THE BASIS PROBLEM REVISITED

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Abstract. Eichler investigated when there is a basis of a space of modular forms consisting of theta series attached to quaternion algebras, and treated squarefree level. Hijikata, Pizer and Shemanske completed the solution to Eichler's basis problem for elliptic modular forms of arbitrary level by tour-de-force trace calculations. We revisit the basis problem using the representation-theoretic perspective of the Jacquet–Langlands correspondence.

Our results include: (i) a simpler proof of the solution to the basis problem for elliptic modular forms, which also allows for more flexibility in the choice of quaternion algebra; (ii) a solution to the basis problem for Hilbert modular forms; (iii) a theory of (local and global) new and old forms for quaternion algebras; and (iv) an explicit version of the Jacquet–Langlands correspondence at the level of modular forms, which is a refinement of the Hijikata–Pizer–Shemanske solution to the basis problem. Both (i) and (ii) have practical applications to computing elliptic and Hilbert modular forms. Moreover, (iii) and (iv) are desired for arithmetic applications—to illustrate, we give a simple application to Eisenstein congruences in level p^3.

Contents

1. Introduction

Let $S_k(N)$ denote the space of elliptic cusp forms of weight k on $\Gamma_0(N)$ with trivial character. Denote by B a definite quaternion algebra over $\mathbb{Q}$ (for the moment), which is characterized by its discriminant D_B, a positive squarefree product of an odd number of primes (the finite places of ramification of B). Denote by $\mathcal{O}$ an order in B.

1.1. The basis problem: a classical history.

In the case $D_B = p$ is prime and $\mathcal{O}$ is a maximal order of B, Hecke (1940) conjectured that the (differences of) theta series associated to a set of one-sided $\mathcal{O}$-ideal class representatives yield a basis for $S_2(p)$. While this conjecture was not quite correct (the number of theta series here equals $1 + \dim S_2(p)$, but they often are linearly dependent), Eichler [Eic55] showed that there is a basis for $S_2(p)$ coming from a larger collection of theta series associated to $\mathcal{O}$. These theta series come from forming Fourier series $\sum a_{ij}(n)q^n$, where $a_{ij}(n)$ is the (i,j)-th entry of the n-th Brandt matrix A_n associated to $\mathcal{O}$. Eichler's proof relies on a comparison of traces of Brandt matrices with traces of Hecke operators on $S_2(p)$ via explicit computation.

More generally, Eichler considered the question of whether the newspace $S_k^{\mathrm{new}}(N)$ of $S_k(N)$ has a basis consisting of theta series attached to an order $\mathcal{O}$ in a definite quaternion algebra, which is called the (Eichler) basis problem. Eichler [Eic73] extended his approach from [Eic55] to show that this question has a positive answer when $N > 1$ is squarefree using what are now known as Eichler orders $\mathcal{O}$ (intersections of at most 2 maximal orders). Note one cannot construct elements of $S_k(1)$ via quaternionic theta series, but there are well-known ways for constructing a basis for $S_k(1)$, e.g., with the Eisenstein series E_4 and E_6. Hence [Eic73] together with Atkin–Lehner theory provides a way of constructing bases for $S_k(N)$ for any squarefree N.

Using more general Eichler orders, Hijikata and Saito [HS73] extended Eichler's results to levels of the form $N = pM$. Here M denotes a positive integer prime to p. This was further generalized by Pizer in [Piz80a] and [Piz80b] where he treated levels of the form $N = p^{2r+1}M$ and $N = p^2M$. However, in the case $N = p^2M$, Pizer needs to assume p is odd, and now it is only true that $S_k^{\mathrm{new}}(N)$ is generated by quaternionic theta series and twists of newforms of levels M and pM (with character). Pizer followed a similar approach to Eichler's, but needed to work with non-Eichler orders. (An Eichler order is maximal at primes dividing $D_B = p$, so there are no Eichler orders of level $p^r M$ for $r > 1$.)

The only levels that remain are those of the form $N = (N')^2$ where no odd prime sharply divides N'. To treat these, Hijikata, Pizer and Shemanske [HPS89a] introduced a generalization of Eichler's and Pizer's orders, called special orders. Then in a tour-de-force calculation [HPS89b], they solved the basis problem using special orders of level $N = p^r M$ in a definite quaternion algebra of discriminant p. The basic argument follows Eichler's original approach, but the necessary calculations with special orders are much more complicated (especially for $N = 2^{2r}M$), and again one needs to consider twists of forms of smaller level (see Corollary 1.3 below).

1.2. **Connection with the Jacquet–Langlands correspondence.** The solution to the basis problem may be viewed as a classical interpretation of the representation-theoretic Jacquet–Langlands correspondence, which is an injective correspondence from (irreducible, infinite-dimensional) automorphic representations π of $B^\times$ to (irreducible, infinite-dimensional) cuspidal automorphic representations π' of GL(2). It is well known how to view $S_k(N)$ as a sum of certain invariant subspaces of appropriate π'. One can also define a space of *quaternionic modular (cusp) forms* $S_k(\mathcal{O})$, which can be viewed as a sum of certain invariant subspaces of appropriate π. Here the analogue of level is played by an order $\mathcal{O}$ in B, and $S_k(\mathcal{O})$ may be viewed as certain (vector-valued if $k > 0$) functions on the set of right $\mathcal{O}$-ideal classes. Then the Jacquet–Langlands correspondence may be viewed "classically" as a non-canonical linear map

(1.1)
$$\mathrm{JL} : S_k(\mathcal{O}) \to S_{k+2}(N) \qquad (\text{where } k \geq 0),$$

which preserves the action of unramified Hecke operators. This applies to any order $\mathcal{O}$ in any B, and some N depending on $\mathcal{O}$. This is related to the basis problem by associating weight $k + 2$ theta series to $S_k(\mathcal{O})$.

However, the proof of the Jacquet–Langlands correspondence does not answer any of the following questions: (1) given $\mathcal{O}$, what can we take for N? (2) what is the kernel of this map? and (3) what is the image of this map? Note that while the map JL in (1.1) is not canonical, if one defines this map so the kernel has minimum possible dimension, then the kernel and image are canonical up to isomorphism as Hecke modules. In fact, one can define JL so that the image is spanned by theta series, which specifies the image exactly, though then it is not clear if the kernel necessarily has minimum possible dimension on the old space.

The first question is essentially asking: (1$'$) how does "level" behave under the local Jacquet–Langlands correspondence? An answer to this will also let us describe the image of this map, using the representation-theoretic characterization of the image of the Jacquet–Langlands correspondence. The solution to the basis problem in [HPS89b] says that, if $D_B = p$ and $\mathcal{O}$ is a special order of level $p^r M$ (of a certain type if r is odd), then one can take $N = p^r M$ and describe the portion of the image lying in $S_{k+2}^{\mathrm{new}}(N)$. However, [HPS89b] does not tell us everything we want to know: it does not completely answer (2) or (3), or the local analogue (1$'$), or handle general B.

Conversely, understanding the "classical" Jacquet–Langlands correspondence (1.1) can be applied to the basis problem, and this may be approached via representation theory. That is the goal of the present work. This applies to arbitrary totally real base fields F, and we obtain both more information about the classical Jacquet–Langlands correspondence and the basis problem when $F = \mathbb{Q}$, as well a solution to the basis problem for Hilbert modular forms.

To our knowledge, this idea was first realized by Shimizu in [Shi72], albeit in a restricted setting. In that work, he gave a theta series proof of the Jacquet–Langlands correspondence (under some conditions), and applied this to the basis problem over totally real fields F for squarefree level $\mathfrak{N}$ under a parity condition on $[F : \mathbb{Q}]$ and $\mathfrak{N}$ (and weight > 2 at each infinite place). Shimizu's condition corresponds to using a maximal order $\mathcal{O}$ in a definite quaternion algebra B/F of discriminant $\mathfrak{N}$. The representation-theoretic approach makes it easy to get a solution to the basis problem more generally when $[F : \mathbb{Q}]$ is even or when $\mathfrak{N}$ is divisible by some prime ideal $\mathfrak{p}$ such that $\mathfrak{p}^2 \nmid \mathfrak{N}$. (Shimizu's

condition on the weight is not essential for our purposes. See also [Wal80, Theorem 3] for a more classical treatment when $[F : \mathbb{Q}]$ is even.) When $F = \mathbb{Q}$, this is the case treated by Hijikata and Saito [HS73]. This extension of Shimizu's application to the basis problem simply comes from working with Eichler orders instead of maximal orders, and using a quaternion algebra ramified at at most one finite place. (Shimizu's theta series are not presented in a classically explicit way like Eichler's—cf. [Geb09]—but ours below will be.)

