissible in weight $\mathbf{k}$.

Lemma 4.7. *Suppose* $\dim \mathcal{M}_\mathbf{0}^\chi(\mathcal{O}) = \dim \mathcal{M}_\mathbf{0}^{\chi'}(\mathcal{O})$ *for any choices of sign patterns* χ, χ' *for* $\mathfrak{M}$. *Then for any* $\mathbf{k}$, *sign pattern* χ *for* $\mathfrak{M}$ *and* $X_i \in \mathrm{Cl}_S(\mathcal{O})$, *we have that* X_i *is* χ-admissible *in weight* $\mathbf{k}$. *Moreover, for fixed* $\mathbf{k}$, *the spaces* $\mathcal{M}_\mathbf{k}^\chi(\mathcal{O})$ *have the same dimension for all* χ.

Proof. We prove this by induction on $\mathfrak{M}$. It is vacuously true for $\mathfrak{M} = \mathfrak{o}_F$, so suppose $\mathfrak{M} = \mathfrak{p}_0\mathfrak{M}_0$ and assume the lemma is true for $\mathfrak{M}_0$. Write $\mathrm{Cl}_S(\mathcal{O}) = \{X_1, \ldots, X_t\}$ and order $x_1, \ldots, x_h$ so $x_i \in X_i$ for $1 \leq i \leq t$. Put $S_0 = \{\mathfrak{p}|\mathfrak{M}_0\}$. The hypothesis in the lemma with χ, χ' taken to be the two sign patterns for $\mathfrak{M}$ which restrict to $+_{\mathfrak{M}_0}$ for $\mathfrak{M}_0$ implies $\mathrm{Cl}_S(\mathcal{O}) = \frac{1}{2}\mathrm{Cl}_{S_0}(\mathcal{O})$ by (3.9). Then for any $X_i \in \mathrm{Cl}_S(\mathcal{O})$, we may write $X_i = Y_i \sqcup Y_i'$ where $Y_i, Y_i' \in \mathrm{Cl}_{S_0}(\mathcal{O})$. By Lemma 4.1, the Γ_j's are the same for all $x_j \in X_i$.

Fix a sign pattern χ for $\mathfrak{M}$ and let χ_0 be the restriction of χ to S_0. Let χ' be the extension of χ_0 to S such that $\chi'_{\mathfrak{p}_0} = -\chi_{\mathfrak{p}_0}$. On one hand, we have

$$\dim \mathcal{M}_\mathbf{k}^{\chi_0}(\mathcal{O}) = 2 \sum_{i=1}^t \dim V_\mathbf{k}^{\Gamma_i}.$$

On the other hand, we have

$$\dim \mathcal{M}_\mathbf{k}^{\chi_0}(\mathcal{O}) = \dim \mathcal{M}_\mathbf{k}^\chi(\mathcal{O}) + \dim \mathcal{M}_\mathbf{k}^{\chi'}(\mathcal{O}).$$

But each of the dimensions on the right is at most $\sum_{i=1}^t \dim V_\mathbf{k}^{\Gamma_i}$, so our previous equation means in fact $\dim \mathcal{M}_\mathbf{k}^\chi(\mathcal{O}) = \dim \mathcal{M}_\mathbf{k}^{\chi'}(\mathcal{O}) = \sum_{i=1}^t \dim V_\mathbf{k}^{\Gamma_i}$. This implies both the admissibility and dimension assertions. $\qquad\square$

Corollary 4.8. *Suppose $F = \mathbb{Q}$ and $M|N$ such that, for each divisor $d|M$ with $d \neq 1$, there exists an odd $p|\frac{N}{M}$ such that $\left(\frac{-d}{p}\right) = 1$. If M is even, we further assume $\frac{N}{M}$ is divisible by a prime $p \equiv 1 \bmod 4$. Then each $X_i \in \mathrm{Cl}_M(\mathcal{O})$ is χ-admissible in weight $\mathbf{k}$ for all weights $\mathbf{k}$ and sign patterns χ for M.*

Proof. By the lemma, we want to know that the sign patterns for M are perfectly equidistributed in the space $\mathcal{M}_{\mathbf{0}}(\mathcal{O})$, i.e., that $\dim S_2^{\mathrm{new}, \varepsilon}(N) = \dim S_2^{\mathrm{new}, -M}(N) + 1$ for all sign patterns ε for M with $\varepsilon \neq -_M$. This is immediate from [Mar2, Thm 3.3] (which also immediately implies the sign patterns for M are perfectly equidistributed in higher weight). $\qquad\square$

Remark 4.1. If M is prime which is 7 mod 8 and N is even, then the conclusion of the corollary also holds by Lemma 4.3.

5. Congruences

Now we prove a congruence result under admissibility hypotheses. In particular, we will find that equidistribution of sign patterns in weight 0 implies sign patterns are in some sense equidistributed in congruence classes in all weights.

Let $F, B, \mathcal{O}, \mathfrak{N}$ be as above. Fix a set of representatives $x_1, \ldots, x_h$ for $\mathrm{Cl}(\mathcal{O})$.

5.1. Integrality. First we describe some notions and properties of integrality.

Recall $\tau_1, \ldots, \tau_d$ are the embeddings of F into $\mathbb{C}$. Let E/F be a totally imaginary quadratic extension which splits B. Then we may fix an embedding of B into $M_2(E)$ so that $\mathcal{O}$ maps into $M_2(\mathfrak{o}_E)$. If v_i is the place of F associated to τ_i, the embedding of B into $M_2(E)$ induces an embedding $\tau_i^B : B_{v_i} \to M_2(\mathbb{C})$ such that $\mathcal{O}$ maps into $M_2(\mathfrak{o}_{E_i})$, where E_i the image of E under an extension of τ_i. We take these embeddings in our definition of $(\rho_{\mathbf{k}}, V_{\mathbf{k}})$. In particular, $\rho_{k_i}(\gamma) \in M_{k_i+1}(\mathfrak{o}_{E_i})$ for $\gamma \in \mathcal{O}$.

Let $R \subset \mathbb{C}$ be a ring such that $\tau_i^B(\mathcal{O}) \subset M_2(R)$ for all $1 \leq i \leq d$. Realizing $V_{\mathbf{k}} = \mathbb{C}^n$, let $V_{\mathbf{k}}(R) = R^n$ be the subspace of "R-integral vectors." We say $\varphi \in \mathcal{M}_{\mathbf{k}}(\mathcal{O})$ is R-integral (with respect to $x_1, \ldots, x_h$) if $\varphi(x_i) \in V_{\mathbf{k}}(R)$ for $1 \leq i \leq h$. Let $\mathcal{M}_{\mathbf{k}}(\mathcal{O}; R)$ be the R-submodule of R-integral forms in $\mathcal{M}_{\mathbf{k}}(\mathcal{O})$ (with respect to $x_1, \ldots, x_h$).

Recall for any Hecke operator $T = T_\alpha$, there exists a finite collection of $\beta_j \in \hat{B}^\times$ such that for any $\varphi \in \mathcal{M}_{\mathbf{k}}(\mathcal{O})$,

$$(T\varphi)(x) = \sum \varphi(x\beta_j).$$

For any $1 \leq i \leq h$, we can write $x_i \beta_j = z_{ij} \gamma_{ij} x_{m_{ij}} u_{ij}$ for some $z_{ij} \in Z(B^\times)$, $\gamma_{ij} \in B^\times \cap \mathcal{O}$, $1 \leq m_{ij} \leq h$, and $u_{ij} \in \hat{\mathcal{O}}^\times$. Then

$$(T\varphi)(x_i) = \sum \rho_{\mathbf{k}}(\gamma_{ij})\varphi(x_{m_{ij}}), \quad 1 \leq i \leq h.$$

By our integrality condition on γ_{ij} and assumptions on R, $T\varphi$ is R-integral when φ is.