1.3. **Summary of results.** Let B be a definite quaternion algebra of discriminant $\mathfrak{D}$ over a totally real number field F. Let $\mathcal{O}$ be a special order of level $\mathfrak{N}$. This means, for $\mathfrak{p}$ a finite prime of F, $\mathcal{O}_\mathfrak{p}$ is an Eichler order for $\mathfrak{p} \nmid \mathfrak{D}$ and $\mathcal{O}_\mathfrak{p}$ contains the ring of integers of a quadratic extension $E_\mathfrak{p}/F_\mathfrak{p}$ for $\mathfrak{p}|\mathfrak{D}$, and that the product of the levels of the local orders is $\mathfrak{N}$. (We use the convention that the maximal order $\mathcal{O}_\mathfrak{p}$ for $\mathfrak{p}|\mathfrak{D}$ has level $\mathfrak{p}$, rather than level $\mathfrak{o}_F$.) Necessarily, $\mathfrak{D}|\mathfrak{N}$. We further assume $E_\mathfrak{p}/F_\mathfrak{p}$ is unramified when $v_\mathfrak{p}(\mathfrak{N})$ is odd. Such special orders exist for all multiples $\mathfrak{N}$ of $\mathfrak{D}$.

Let $d = [F : \mathbb{Q}]$ and $\mathbf{k} = (k_1, \ldots, k_d)$ with each $k_i \in \mathbb{Z}_{\geq 0}$. Let $M_\mathbf{k}(\mathcal{O})$ (resp. $S_\mathbf{k}(\mathcal{O})$) be the space of quaternionic modular (resp. cusp) forms of weight $\mathbf{k}$, level $\mathcal{O}$, and trivial character. (These spaces will be denoted $M_\mathbf{k}(\mathcal{O}, 1, 1)$ and $S_\mathbf{k}(\mathcal{O}, 1, 1)$ in Section 4.1, where we treat quaternionic modular forms with character.) In fact $S_\mathbf{k}(\mathcal{O}) = M_\mathbf{k}(\mathcal{O})$ unless $\mathbf{k} = \mathbf{0} = (0, \ldots, 0)$. In Section 4, we define Hecke operators $T_\mathfrak{n}$ on $M_\mathbf{k}(\mathcal{O})$, and associated Brandt matrices which realize their action with respect to a suitable basis.

We first develop a theory of quaternionic newforms and oldforms along the lines of Casselman's approach [Cas73] to Atkin–Lehner theory. This relies on a decomposition of $M_\mathbf{k}(\mathcal{O})$ as a direct sum of invariant subspaces π_f^K, where $\pi = \otimes \pi_v$ runs over automorphic representations of $B^\times$ of "weight $\mathbf{k}$" and $K = \hat{\mathcal{O}}^\times = \prod_\mathfrak{p} \mathcal{O}_\mathfrak{p}^\times$.

1.3.1. *Local results.* In Section 3, we study the dimensions of $\pi_\mathfrak{p}^{K_\mathfrak{p}}$ for $\mathfrak{p}|\mathfrak{D}$ (and to treat general characters, more generally the restriction of $\pi_\mathfrak{p}$ to $\mathcal{O}_\mathfrak{p}^\times$). Since $\mathcal{O}_\mathfrak{p}^\times \supset \mathfrak{o}_{E_\mathfrak{p}}^\times$, this is closely related to the restriction problem for $(B_\mathfrak{p}^\times, E_\mathfrak{p}^\times)$, which was solved by Tunnell [Tun83] and Saito [Sai93] in terms of epsilon factors. In particular, $\dim \pi_\mathfrak{p}^{K_\mathfrak{p}} \leq e(E_\mathfrak{p}/F_\mathfrak{p})$, the ramification index. We get an essentially complete answer when $E_\mathfrak{p}/F_\mathfrak{p}$ is unramified, but only a partial answer when $E_\mathfrak{p}/F_\mathfrak{p}$ is ramified. The former case gives a simple formula for epsilon factors in certain situations (Remark 3.2). In the latter case, the obstruction to a complete answer is that one cannot prove such a simple formula for relevant epsilon factors.

In contrast to local newform theory for $\mathrm{GL}(2)$, there are two new phenomena here. First, for fixed $\pi_\mathfrak{p}$, $\dim \pi_\mathfrak{p}^{K_\mathfrak{p}}$ essentially does not increase upon raising the level of $K_\mathfrak{p}$ beyond the "conductor" of $\pi_\mathfrak{p}$. This is perhaps not surprising as these representations are finite dimensional. Second, at the lowest level $K_\mathfrak{p}$ for which $\pi_\mathfrak{p}^{K_\mathfrak{p}}$ is nonzero, $\pi^{K_\mathfrak{p}}$ may be 2-dimensional when $E_\mathfrak{p}/F_\mathfrak{p}$ is ramified (and it is when $\pi_\mathfrak{p}$ corresponds to a minimal supercuspidal of $\mathrm{GL}(2)$ of even conductor).

A consequence of our study of local representations is a description of how "level" behaves along the Jacquet–Langlands correspondence, i.e., an answer to question $(1')$, and thus also (1), above. In general, for functorial transfers, one often knows depth is preserved, but the behavior of level is more mysterious. To our knowledge, this is the

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first complete description of the behavior of level for local functorial transfer between two groups G and G' where one group is not quasi-split. Thus our local results may be viewed as a baby case of this general problem.

For general groups G, there is no canonical way to define level (conductor), particularly for non-quasi-split groups, so the above choice of $K_{\mathfrak{p}}$ may suggest reasonable analogues to consider in rather general situations. We venture that a relevant property of the compact open subgroup $K_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}}^{\times}$ here is that it contains a maximal compact subgroup (here $\mathfrak{o}_{E_{\mathfrak{p}}}^{\times}$) of a subgroup (here $E_{\mathfrak{p}}^{\times}$) of $G = B_{\mathfrak{p}}^{\times}$ which possesses the multiplicity one property for restriction of representations.

1.3.2. *Global results.* Using our local results, we give an Atkin–Lehner type global decomposition of the space of quaternionic modular forms into new spaces of smaller levels (see Corollary 4.5 and (4.10) for cusp forms, and (4.11) for Eisenstein series).

With this decomposition of $S_{\mathbf{k}}(\mathcal{O})$, and our study of level in the local Jacquet–Langlands correspondence (1.1), we give a description of both the kernel and the image (questions (2) and (3) above) of the classical Jacquet–Langlands correspondence. However, due to our incomplete local results when $E_{\mathfrak{p}}/F_{\mathfrak{p}}$ is ramified, we do not get a complete description in all situations. Things are simpler when we restrict to the new space, where the hypothesis we need to get a complete description is that any dyadic $\mathfrak{p}|\mathfrak{D}$ satisfies $v_{\mathfrak{p}}(\mathfrak{N})$ is odd, or 2, or sufficiently large (see Theorem 5.4 and Remark 5.3). For the full space $S_{\mathbf{k}}(\mathcal{O})$, we give a complete description of JL when any prime $\mathfrak{p}|\mathfrak{D}$ satisfies $v_{\mathfrak{p}}(\mathfrak{N})$ is odd or 2 (see Corollary 5.5). This condition means that every π appears in $S_{\mathbf{k}}(\mathcal{O})$ will be locally minimal or 1-dimensional for each $\mathfrak{p}|\mathfrak{D}$.

For simplicity, we only properly state some of our results now (see Proposition 5.3 and Corollary 5.5 for more general statements). Let $\mathbf{2} = (2, \ldots, 2)$, and write $\mathfrak{N} = \mathfrak{N}'\mathfrak{M}$, where $\mathfrak{M}$ is the part of $\mathfrak{N}$ coprime to $\mathfrak{D}$. For $\mathfrak{d}|\mathfrak{N}$, let $S_{\mathbf{k}}^{\mathfrak{d}\text{-new}}(\mathfrak{N})$ be the subspace of $S_{\mathbf{k}}(\mathfrak{N})$ consisting of forms which are $\mathfrak{p}$-new for all $\mathfrak{p}|\mathfrak{d}$. We also say a form is $\mathfrak{p}$-primitive if there is no twist which lowers the level at $\mathfrak{p}$.