Moreover, viewing φ as a vector in $\mathbb{C}^{nh}$ formed by concatenating the vectors $\varphi(x_i) \in \mathbb{C}^n$ for $1 \leq i \leq h$, we can think of T as given by a $nh \times nh$ Brandt matrix with entries in R (in fact in $\mathbb{Z}_{\geq 0}$ when $\mathbf{k} = \mathbf{0}$). Since there exists a Hecke operator $T = T_\alpha$ with distinct eigenvalues, a basis of eigenforms of $\mathcal{M}(\mathcal{O})$ can be described as a complete set of eigenvectors for some R-integral matrix T. Thus $\mathcal{M}(\mathcal{O})$ has a basis consiting of R-integral eigenforms for some integer ring R.

For integral $\varphi, \varphi' \in \mathcal{M}_{\mathbf{k}}(\mathcal{O}; R)$ and an ideal ℓ of R, we write $\varphi \equiv \varphi' \bmod \ell$ if the vectors $\varphi(x_i)$ and $\varphi'(x_i)$ are coordinate-wise congruent mod ℓ for all i.

5.2. Congruences under admissibility.
Let $\mathfrak{M}|\mathfrak{N}$ and $S = \{\mathfrak{p}|\mathfrak{M}\}$.

Theorem 5.1. *Let $\varphi \in \mathcal{M}_{\mathbf{k}}(\mathcal{O})$ be an eigenform, χ a sign pattern for $\mathfrak{M}$, and $\ell|2$ a prime of $\bar{\mathbb{Q}}$. Suppose each $X \in \mathrm{Cl}_S(\mathcal{O})$ is χ-admissible in weight $\mathbf{k}$. Then there exists an eigenform $\varphi' \in \mathcal{M}_{\mathbf{k}}^{\chi}(\mathcal{O})$ such that $a_{\mathfrak{p}}(\varphi) \equiv a_{\mathfrak{p}}(\varphi') \bmod \ell$ for all primes $\mathfrak{p}$ of F.*

Proof. Let K be sufficiently large number field. Namely, assume K contains the rationality fields for all eigenforms in $\mathcal{M}_{\mathbf{k}}(\mathcal{O})$ and $\tau_i^B(\mathcal{O}) \subset M_2(\mathfrak{o}_K)$ for all $1 \le i \le d$. We may assume φ is $\mathfrak{o}_K$-integral with respect to $x_1, \dots, x_h$.

Let $\mathcal{I}$ be a prime ideal of $\mathfrak{o}_K$ under ℓ, and R the localization of $\mathfrak{o}_K$ at $\mathcal{I}$. A priori, if $\mathcal{I}$ is not principal, it may not be possible to scale the values of φ so that φ is R-integral and $\varphi \not\equiv 0 \bmod \mathcal{I}$, but we can pass to a finite extension of K (i.e., enlarge K if necessary) that principalizes $\mathcal{I}$ to assume this.

Let ε be the sign pattern for $\mathfrak{M}$ such that $\varphi \in \mathcal{M}_{\mathbf{k}}^{\varepsilon}(\mathcal{O})$. Write $\mathrm{Cl}_S(\mathcal{O}) = \{X_1, \dots, X_t\}$ and order $x_1, \dots, x_h$ so that $x_i \in X_i$ for $1 \le i \le t$. For each $\mathfrak{p}|\mathfrak{M}$ and $1 \le i \le h$, let $\gamma_{i,\mathfrak{p}} \in \Gamma^{\sigma_{\mathfrak{p}}}(x_i)$. Then φ is determined by $\varphi(x_1), \dots, \varphi(x_t)$ and $\varphi(x_i) = \varepsilon_{\mathfrak{p}} \rho_{\mathbf{k}}(\gamma_{i,\mathfrak{p}}) \varphi(\sigma_{\mathfrak{p}}(x_i))$ for all $i, \mathfrak{p}$.