Theorem 1.1. *There is a Hecke-module homomorphism* $\mathrm{JL} : S_{\mathbf{k}}(\mathcal{O}) \to S_{\mathbf{k}+\mathbf{2}}(\mathfrak{N})$ *such that*

(i) any newform $f \in S_{\mathbf{k}+\mathbf{2}}(\mathfrak{N})$ which is $\mathfrak{p}$-primitive for $\mathfrak{p}|\mathfrak{D}$ is contained in the image;
(ii) if $v_{\mathfrak{p}}(\mathfrak{N})$ is odd for all $\mathfrak{p}|\mathfrak{D}$, then JL is injective and yields an isomorphism

$$S_{\mathbf{k}}(\mathcal{O}) \simeq \bigoplus_{\mathfrak{d}} S_{\mathbf{k}+\mathbf{2}}^{\mathfrak{d}\text{-new}}(\mathfrak{d}\mathfrak{M}),$$

where $\mathfrak{d}$ runs over all divisors of $\mathfrak{N}'$ such that $v_{\mathfrak{p}}(\mathfrak{d})$ is odd for all $\mathfrak{p}|\mathfrak{D}$.

The (notational) complication in the description when $v_{\mathfrak{p}}(\mathfrak{N}) = 2$ for some $\mathfrak{p}|\mathfrak{D}$ is that the kernel is generally nonzero, and its description depends not just on the conductor of $\pi_{\mathfrak{p}}$, but whether that representation is 1-dimensional or not. For instance, in the simple case that $F = \mathbb{Q}$, $D_B = p$, and $\mathcal{O}$ has level p^2, then we have

$$S_k(\mathcal{O}) \simeq S_{k+2}^{\mathrm{new}}(p) \oplus S_{k+2}^{\mathrm{new\text{-}sp}}(p^2) \oplus 2S_{k+2}^{\mathrm{new\text{-}sc}}(p^2),$$

where the second (resp. third) space on the right denotes the span of newforms whose local representation is special (resp. supercuspidal) at p. These first two spaces on the right correspond to the associated local representation π_p of $B_p^{\times}$ being 1-dimensional.

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The issue when $v_{\mathfrak{p}}(\mathfrak{N})$ is some higher even power for a $\mathfrak{p}|\mathfrak{D}$ is that we cannot say what forms f whose $\mathfrak{p}$-power level is odd appear in the image of the Jacquet–Langlands correspondence. If such a $\mathfrak{p}$ is dyadic, we also cannot always say what non-minimal $\mathfrak{p}$-new forms appear. Modulo these issues, we can describe $S_{\mathbf{k}}(\mathcal{O})$ completely as a Hecke module in terms of Hilbert modular forms for an arbitrary $\mathfrak{N}$, and consequently the map JL.

We note that, when restricted to new spaces, the image of JL is always canonical. However, even restricted to new spaces, the kernel of JL is not canonical when it is nonzero. Note the kernel on the new space is in general nonzero in the above level p^2 example, or more generally when primes occur to even powers in $\mathfrak{N}'$. See Remark 6.1 for the dimension of the kernel restricted to the new space.

Part (ii) of the above theorem specializes to the known extension of [Shi72] when $v_{\mathfrak{p}}(\mathfrak{N}) = 1$ for all $\mathfrak{p}|\mathfrak{D}$. If further $F = \mathbb{Q}$, $k = 0$ and $\mathcal{O}$ is maximal, a different proof follows from [Pon09]. (The latter paper works with theta series, but its result is equivalent to the above type of statement via Proposition 6.1 and Theorem 6.3.)

Finally, in Section 6, we explain how to associate spaces of theta series to spaces of quaternionic modular forms via Brandt matrices. This gives a realization of the map JL in terms of classical theta series. Our treatment of theta series is also rather representation theoretic, as opposed to Eichler's treatments in [Eic73] and [Eic77]. A key ingredient is a comparison of two different definitions of local ramified Hecke operators (both for $\mathfrak{p}|\mathfrak{D}$ and $\mathfrak{p} \nmid \mathfrak{D}$). This leads to our solution to the basis problem. Again, for simplicity of exposition, we will not state our results here in full generality or precision.

First, we have the following "weak solution" to the basis problem (see Corollary 6.4 and the subsequent discussion), generalizing the $F = \mathbb{Q}$ case of [HPS89b].

Theorem 1.2. *The space $S_{\mathbf{k+2}}^{\mathrm{new}}(\mathfrak{N})$ is linearly generated by theta series associated to $S_{\mathbf{k}}^{\mathrm{new}}(\mathcal{O})$ together with twists of Hilbert modular forms (with character) of lower level.*

By suitably varying our quaternion algebra B and order $\mathcal{O}$, we get the following solution to the basis problem (see the discussion after Corollary 6.4).

Corollary 1.3. *Any space $S_{\mathbf{k}}^{\mathrm{new}}(\mathfrak{N})$ is linearly generated by twists of suitable quaternionic theta series, unless $[F : \mathbb{Q}]$ is odd and $\mathfrak{N}$ is a perfect square, in which case the space is generated by twists of quaternionic theta series together with twists of forms of level 1 and nebentypus conductor $\mathfrak{N}^{1/2}$.*

We note that, like the solution in [HPS89b], this requires using quaternionic theta series "with character" for lower levels, so we must work with quaternionic modular forms and Brandt matrices with character. At the end of Section 6, we also explain what one can say about the basis problem for Hilbert modular forms with character.

Finally, we describe two applications. First, even when $F = \mathbb{Q}$, Theorem 1.1, and the corresponding application to the basis problem, is new. Namely we do not restrict to quaternion algebras of prime discriminant. Computationally, say to use Brandt matrices to compute spaces of modular forms as in [Piz80a] or [DV13], this is desirable. E.g., if we want to compute the newforms of level $N = (pqr)^3$, where p, q, r are distinct primes, we can use an order of level N in the quaternion algebra of discriminant pqr. This will only pick up newforms of level $p^e q^f r^h$, where $e, f, h \in \{1, 3\}$, and each of them only once. On the other hand if we were to use a quaternion algebra ramified only at p, we will pick up many more old forms at q and r, complicating the calculations.

More generally, if one wants to focus on studying newforms, it is often difficult to isolate them analytically on GL(2). However by working with quaternion algebras which are "as ramified as possible" one can eliminate most old forms.

Second, many arithmetic properties of modular forms such as congruences and L-values are studied by using (definite and indefinite) quaternion algebras and the Jacquet–Langlands correspondence. For some problems, understanding the representation-theoretic Jacquet–Langlands correspondence suffices, but for others one wants to understand it at the level of modular forms. One example of the latter is our construction of Eisenstein congruences via quaternionic modular forms [Mar17]. One specific result there is that, for p odd, there is always a mod p Eisenstein congruence in $S_2(p^3)$, but it was not clear to us at the time how to prove there is such a congruence in $S_2^{\text{new}}(p^3)$. In Section 5.4, we show how to deduce this from Theorem 1.1 and [Mar17]. We confess that our original motivation arose, not from an interest the basis problem, but from trying to understand spaces of quaternionic modular forms for applications to congruences and L-values.

2. LOCAL PRELIMINARIES

In this section and the next, we keep the following notation.

Let F be a finite extension of $\mathbb{Q}_p$ with residue degree q. For a finite extension E/F of fields, we let $\mathfrak{o}_E$ denote the ring of integers, $\mathfrak{p}_E$ the prime ideal, $\mathfrak{u}_E^0 = \mathfrak{o}_E^\times$ the unit group, $\mathfrak{u}_E^n = 1 + \mathfrak{p}_E^n$ the n-th higher unit group ($n \geq 1$), ϖ_E a uniformizer, v_E the (exponential) valuation normalized so $v_E(\varpi_E) = 1$, $N = N_{E/F}$ the norm map, and $\text{tr} = \text{tr}_{E/F}$ the trace map. For a character χ of $E^\times$ (or even just of $\mathfrak{o}_E^\times$), let $c(\chi)$ denote its conductor, i.e., the minimal $n \geq 0$ such that $\mathfrak{u}_E^n \subset \ker \chi$. When $E = F$, we usually omit the subscript F.

Denote by B the unique quaternion division algebra over F. Let $N = N_{B/F}$ and $\text{tr} = \text{tr}_{B/F}$ denote the reduced norm and trace maps. The valuation v_F induces a valuation v_B on B such that $v_B(\alpha) = v_F(N(\alpha))$ for $\alpha \in B$. Then $\mathcal{O}_B = \{\alpha \in B : v_B(\alpha) \geq 0\}$ is the unique maximal order of B with (2-sided) prime ideal $\mathfrak{P} = \{\alpha \in B : v_B(\alpha) \geq 1\} = \varpi_B \mathcal{O}_B$, where ϖ_B denotes a uniformizer of B, i.e., an element of B with valuation 1. We also define the higher unit groups $\mathcal{U}^0 = \mathcal{O}_B^\times$ and, for $n \geq 1$, $\mathcal{U}^n = 1 + \mathfrak{P}^n$. Note $\mathcal{U}^n$ is a normal subgroup of $\mathcal{O}_B^\times$ as conjugation stabilizes $\mathfrak{P}^n$, and the collection $\{\mathcal{U}^n\}$ forms a neighborhood basis of compact open sets for the identity in $B^\times$.