We define a function φ' on $\mathrm{Cl}(\mathcal{O})$ as follows. For $1 \le i \le t$, let $\varphi'(x_i) = \varphi(x_i)$. Extend φ' to $\mathrm{Cl}(\mathcal{O})$ by requiring $\varphi(x_i) = \chi_{\mathfrak{p}} \rho_{\mathbf{k}}(\gamma_{i,\mathfrak{p}}) \varphi(\sigma_{\mathfrak{p}}(x_i))$ for all $i, \mathfrak{p}$. Then $\varphi' \in \mathcal{M}_{\mathbf{k}}^{\chi}(\mathcal{O})$ by (4.9), and $\varphi'(x_i) = \pm\varphi(x_i)$ for $1 \le i \le h$. Thus $\varphi' \equiv \varphi \bmod 2$ with respect to $x_1, \dots, x_h$. However, this φ' need not be an eigenform.

Take a basis of $\mathcal{M}_{\mathbf{k}}(\mathcal{O})$ consisting of eigenforms $\varphi_1, \dots, \varphi_m \in \mathcal{M}_{\mathbf{k}}(\mathcal{O}; K)$ such that $\varphi_1 = \lambda\varphi$ for some $\lambda \in K^{\times}$. Since $\varphi' \in \mathcal{M}_{\mathbf{k}}(\mathcal{O}; R)$ we have that $\varphi' = \sum c_j \varphi_j$ for some $c_j \in K$. By rescaling our basis vectors if necessary, we may assume φ' and φ are R-linear combinations of $\varphi_1, \dots, \varphi_m$.

Let M be the R-module generated by $\varphi_1, \dots, \varphi_m$, and M^{χ} be the submodule generated by the collection of φ_j's which lie in $\mathcal{M}_{\mathbf{k}}^{\chi}(\mathcal{O})$. Then $\varphi' \in M^{\chi}$.

Then the integrality property of Hecke operators implies each Hecke operator T_{α} acts on M as well as $M/\mathcal{I}M$. Now $\varphi' \equiv \varphi \bmod \mathcal{I}$, so the image of φ' in $M^{\chi}/\mathcal{I}M^{\chi}$ is a (nonzero) mod $\mathcal{I}$ eigenvector of each T_{α}. The Deligne–Serre lifting lemma [DS74, Lem 6.11] now tells us there is an eigenform $\varphi'' \in M^{\chi}$, i.e. some φ_j, which has the same mod $\mathcal{I}$ Hecke eigenvalues as φ', and thus φ. (Note the Deligne–Serre lemma does not tell us that we may take $\varphi'' \equiv \varphi \bmod \mathcal{I}$—cf. [Mar1, (3.3)].) $\square$

Let $S_{\mathbf{k}}^{\mathrm{new}}(\mathfrak{N})^*$ be the space $S_{\mathbf{k}}^{\mathrm{new}}(\mathfrak{N})$ if $\mathbf{k} \ne \mathbf{2}$ and $S_{\mathbf{2}}^{\mathrm{new}}(\mathfrak{N}) \oplus \mathbb{C}E_{2,\mathfrak{N}}$ if $\mathbf{k} = \mathbf{2}$. Similarly, for a sign pattern ε for $\mathfrak{M}$, let $S_{\mathbf{k}}^{\mathrm{new},\varepsilon}(\mathfrak{N})^*$ be $S_{\mathbf{k}}^{\mathrm{new},\varepsilon}(\mathfrak{N})$ unless $\mathbf{k} = \mathbf{2}$ and $\varepsilon = -\mathfrak{m}$, in which case it is $S_{\mathbf{2}}^{\mathrm{new},-\mathfrak{m}}(\mathfrak{N}) \oplus \mathbb{C}E_{2,\mathfrak{N}}$.