We will need some facts about quotients of unit groups. First note that $\mathcal{O}_B/\mathfrak{P}$ is a finite division ring, and thus a field. It has order q^2. Hence $\mathcal{U}^0/\mathcal{U}^1$ is commutative and cyclic of order $q^2 - 1$. For $n \geq 1$, any successive quotient $\mathcal{U}^n/\mathcal{U}^{n+1}$ has order q^2, and is also abelian. In fact, since

$$(2.1) \qquad (1 + \varpi_B^n x)(1 + \varpi_B^n y) \in 1 + \varpi_B^n(x + y) + \mathfrak{P}^{2n}, \quad x, y \in \mathcal{O}_B,$$

we see that $\mathcal{U}^n/\mathcal{U}^{2n}$ is abelian of order q^{2n} for each n.

Suppose E/F is an extension which embeds in B. This means $E = F$ or E is any quadratic extension of F. If $e = e(E/F)$ denotes the ramification index of E/F, then we have $\mathfrak{P}^n \cap E = \mathfrak{p}_E^{\lceil ne/2 \rceil}$, where $\lceil x \rceil$ denotes the least integer (ceiling) function. Thus $\mathcal{U}^n \cap E = \mathfrak{u}_E^{\lceil ne/2 \rceil}$.

If $x \in \mathfrak{P}^n$, then $N(1 + x) \in 1 + \operatorname{tr} x + \mathfrak{P}^{2n} \subset 1 + \mathfrak{p}^{\lceil n/2 \rceil}$. If E/F is the unramified quadratic extension, then $\mathfrak{p}_E^{\lceil n/2 \rceil} \subset \mathfrak{P}^n$ and since $N_{E/F}$ is surjective on higher unit groups, we see that in fact $N(\mathcal{U}^n) = \mathfrak{u}^{\lceil n/2 \rceil}$.

If E/F is a quadratic extension, let $t = t(E/F) = v_E(\bar{x} - x) - 1$ where $x \in E$ is such that $\mathfrak{o}_E = \mathfrak{o}_F + x\mathfrak{o}_F$. Then $t = -1$ if and only if E/F is unramified, and $t = 0$ if and only if E/F is ramified with odd residual characteristic. If E/F is unramified, we have $N(\mathfrak{u}_E^n) = \mathfrak{u}^n$ for each $n \geq 0$. If E/F is ramified, then for each $n \geq 0$, we have

$$(2.2) \qquad N(\mathfrak{u}_E^{t+2n+1}) = N(\mathfrak{u}_E^{t+2n+2}) = \mathfrak{u}^{t+n+1}.$$

In particular, if E/F is ramified with odd residual characteristic, we have $N(\mathfrak{u}_E^n) = \mathfrak{u}^{\lceil n/2 \rceil}$ for all $n \geq 1$. See [HPS89a] for more details.

2.1. Special orders.

Let E/F denote a quadratic extension in B. In [HPS89a], Hijikata, Pizer and Shemanske considered the local orders

$$(2.3) \qquad \mathcal{O}_r(E) = \mathfrak{o}_E + \mathfrak{P}^{r-1} \qquad (r \geq 1),$$

which they termed *special*. The *level* of a special order $\mathcal{O}$ is $\mathfrak{p}^r$ if r is minimal such that $\mathcal{O} \simeq \mathcal{O}_r(E)$ for some E. Observe that, for any E, $\mathcal{O}_1(E) = \mathcal{O}_B$ is the maximal order. Also note that $\mathcal{O}_r(E)^\times = \mathfrak{o}_E^\times \mathcal{U}^{r-1}$.

We recall a few facts about special orders from *op. cit.*

First, an order $\mathcal{O}$ of B is of the form $\mathcal{O}_r(E)$ for some r if and only if $\mathfrak{o}_E$ embeds in $\mathcal{O}$.

If E/F is unramified, then $\mathcal{O}_{2m-1}(E) = \mathcal{O}_{2m}(E) \neq \mathcal{O}_{2m+1}(E)$ for any $m \geq 1$. In particular, $\mathfrak{o}_E^\times \mathcal{U}^{2m} = \mathfrak{o}_E^\times \mathcal{U}^{2m+1}$, which in fact holds for $m \geq 0$. For E/F unramified, the level of $\mathcal{O}_r(E)$ always has odd valuation.

If E/F is ramified, then the orders $\mathcal{O}_r(E)$ are distinct for all $r \geq 1$. Note that $\mathfrak{o}_E = \mathfrak{o}_F + \mathfrak{p}_E$ as the index of the latter in the former is $[\mathfrak{o}_E : \mathfrak{p}_E]/[\mathfrak{o}_F : \mathfrak{p}_F] = q/q = 1$. So if K/F is another ramified quadratic extension in B, then $\mathfrak{o}_E \subset \mathcal{O}_2(K)$ as $\mathfrak{o}_F \subset \mathfrak{o}_K$ and $\mathfrak{p}_E \subset \mathfrak{P}$. Hence $\mathcal{O}_2(E) \subset \mathcal{O}_2(K)$ and conversely, i.e., $\mathcal{O}_r(E) = \mathcal{O}_r(K)$ for $r = 1, 2$. On the other hand, when q is odd, $\mathfrak{o}_E$ does not embed in $\mathcal{O}_3(K)$, so $\mathcal{O}_r(E)$ and $\mathcal{O}_r(K)$ are non-isomorphic for all $r \geq 3$ when $K \not\simeq E$. (See [HPS89a, Lemma 3.9] for the case of q even.)

2.2. Local representations.

Next we recall some facts about the representation theory of $B^\times$, most of which can be found in, or easily derived from, some combination of [BH06], [Tun78] and [Tun83]. In any case, we at least indicate the proof for facts not explicitly stated in [BH06] (e.g., facts about conductors and dimension formulas).

Let π be a smooth irreducible complex representation of $B^\times$. Let ω_π be the central character of π. For a character μ of $F^\times$, denote by $\pi \otimes \mu$ the twist $(\mu \circ N_{B/F}) \cdot \pi$, which has central character $\omega_\pi \mu^2$. So we can choose a twist $\pi \otimes \mu$ which is trivial on $\langle \varpi \rangle$. Hence, up to twisting, we can view π as a representation of the compact quotient $B^\times / \langle \varpi \rangle$, and thus π is finite dimensional.

By smoothness there is some n such that π restricted to $\mathcal{U}^n$ acts trivially, i.e., $\mathcal{U}^n \subset \ker \pi$. The minimal integer $\ell \geq 0$ such that $\mathcal{U}^{\ell+1} \subset \ker \pi$ is called the (unnormalized) level of π, and is denoted by $\ell = \ell(\pi)$.

Denote by $\pi' = \mathrm{JL}(\pi)$ the Jacquet–Langlands transfer of π to $\mathrm{GL}_2(F)$, which is a discrete series representation of $\mathrm{GL}_2(F)$ with the same central character as that of π. Set $c(\pi) = c(\pi')$, where $c(\pi')$ is the conductor of π'. Alternatively, we can define $c(\pi)$ to be the minimal $n \geq 1$ such that π is trivial on $\mathcal{U}^{n-1}$, and thus $\ell(\pi) = \max\{c(\pi) - 2, 0\}$. That these definitions are the same is easy to see when $\dim \pi = 1$ (see below). For general π, it follows from the fact that $\ell(\pi') = \frac{1}{2}\ell(\pi)$, using the normalization of level for $\mathrm{GL}_2(F)$ as in [BH06] (see Section 56.1), and comparing $\ell(\pi')$ with $c(\pi')$ via ε-factor relations. Note that ω_π must be trivial when restricted to $\mathcal{U}^{c(\pi)-1} \cap F$, so $c(\omega_\pi) \leq \lfloor \frac{c(\pi)}{2} \rfloor$.

We say π is minimal if $c(\pi) \leq c(\pi \otimes \mu)$ for all μ; equivalently, if $\ell(\pi) \leq \ell(\pi \otimes \mu)$ for all μ and $c(\pi) = 1$ if $\dim(\pi) = 1$. If π is minimal, then $c(\pi \otimes \mu) = \max\{c(\pi), 2c(\mu)\}$ [Tun78, Proposition 3.4]. Hence any non-minimal representation has even conductor, and thus even level.