Corollary 5.2. *Let $\mathfrak{M}|\mathfrak{N}$. Suppose all sign patterns ε for $\mathfrak{M}$ are equidistributed in the space $S_{\mathbf{2}}^{\mathrm{new}}(\mathfrak{N})^*$, i.e. $\dim S_{\mathbf{2}}^{\mathrm{new},\varepsilon}(\mathfrak{N})^*$ is independent of ε. Let $\mathbf{k} \in (2\mathbb{N})^d$, f an eigenform in $S_{\mathbf{k}}^{\mathrm{new}}(\mathfrak{N})^*$ and ℓ a prime of $\bar{\mathbb{Q}}$ above 2. Then for any sign pattern ε for $\mathfrak{M}$, there exists an eigenform $g \in S_{\mathbf{k}}^{\mathrm{new},\varepsilon}(\mathfrak{N})^*$ such that $a_{\mathfrak{p}}(f) \equiv a_{\mathfrak{p}}(g) \bmod \ell$ for all primes $\mathfrak{p}$. In particular, there are at least $2^{\omega(\mathfrak{M})}$ eigenforms in $S_{\mathbf{k}}^{\mathrm{new}}(\mathfrak{N})^*$ which are congruent to $f \bmod \ell$.*

Proof. Use the Jacquet–Langlands correspondence, Lemma 4.7, and the above theorem. $\square$

By Corollary 4.8, this gives Theorem 1.1 when $F = \mathbb{Q}$ excepting the assertion that we can take $g \in S_2^{\text{new},-M}(N)$ when the weight $k = 2$, $\varepsilon = -_M$ and N is not an even product of three primes. We handle this below.

Since, in weight $\mathbf{0}$, all quaternionic S-ideal classes are $+_{\mathfrak{M}}$-admissible, the above theorem also gives the following.

Corollary 5.3. *Let $f \in S_2(\mathfrak{M})$ be a newform and ℓ a prime of $\bar{\mathbb{Q}}$ above 2. Then there exists an eigenform $g \in S_2(\mathfrak{M}) \oplus \mathbb{C}E_{2,\mathfrak{M}}$ such that $a_{\mathfrak{p}}(f) \equiv a_{\mathfrak{p}}(g) \bmod \ell$ for all $\mathfrak{p}$ and $a_{\mathfrak{p}}(g) = +1$ for all $\mathfrak{p}|\mathfrak{M}$.*

This gives Theorem 1.3 when $F = \mathbb{Q}$ excepting the assertion about when we may take g cuspidal.

5.3. **Eisenstein and non-Eisenstein congruences.** Here we will refine the latter corollary to show that when $F = \mathbb{Q}$, we can take $g \in S_2^{\text{new},-M}(N)$ if N is not an even product of three primes, which will finish the proof of both Theorem 1.1 and Theorem 1.3.

First we refine the main theorem of [Mar1] in the setting $\ell = 2$ and $h_F^+ = 1$. (The proof is also similar.)

Proposition 5.4. *Suppose the numerator of $2^{1-d}|\zeta_F(-1)|N(\mathfrak{M})\prod_{p|\mathfrak{M}}(1-N(\mathfrak{p})^{-1})$ is even and the type number $t_B > 1$. Then there exists an newform $g \in S_2(\mathfrak{M})$ such that $w_{\mathfrak{p}}(g) = -1$ for all $\mathfrak{p}|\mathfrak{M}$ and $a_{\mathfrak{p}}(g) \equiv a_{\mathfrak{p}}(E_{2,\mathfrak{M}}) \bmod 2$ for all $\mathfrak{p}$.*

Proof. Consider the graph Σ_χ described above when χ is the sign pattern $+_{\mathfrak{M}}$ for $\mathfrak{M}$ with components $X_1, \ldots, X_t$. Let $n_j = |X_j|$ for $1 \leq j \leq t$. Recall $t = t_B$, and $t > 1$ means $\mathcal{S}_{\mathbf{0}}^{+\mathfrak{M}}(\mathcal{O}) \neq 0$. By Lemma 4.3, for a fixed X_j the coefficients $\frac{|\mathfrak{o}_F^\times|}{|\Gamma_i|}$ appearing in (3.3) are identical for all i with $x_i \in X_j$. Let c_j be this number for X_j. Then $(\mathbb{1}, \mathbb{1}) = \sum_{i=1}^h \frac{|\mathfrak{o}_F^\times|}{|\Gamma_i|} = \sum_{j=1}^t c_j^{-1}n_j$. This number is the mass $m(\mathcal{O})$ of $\mathcal{O}$ studied by Eichler, and equals $2^{1-d}|\zeta_F(-1)|N(\mathfrak{M})\prod_{p|\mathfrak{M}}(1-N(\mathfrak{p})^{-1})$ (see, e.g., [Mar1]).