1-dimensional representations. Any 1-dimensional representation of $B^\times$ is of the form $\pi = \mu \circ N$ for some character μ of $F^\times$. Then $\omega_\pi = \mu^2$. Recalling that $N(\mathcal{U}^n) = \mathfrak{u}^{\lceil n/2 \rceil}$ we see that $\ell(\pi) = \max\{2c(\mu) - 2, 0\}$. Since $\pi' = \mathrm{St} \otimes \mu$, where St is the Steinberg representation, we see that $c(\pi) = c(\pi') = \max\{2c(\mu), 1\}$. Hence $\ell(\pi) = \max\{c(\pi) - 2, 0\}$.

Note if $c(\pi) = 1$, then μ is unramified so π is trivial on $\mathcal{U}^0$. Otherwise, $\ell(\pi) = c(\pi) - 2$. Hence we always have that π is trivial on $\mathcal{U}^{c(\pi)-1}$, as asserted above.

Nonabelian representations. Now we describe the higher-dimensional representations of $B^\times$ in terms of induction from certain finite-index subgroups.

Suppose $\dim \pi > 1$. Since $B^\times / \mathcal{O}_B^\times \simeq \mathbb{Z}$, π must be nontrivial on $\mathcal{O}_B^\times$, so $c(\pi) \geq 2$. In fact, π' is supercuspidal. Let $\ell = \ell(\pi) = c(\pi) - 2$, so $\mathcal{U}^{\ell+1}$ is the largest unit group in $\ker \pi$. Recalling that $\mathcal{U}^{\lceil (\ell+1)/2 \rceil} / \mathcal{U}^{\ell+1}$ is abelian, π restricted to $\mathcal{U}^{\lceil (\ell+1)/2 \rceil}$ must break up as a sum of characters.

Fix an additive character $\psi : F \to \mathbb{C}^\times$ of level 1, i.e., ψ is trivial on $\mathfrak{p}$ but not $\mathfrak{o}_F$. For $\alpha \in B$, define
$$\psi_\alpha(x) = \psi(\mathrm{tr}(\alpha(x-1))), \quad x \in B.$$
By essentially the same calculation as in (2.1), we can see that ψ_α is a character of $\mathcal{U}^{\lceil (m+1)/2 \rceil}$ which is nontrivial on $\mathcal{U}^m$ but trivial on $\mathcal{U}^{m+1}$, where $m = -v_B(\alpha) > 0$. Furthermore, any character of $\mathcal{U}^{\lceil (m+1)/2 \rceil} / \mathcal{U}^{m+1}$ is of the form ψ_α for some $\alpha \in \mathfrak{P}^{-m}$.

We say $\alpha \in B^\times$ is minimal if it has odd valuation or if the characteristic polynomial of $\varpi^{-v_B(\alpha)/2} \alpha$ is irreducible mod $\mathfrak{p}$.

Odd-level representations. First suppose $c(\pi) = 2n + 1 \geq 3$, i.e., $\ell(\pi) = 2n - 1$. Necessarily, π is minimal. Since we may view $\pi|_{\mathcal{U}^n}$ as a representation of the abelian group $\mathcal{U}^n / \mathcal{U}^{2n}$, it is a sum of characters ψ_α for some collection of $\alpha \in \mathfrak{P}^{1-2n}$. By Mackey theory and normality of $\mathcal{U}^n$, all of these ψ_α's are conjugate. Fix one such α. Then $v_B(\alpha) = 1 - 2n$.

Let $E = F[\alpha] \subset B$, which is a ramified quadratic extension of F. Then conjugation by some $x \in B^\times$ fixes ψ_α if and only if $x \in J := E^\times \mathcal{U}^n$. From this, one can deduce that $\pi \simeq \mathrm{Ind}_J^{B^\times} \Lambda$ for some character Λ of J such that $\Lambda|_{\mathcal{U}^n} = \psi_\alpha$.

Since $B^\times / J$ has cardinality $|\mathcal{O}_B^\times / \mathfrak{o}_E^\times \mathcal{U}^n| = q^{n-1}(q+1)$, this is the dimension of π.

Even-level representations. Now suppose $c(\pi) = 2n \geq 2$, i.e., $\ell(\pi) = 2n - 2$. We further assume π is minimal. As above, $\pi|_{\mathcal{U}^n}$ is a representation of the abelian group $\mathcal{U}^n/\mathcal{U}^{2n-1}$. Assuming $n \geq 2$, the restriction to $\mathcal{U}^n$ contains a character ψ_α, where α is minimal of valuation $2 - 2n$.

Let E be the unramified quadratic extension of F. For any $n \geq 1$, set $J = E^\times \mathcal{U}^{n-1}$. In this case, we can write $\pi \simeq \operatorname{Ind}_J^{B^\times} \Lambda$ where Λ is a representation of $J/\mathcal{U}^{2n-1}$ of dimension 1 (resp. q) if n is odd (resp. even). Specifically, if $n \geq 2$, then $\Lambda|_{\mathcal{U}^n} \simeq c\psi_\alpha$, for some $\alpha \in E$ as above, where c is 1 or q according to the parity of n (see [BH06, Section 54.7]). If $n = 1$ (i.e., π has level zero), we may take Λ to be a character of $J = E^\times \mathcal{U}^1 = E^\times \mathcal{O}_B^\times$ which is trivial on $\mathcal{U}^1$ with $\Lambda|_{E^\times}$ a character of $E^\times$ of conductor 1. Hence for all $n \geq 1$, we have that $\Lambda|_{E^\times}$ is a sum of characters which are nontrivial on $\mathfrak{u}_E^{n-1}$.

Again, computing $|B^\times/J| = 2|\mathcal{O}_B^\times/\mathfrak{o}_E^\times \mathcal{U}^{n-1}| = 2q^{2\lfloor(n-1)/2\rfloor}$ (times $\dim \Lambda$) gives the dimension of π, and we summarize the even and odd conductor cases together for future reference:

Lemma 2.1. *Suppose π is a minimal representation of $B^\times$ of dimension > 1. If $c(\pi) = 2n + 1$, then $\dim \pi = q^{n-1}(q + 1)$. If $c(\pi) = 2n$, then $\dim \pi = 2q^{n-1}$.*

This is a special case of [Car84, Proposition 6.5], which also covers higher degree division algebras.

Remark 2.1. Since the formal degree $d(\pi')$ of a discrete series representation π' of $\operatorname{GL}_2(F)$, normalized so that $d(\operatorname{St}) = 1$, is simply $\dim \pi$ when $\pi' = \operatorname{JL}(\pi)$, the above lemma gives a formula for the formal degree of a supercuspidal representation of $\operatorname{GL}_2(F)$ in terms of the conductor of a minimal twist.

3. Local new and old forms

We keep the local notation of the previous section.

Let E/F be a quadratic extension, $r \geq 1$, and consider a special order $\mathcal{O} = \mathcal{O}_r(E)$. Let π be a smooth irreducible representation of $B^\times$ with central character ω. Let Ω be a character of $\mathcal{O}^\times = \mathfrak{o}_E^\times \mathcal{U}^{r-1}$ which is trivial on $\mathcal{U}^{r-1}$. Denote by $c_E(\Omega)$ the conductor of $\Omega|_{\mathfrak{o}_E^\times}$. Necessarily, Ω is trivial on $\mathcal{U}^{r-1} \cap E^\times = \mathfrak{u}_E^{\lceil e(r-1)/2 \rceil}$ where $e = e(E/F)$, i.e., $c_E(\Omega) \leq \lceil e(r-1)/2 \rceil$. Consider the subspace of Ω-equivariant vectors

$$(3.1) \qquad \pi^\Omega = \{v \in \pi : \pi(g)v = \Omega(g)v, \ g \in \mathcal{O}^\times\}.$$

The goal of this section is to determine $\dim \pi^\Omega$. We first note the following obvious necessary condition for existence of equivariant vectors.

Lemma 3.1. *We have $\pi^\Omega = 0$ unless Ω and ω agree on $\mathfrak{o}_F^\times$ and $c(\pi) \leq r$.*

Proof. Compatibility of Ω and ω is obviously necessary for $\pi^\Omega \neq 0$. If $c(\pi) > r$, then π is nontrivial on $\mathcal{U}^{r-1}$. Then irreducibility implies π cannot have any vectors fixed by the normal subgroup $\mathcal{U}^{r-1}$. $\qquad\square$

From now on we assume $\Omega|_{\mathfrak{o}_F^\times} = \omega|_{\mathfrak{o}_F^\times}$. If we also have $c(\pi) \leq r$, then we simply have

$$(3.2) \qquad \pi^\Omega = \{v \in \pi : \pi(t) = \Omega(t)v, \ t \in \mathfrak{o}_E^\times\}.$$

We also have the following easy bound on dimension.

Lemma 3.2. *We have* $\dim \pi^\Omega \le e(E/F)$.

Proof. It is well-known that multiplicity one holds for $(B^\times, E^\times)$, meaning in the irreducible decomposition of $\pi|_{E^\times}$, each character χ of $E^\times$ occurs with multiplicity $m(\pi, \chi)$ at most one. Furthermore, each χ appearing must satisfy $\chi|_{F^\times} = \omega$.

We may as well assume $c(\pi) \le r$, so then we have (3.2). Consequently $\pi|_{E^\times}$ acting on the subspace π^Ω is simply the sum over characters χ such that $\chi|_{\mathfrak{o}_E^\times} = \Omega|_{\mathfrak{o}_E^\times}$ and $m(\pi, \chi) = 1$. If E/F is unramified, $F^\times \mathfrak{o}_E^\times = E^\times$ so the compatibility with ω and Ω determines χ uniquely. If E/F is ramified, there are two possible χ which are compatible with both ω and Ω—these are determined by choosing $\chi(\varpi_E)$ so that $\chi(\varpi_E)^2 = \omega(\varpi)$, assuming ϖ_E is chosen to square to ϖ. $\qquad\square$

Let χ be a character of $E^\times$ which is compatible with ω and Ω as in the above proof. By Tunnell [Tun83] and Saito [Sai93], we know χ occurs in $\pi|_{E^\times}$ (and thus contributes a line to π^Ω if $c(\pi) \le r$) if and only if $\varepsilon(1/2, \pi_E \otimes \chi) = -\omega(-1)$, where π_E denotes the base change of π to $(B \otimes E)^\times \simeq \mathrm{GL}_2(E)$. Hence we have the formula

$$(3.3) \qquad \dim \pi^\Omega = -\omega(-1) \sum_{i=1}^{e(E/F)} \frac{\varepsilon(1/2, \pi_E \otimes \chi_i) - \omega(-1)}{2}$$

when $c(\pi) \le r$ and χ_i runs over compatible characters χ.

The local root numbers $\varepsilon(1/2, \pi_E \otimes \chi)$ were calculated in [Tun83] when π' is a dihedral supercuspidal representation. (In the case of odd residual characteristic, all supercuspidals π' are dihedral—[Sai93] reproved Tunnell's main result without computing local root numbers in a way that also works in even characteristic.) For minimal representations, there 4 basic situations, according to whether the level of π is odd or even (i.e., the inducing subgroup $J = J_\pi$ for π as in Section 2.2 contains a ramified or unramified quadratic extension) and whether E/F is unramified or ramified. Half of the time (when the quadratic extension contained in J_π has opposite ramification type as E/F) the description of the characters in $\pi|_{E^\times}$ is simple and depends only on conductors (and compatibility with ω), and thus $\varepsilon(1/2, \pi_E \otimes \chi)$ is easily described. However, the other half of the time, the description of $\varepsilon(1/2, \pi_E \otimes \chi)$ is complicated.

So instead of trying to use (3.3) to compute $\dim \pi^\Omega$, we will examine $\pi|_{\mathfrak{o}_E^\times}$ directly using the description of π as $\mathrm{Ind}_J^{B^\times} \Lambda$ for suitable J, Λ. Things are somewhat simplified by the fact that for our applications we do not need to consider arbitrary ramification of Ω (and to some extent ω). In addition, it seems that the description of $\pi|_{E^\times}$ is simpler when given in terms of the inducing data (J, Λ) rather than the description of π' as dihedrally induced. (It also has the advantage of being applicable when π' is not dihedral.) Hence this approach provides an alternate way to compute root numbers $\varepsilon(1/2, \pi_E \otimes \chi)$ via Mackey theory. This is similar in spirit to the use of Mackey theory on $\mathrm{GL}_2(F)$ in [FMP17, Section 5], though the main goal there was determination of test vectors. That said, we will focus on computing $\dim \pi^\Omega$ when the description is simple enough to give clean global statements (essentially, when it only depends upon conductors), but see Remark 3.3 for more discussion about this.

For 1-dimensional representations, the following is clear.

Lemma 3.3. *Suppose $\pi = \mu \circ N_{B/F}$ and $\mathcal{O} = \mathcal{O}_r(E)$. Then $\pi^\Omega \neq 0$ if and only if $c(\pi) \leq r$ and $\Omega|_{\mathfrak{o}_E^\times} = (\mu \circ N_{E/F})|_{\mathfrak{o}_E^\times}$.*

Next we observe non-minimal representations often do not have Ω-equivariant vectors.

Proposition 3.4. *Suppose π is non-minimal, $\dim \pi > 1$, and write $\pi = \tau \otimes \mu$ where τ is minimal. Let $t = t(E/F)$. Then $\pi^\Omega = 0$ if one of the following conditions is satisfied:*

(i) E/F is unramified and $c_E(\Omega) < \frac{c(\pi)}{2}$; or
(ii) E/F is ramified, q is odd, and $c_E(\Omega) < c(\pi) - 1$; or
(iii) E/F is ramified, $c_E(\Omega) < c(\pi) - t - 1$, $c(\pi) \geq 2t + 4$, and $c(\pi) - c(\tau) > t$.

Proof. Recall $c(\pi) = 2c(\mu) > c(\tau)$. Note that $\pi^\Omega = \tau^{\Omega \otimes \mu^{-1}}$, where $\Omega \otimes \lambda = (\lambda \circ N_{B/F})|_{\mathcal{O}^\times} \cdot \Omega$. So for $\pi^\Omega \neq 0$ we need that $\Omega \otimes \mu^{-1}$ is trivial on $\mathcal{U}^{c(\tau)-1} \subset \ker \tau$. Hence it suffices to show that any of the above conditions imply that $c_E(\Omega \otimes \mu^{-1})$ is strictly bigger than $\lceil e(c(\tau) - 1)/2 \rceil$.

First, suppose E/F is unramified. Note $c(\mu^{-1} \circ N_{E/F}) = c(\mu)$. So if $c_E(\Omega) < c(\mu)$, then $c_E(\Omega \otimes \mu^{-1}) = c(\mu \circ N_{E/F}) = \frac{c(\pi)}{2} > \frac{c(\tau)}{2} \geq \lceil (c(\tau) - 1)/2 \rceil$.

Now suppose E/F is ramified. Since $\dim \pi > 1$ and π non-minimal implies $c(\pi) \geq 4$, (ii) follows from (iii), so assume (iii) holds. By (2.2), $c(\mu^{-1} \circ N_{E/F}) = 2c(\mu) - t - 1$ if $c(\mu) \geq t + 2$, which is equivalent to our assumption $c(\pi) \geq 2t + 4$. Then the conditions on $c_E(\Omega)$ and $c(\pi) - c(\tau)$ imply $c_E(\Omega \otimes \mu^{-1}) = c_E(\mu^{-1} \circ N_{E/F}) > c(\tau) - 1$. $\qquad\square$

Remark 3.1. The conditions in the lemma are in fact necessary. For instance, suppose E/F is unramified, and $\omega_\tau = 1$ and $c(\tau)$ is odd. If we take $\Omega = \mu \circ N|_{\mathcal{O}^\times}$, then $\pi^\Omega = \tau^{\mathcal{O}^\times}$, which we will see is 1-dimensional in Theorem 3.5.

Moreover, we really do need the more complicated condition (iii)—that π be sufficiently more ramified than τ—when E/F is ramified and q is even. For instance, the global calculations in [HPS89b, Examples 10.6, 10.7] imply π^Ω can be nonzero when E is a ramified quadratic extension of $F = \mathbb{Q}_2$, $\Omega = 1$, $c(\pi) = 6$, and $c(\tau) = 5$.

We say Ω is a minimally ramified extension of $\omega|_{\mathfrak{o}_F^\times}$ to $\mathcal{O}^\times$ if $c_E(\Omega) = c(\omega)$ when E/F is unramified and $c_E(\Omega) = \max\{2c(\omega) - 1, 0\}$ when E/F is ramified. It is easy to see that a smaller value of $c_E(\Omega)$ is not possible. We will explain in Section 4.1 how to construct a minimally ramified extension Ω of ω under suitable bounds on $c(\omega)$.