Let us define $\varphi' \in \mathcal{M}_{\mathbf{0}}^{+\mathfrak{M}}(\mathcal{O})$ by $\varphi'(x_i) = a_j$ for all $x_i \in X_j$, where $a_j \in 2\mathbb{Z} + 1$ for $1 \leq j \leq t$. Then $\varphi' \equiv \mathbb{1} \bmod 2$, and the argument with the Deligne–Serre lemma above will give our proposition if we can choose $\varphi' \in \mathcal{S}_{\mathbf{0}}^{+\mathfrak{M}}(\mathcal{O})$. By (3.3), this means we want to show there is a solution to $\sum_{j=1}^t c_j^{-1}a_jn_j = 0$ in the a_j's. We can scale the quantities $\frac{n_j}{c_j}$ by some $\lambda \in \mathbb{Q}^\times$ so that $m_j = \lambda\frac{n_j}{c_j} \in \mathbb{Z}$ for $1 \leq j \leq t$ and $\gcd(m_1, \ldots, m_t) = 1$.

The hypothesis that $m(\mathcal{O})$ is even means $\sum m_j$ also is. Writing $a_j = 2b_j + 1$ for all j, our desired (scaled) linear equation is that $\sum m_j 2b_j = -\sum m_j$, i.e., $\sum m_j b_j = -\frac{1}{2}\sum m_j$, which has a solution as $\gcd(m_1, \ldots, m_t) = 1$. $\square$

Hence if the hypotheses of this proposition are satisfied, we can take g to be a newform in $S_2(\mathfrak{M})$ in Corollary 5.3.

From now on, assume $F = \mathbb{Q}$.

Then the mass $m(\mathcal{O})$ is just $\frac{\varphi(N)}{12}$, where $\varphi(N) = \prod_{p|N}(p-1)$, so Proposition 5.4 tells us we can take g to be a cusp form if $8|\varphi(N)$ and $t_B > 1$. Recall $t_B \geq 2^{-\omega(N)}h_B$ and $h_B \geq m(\mathcal{O})$, so $t_B > 1$ whenever $\varphi(N) > 12 \cdot 2^{\omega(N)}$. This is automatic if N has at least 5 prime divisors, in which case we also have $8|\varphi(N)$. Hence if $\omega(N) > 3$, we may take g to be a cusp form in Theorem 1.3.

Suppose $N = p_1p_2p_3$ with $2 < p_1 < p_2 < p_3$. Automatically $8|\varphi(N)$. Also, if $p_1 \geq 5$ or $p_1 = 3$, $p_2 \geq 7$, or $p_1 = 3$, $p_2 = 5$, $p_3 \geq 17$ then the above reasoning shows

$t_B > 1$. The remaining possibilities are $N = 3 \cdot 5 \cdot 7$, $N = 3 \cdot 5 \cdot 11$ or $N = 3 \cdot 5 \cdot 13$, and in fact one checks that $t_B > 1$ in these three cases as well. Hence if N is an odd product of 3 primes, we can take g to be a cusp form in Theorem 1.3.

To finish the theorems in the introduction, it thus remains treat N prime.

Proposition 5.5. *Let $f \in S_2^{\mathrm{new}, \chi}(N)$ be a newform. Suppose there exists $p|N$ such that $w_p(f) = +1$ but p does not satisfy any of the following conditions:*

(a) $p \equiv 7 \bmod 8$ and N is even; or
(b) $p \neq 2$ and $\left(\frac{-p}{q}\right) = 1$ for some odd prime $q|N$; or
(c) $p = 2$, N is divisible by a prime which is $1 \bmod 4$ as well a (not necessarily different) prime q such that $\left(\frac{-2}{q}\right) = 1$.