Theorem 3.5. *Suppose E/F is unramified, $\mathcal{O} = \mathcal{O}_{2r+1}(E)$, π is minimal, and Ω is a minimally ramified extension of $\omega|_{\mathfrak{o}_F^\times}$. Then $\pi^\Omega = 0$ unless $c(\pi) \leq 2r + 1$ and one of the following holds:*

(i) $\dim \pi = 1$, in which case $\dim \pi^\Omega = 1$; or
(ii) $\dim \pi > 1$ and $c(\pi)$ is odd, in which case $\dim \pi^\Omega = 1$; or
(iii) $c(\pi)$ is even and $c(\omega) = \frac{c(\pi)}{2}$.

Proof. By Lemma 3.1, we may assume $c(\pi) \leq 2r + 1$. Then by (3.2) and Lemma 3.2, $\dim \pi^\Omega = 1$ if $\Omega|_{\mathfrak{o}_E^\times}$ appears in $\pi|_{\mathfrak{o}_E^\times}$, and $\pi^\Omega = 0$ otherwise.

If $\dim \pi = 1$, then π minimal means $\pi = \mu \circ N$ with μ is unramified (so ω and thus Ω are unramified), and the statement is clear. So assume $\dim \pi > 1$. Let χ denote a character of $E^\times$ whose restriction to $F^\times$ is ω.

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First suppose $c(\pi) = 2n + 1$, so $\dim \pi = q^{n-1}(q+1)$ by Lemma 2.1. Note for χ to appear in $\pi|_{E^\times}$, we need χ to be trivial on $\mathcal{U}^{2n} \cap E^\times = \mathfrak{u}_E^n$. But since the number of χ compatible with ω such that $c(\chi) \leq n$ is $q^{n-1}(q+1)$, by multiplicity one we see that χ appears in π if and only if $c(\chi) \leq n$. (This argument is already in [Tun83].) Hence $\dim \pi^\Omega = 1$.

Now suppose $c(\pi) = 2n$. Then we can write π in the form $\mathrm{Ind}_J^{B^\times} \Lambda$, where $J = E^\times \mathcal{U}^{n-1}$, for a suitable embedding of E into B and representation Λ of J. For $s \in B^\times$, put $J^s = sJs^{-1} \cap \mathfrak{o}_E^\times$ and $\Lambda^s(x) = \Lambda(s^{-1}xs)$. By Mackey theory, we have

$$(3.4) \qquad \pi|_{\mathfrak{o}_E^\times} = \bigoplus_{s \in \mathfrak{o}_E^\times \backslash B^\times / J} \mathrm{Ind}_{J^s}^{\mathfrak{o}_E^\times} \Lambda^s.$$

Since $\mathcal{U}^{n-1} \subset sJs^{-1}$ for any $s \in B^\times$, we have $\mathcal{U}^n \cap \mathfrak{o}_E^\times = \mathfrak{u}_E^{\lceil n/2 \rceil}$ is contained in J^s. In fact since $\mathfrak{o}_E^\times \mathcal{U}^{n-1} = \mathfrak{o}_E^\times \mathcal{U}^{n-2}$ when n is even, we have $\mathfrak{u}_E^{m/2} \subset J^s$ where $m = n - 2$ if n is even and $m = n - 1$ if n is odd. Note m is minimal such that $J = \mathfrak{o}_E^\times \mathcal{U}^m$.

Write

$$B = \{ \begin{pmatrix} x & \varpi y \\ \bar{y} & \bar{x} \end{pmatrix} : x, y \in E \}.$$

Thus $E^\times$ embeds diagonally in $B^\times$ and $\begin{pmatrix} 0 & \varpi \\ 1 & 0 \end{pmatrix}$ is a uniformizer for B. Also, $\mathcal{O}_B$ is the set of elements of the above form with $x, y \in \mathfrak{o}_E$, and $\mathcal{U}^{2j}$ is the set of elements of the above form with $x \in 1 + \mathfrak{p}_E^j$, $y \in \mathfrak{p}_E^j$ for any $j \geq 0$. Hence, we can take a set of representatives for $B^\times / \mathfrak{o}_E^\times \mathcal{U}^m$ to be

$$\{ \begin{pmatrix} 1 & \varpi y \\ \bar{y} & 1 \end{pmatrix} : y \in \mathfrak{o}_E / \mathfrak{p}_E^{m/2} \} \cup \{ \begin{pmatrix} 1 & \varpi y \\ \bar{y} & 1 \end{pmatrix} \begin{pmatrix} 0 & \varpi \\ 1 & 0 \end{pmatrix} : y \in \mathfrak{o}_E / \mathfrak{p}_E^{m/2-1} \}.$$

(If $m = 0$, we interpret $y \in \mathfrak{o}_E / \mathfrak{p}_E^{-1}$ to mean $y = 0$.) It is easy to see this is also a set of representatives for $\mathfrak{o}_E^\times \backslash B^\times / J$.

If $s = \begin{pmatrix} 1 & \varpi y \\ \bar{y} & 1 \end{pmatrix}$, and $t \in E^\times$, we compute

$$s \begin{pmatrix} t & \\ & \bar{t} \end{pmatrix} s^{-1} = \frac{1}{1 - \varpi y \bar{y}} \begin{pmatrix} t - \varpi \bar{t} y \bar{y} & \varpi y(\bar{t} - t) \\ \bar{y}(t - \bar{t}) & \bar{t} - \varpi t y \bar{y} \end{pmatrix} = \begin{pmatrix} t & \\ & \bar{t} \end{pmatrix} + \frac{(t - \bar{t})y}{1 - \varpi y \bar{y}} \begin{pmatrix} \varpi \bar{y} & -\varpi \\ -1 & \varpi y \end{pmatrix}.$$

If $s = \begin{pmatrix} 0 & \varpi \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \varpi y \\ \bar{y} & 1 \end{pmatrix}$, the calculation is the same as above with only the effect of exchanging t and $\bar{t}$ on the right.

Since ψ_α is nontrivial on $\mathfrak{u}_E^{n-1}$, we see $\Lambda|_{\mathfrak{o}_E^\times}$ is a sum of characters which are nontrivial on $\mathfrak{u}_E^{n-1}$ but trivial on $\mathfrak{u}_E^n$. Note $t - \bar{t} \in \mathfrak{p}^j$ for $t \in \mathfrak{u}_E^j$. So the above calculation means either $\Lambda^s(1+x) = \Lambda(1+x)$ or $\Lambda^s(1+x) = \Lambda(1+\bar{x})$ for $x \in \mathfrak{p}_E^{n-1}$. Hence any character appearing in $\pi|_{E^\times}$ has conductor n, whence π^Ω can only be nonzero if $c_E(\Omega) = c(\omega) = n$. $\qquad \square$

Remark 3.2. Gross [Gro88] calculated $\varepsilon(1/2, \pi_E \otimes \chi) = (-1)^{c(\pi)}$ when E/F, χ and ω are all unramified. This combined with (3.3) gives an alternative proof of the above theorem in the case that ω is unramified (so $\Omega = 1$). Conversely, our theorem says that $\varepsilon(1/2, \pi_E \otimes \chi) = (-1)^{c(\pi)}$ when E/F is unramified, π is minimal and $c(\chi) < \frac{c(\pi)}{2}$.

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Remark 3.3. The only case of the theorem where things are not completely settled is (iii). Suppose $c(\pi) = 2n$ and Λ as in the proof. Then $\Lambda|_{\mathfrak{u}_E^{n-1}} \simeq c\lambda$ for some nontrivial character λ of $\mathfrak{u}_E^{n-1}/\mathfrak{u}_E^n$. The proof shows that any χ appearing in $\pi|_{E^\times}$ agrees with λ or $\tilde{\lambda}$ on $\mathfrak{u}_E^{n-1}$, and such χ must also restrict to ω on $F^\times$, where $\tilde{\lambda}(x) = \lambda(\bar{x})$. There are 2 such characters if $n = 1$, and $2(q+1)q^{n-2}$ if $n \geq 2$. On the other hand $\dim \pi = 2q^{n-1}$. So when $n = 1$ this completely characterizes the characters appearing in $\pi|_{E^\times}$ (namely λ and $\tilde{\lambda}$), and when $n \geq 2$ it "almost" does. This supports the idea that $\varepsilon(1/2, \pi_E \otimes \chi)$ may often have a simple description in terms of the inducing data (J, Λ).