Then there exists a newform $g \in S_2^{\mathrm{new}, -N}(N)$ such that $f \equiv g \bmod 2$.

Proof. Let φ be an associated integral newform to f. By Lemma 4.3, the hypothesis on p implies σ_p has at least one fixed point $x_i \in \mathrm{Cl}(\mathcal{O})$. The condition $T_p\varphi = -\varphi$ then implies $\varphi(x_i) = 0$. Now the construction of $\varphi' \in \mathcal{M}_0^{+N}(\mathcal{O})$ in Theorem 5.1/Corollary 5.3 with $\varphi' \equiv \varphi \bmod 2$ also means $\varphi'(x_i) = 0$.

Write $\varphi' = \sum a_j \varphi_j$ as a sum of eigenforms. Note $a_j = 0$ unless $\varphi_j \in \mathcal{M}_0^{+N}(\mathcal{O})$ since $\varphi' \in \mathcal{M}_0^{+N}(\mathcal{O})$. While we used the Deligne–Serre lemma above to get an eigenform φ'' with the same Hecke eigenvalues as $\varphi' \bmod 2$, we gave a different argument for this type of result in the proof of Theorem 2.1 of [Mar1]. That argument tells us that (possibly upon replacing φ' with a different form in $\mathcal{M}_0^{+N}(\mathcal{O})$ which is congruent to $\varphi' \bmod 2$) the Hecke eigenvalues of φ_j are congruent to the Hecke eigenvalues of $\varphi' \bmod 2$ for all j such that $a_j \neq 0$. Say $\varphi_m = \mathbb{1}$ is the constant function generating $\mathcal{E}_2(\mathcal{O})$. Since $\varphi'(x_i) = 0 \neq \varphi_1(x_i)$, it is not possible that $a_j = 0$ for all $j \neq m$. This gives (at least one) $\varphi_j \in \mathcal{S}_0^{+N}(\mathcal{O})$ congruent to $\varphi \bmod 2$, which gives our desired g by the Jacquet–Langlands correspondence. $\square$

Let us finish by explicating additional conditions when $F = \mathbb{Q}$ and $k = 2$ where we can take g to be a cusp form in Theorem 1.3. Assume $N = 2p_1 p_2$ with $2 < p_1 < p_2$.

First note that $8|\varphi(N)$ if p_1 or p_2 is $1 \bmod 4$. Here $t_B > 1$ if $p_1 \geq 11$ or $p_1 = 7$, $p_2 \geq 19$ or $p_1 = 5$, $p_2 \geq 29$ or $p_1 = 3$, $p_2 \geq 53$. Thus $t_B > 1$ if $N > 294$ by this reasoning, and exact calculation of class numbers shows in fact $t_B > 1$ if $N > 258$. Hence if $N > 258$ is an even product of 3 primes, at least one of which is $1 \bmod 4$, then we can take g to be a cusp form in Theorem 1.3 by Proposition 5.4.

On the other hand, suppose $p_1 \equiv p_2 \equiv 3 \bmod 4$. Then $p = 2$ never satisfies (a), (b) or (c) of Proposition 5.5. Now p_1, p_2 never satisfy (c), and satisfy (a) if and only if they are $7 \bmod 8$. By quadratic reciprocity, $\left(\frac{-p_1}{p_2}\right) = (-1)\left(\frac{-p_2}{p_2}\right)$, so (b) of Theorem 1.3 will be satisfied for exactly one of $p = p_1$ and $p = p_2$. Hence Proposition 5.5 cannot be used to guarantee g is a cusp form in the remaining cases of $N = 2p_1 p_2$ for f with arbitrary signs. However, it can be used to say we can take g to be a cusp form if $w_{p_1}(f) = w_{p_2}(f) = +1$ (or just $w_{p_i}(f) = +1$ for whichever p_i does not satisfy (b)) and $p_1 \equiv p_2 \equiv 3 \bmod 8$.

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