Theorem 3.6. *Suppose E/F is ramified, $\mathcal{O} = \mathcal{O}_{2r}(E)$ for $r \geq 1$, π is minimal, and Ω is a minimally ramified extension of $\omega|_{\mathfrak{o}_F^\times}$. Then $\pi^\Omega = 0$ unless $c(\pi) \leq 2r$, in which case we have:*

(i) $\dim \pi^\Omega = 1$ if $\dim \pi = 1$;
(ii) $\dim \pi^\Omega = 2$ if $\dim \pi > 1$ and $c(\pi)$ is even.

Proof. As before, we may assume $c(\pi) \leq 2r$, and let χ denote a character of $E^\times$ such that $\chi|_{F^\times} = \omega$. Again, the case $\dim \pi = 1$ is evident, so assume $\dim \pi > 1$.

First suppose $c(\pi) = 2n$, so $\ell(\pi) = 2n - 2$. For χ to appear in $\pi|_{E^\times}$, we need χ to be trivial on $\mathcal{U}^{2n-1} \cap \mathfrak{o}_E^\times = \mathfrak{u}_E^{2n-1}$, i.e., $c(\chi) \leq 2n - 1$. The number of such χ which are compatible with ω is $2|\mathfrak{o}_E^\times/\mathfrak{o}_F^\times \mathfrak{u}_E^{2n-1}| = 2q^{n-1} = \dim \pi$. Hence $\pi|_{E^\times}$ is simply the sum of all χ compatible with ω such that $c(\chi) \leq 2n - 1$. (Again, this argument is in [Tun83].) In particular, there are two such χ which agree with Ω on $\mathfrak{o}_E^\times$, which gives (ii). $\square$

Remark 3.4. Globally, over $\mathbb{Q}$, one can compare class number formulas for special orders from [HPS89b] to dimensions of spaces of weight 2 newforms of level p^{2n+1}. This comparison suggests, at least for q odd and $\Omega = 1$, that if E/F is ramified with $\mathcal{O} = \mathcal{O}_{2r}(E)$, then $\dim \pi^\Omega = 1$ when $c(\pi) < 2r$ is odd. When $q = 2$, $\omega = \Omega = 1$ and $c(\pi) = 3 < 2r$, [Tun83, Proposition 3.7] tells us indeed $\dim \pi^\Omega = 1$.

4. Quaternionic modular forms and Brandt matrices

Let F be a totally real number field of degree d, with integer ring $\mathfrak{o} = \mathfrak{o}_F$, adele ring $\mathbb{A} = \mathbb{A}_F$, and infinite places $\nu_1, \ldots, \nu_d$. Let B a (totally) definite quaternion algebra over F, so each B_{ν_i} is isomorphic to Hamilton's quaternions $\mathbb{H}$. Let $\mathfrak{D}$ be the discriminant of B, which is the product over finite primes $\mathfrak{p}$ of F at which B ramifies. Also fix a finite-order idele class character $\omega = \bigotimes \omega_v$ of $\mathbb{A}^\times$. Then each ω_{ν_i} is either trivial or the sign character sgn.

Let $L^2(B^\times \backslash B^\times(\mathbb{A}), \omega)$ be the space of square integrable (mod center) functions on $B^\times \backslash B^\times(\mathbb{A})$ which transform by ω on $\mathbb{A}^\times$. Now $B^\times(\mathbb{A})$ acts by right translation R on $L^2(B^\times \backslash B^\times(\mathbb{A}), \omega)$. As $B^\times(\mathbb{A})$ is compact mod center, R breaks up as a direct sum of irreducible representations, which are the irreducible automorphic representations of $B^\times(\mathbb{A})$ with central character ω.

Consider an irreducible automorphic representation $\pi = \bigotimes \pi_v$ of $B^\times(\mathbb{A})$ with central character ω. For $v|\infty$, fix an embedding of $B_v^\times \simeq \mathbb{H}^\times$ into $GL_2(\mathbb{C})$, and for $k \geq 0$, let Sym^k denote the composition of this embedding with k-th symmetric power of the standard representation of $GL_2(\mathbb{C})$. Then any irreducible representation of $B_v^\times$ is equivalent to some twist of Sym^{k_v}, i.e., of the form $\mathrm{Sym}^{k_v} \otimes (\lambda_v \circ N_{B_v/F_v})$, where λ_v is a character of

$\mathbb{R}_{>0}$, since all characters of $B_v^\times$ factor through the reduced norm. Note such a π_v has central character $t \mapsto t^{k_v}\lambda_v(t^2)$, which must be 1 or sgn. Since λ_v is a character of $\mathbb{R}_{>0}$, this forces $\lambda_v(t) = t^{-\frac{k_v}{2}}$, and we see π_v has central character $\omega_v = \text{sgn}^{k_v}$. We call k_v the weight of π_v, $\mathbf{k} = (k_{\nu_1}, \ldots, k_{\nu_d})$ the weight of π, and $\pi_\infty = \bigotimes_{v|\infty} \pi_v$ the infinity type of π. Denote this infinity type π_∞ by $(\rho_\mathbf{k}, V_\mathbf{k})$.

We use the following notation at a finite place v: $\mathfrak{o}_v = \mathfrak{o}_{F,v}$ is the integer ring of F_v; ϖ_v is a uniformizer in F_v; $\mathfrak{p}_v$ is the associated prime ideal of F (and by abuse of notation also the prime of F_v); $\mathcal{O}_{B,v}$ is the unique maximal order of B_v if B_v is division and $M_2(\mathfrak{o}_v)$ otherwise; $\mathfrak{P}_v$ the unique maximal ideal of $\mathcal{O}_{B,v}$ if B_v is division and $\varpi_v \mathcal{O}_{B,v}$ else; and $\mathcal{U}_v^r = 1 + \mathfrak{P}_v^r$. In addition, to uniformize terminology, by a special order of level $\mathfrak{p}_v^r$ when B_v splits, we simply mean an Eichler order of level $\mathfrak{p}_v^r$, i.e., a conjugate of

$$R_0(\mathfrak{p}_v^r) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathfrak{o}_v) : c \equiv 0 \bmod \mathfrak{p}_v^r \}.$$

Denote by $\hat{B}^\times$ the finite part of $B^\times(\mathbb{A})$, which we also regard as the subgroup of $B^\times(\mathbb{A})$ with trivial components at infinity (and will do similarly for $\hat{\mathcal{O}}^\times = \prod_{v<\infty} \mathcal{O}_v^\times$ for an order $\mathcal{O}$). Let $K = \prod_{v<\infty} K_v$ be a compact open subgroup of $\hat{B}^\times$ such that $K_v \simeq \text{GL}_2(\mathfrak{o}_{F,v})$ for almost all v. Let $\Omega = \bigotimes \Omega_v$ be a unitary extension of $\omega|_{\hat{F}^\times \cap K}$ to K such that Ω_v is trivial for almost all v. Let π^Ω be the subspace of π on which K acts by Ω, and similarly for $\pi_v^{\Omega_v}$, $v < \infty$. Then $\dim \pi_v^{\Omega_v} < \infty$ for $v < \infty$, $\dim \pi_v < \infty$ for $v|\infty$, and these dimensions are 1 for almost all v. Hence $\dim \pi^\Omega < \infty$. Moreover, we can choose K sufficiently small and Ω suitably so that $\pi^\Omega \neq 0$. In fact, by our local results for $v < \infty$ we may take $K_v = \mathcal{O}_v^\times$ for some special order $\mathcal{O}_v$ of level $\mathfrak{p}_v^{c(\pi_v)}$, assuming π_v is minimal when B_v is ramified.

Note that if $f \in \pi^\Omega$, then $\pi_\infty(g)f \in \pi^\Omega$ for all $g \in B_\infty^\times$. Denote by $\pi_\mathbf{k}^\Omega$ the subspace of π^Ω consisting of the vectors of highest weight for π_∞. Given any nonzero $f_0 \in \pi_\mathbf{k}^\Omega$, it generates an irreducible π_∞-module V_{f_0}. Since (π_∞, V) is self-dual, we may identify the contragredient $(\pi_\infty^*, V_{f_0}^*)$ with $(\rho_\mathbf{k}, V_\mathbf{k})$. This identification is unique up to scaling. Now fix some $x_0 \in \hat{B}^\times$ such that $f_0(x_0) \neq 0$. To f_0, we associate the map $\varphi : B^\times \backslash B^\times(\mathbb{A}) \to V_\mathbf{k} = V_{f_0}^*$ given by

$$(4.1) \qquad \langle \varphi(x), f \rangle = f_0(x_0)f(x), \quad x \in B^\times(\mathbb{A}), \ f \in V_{f_0}$$

One readily sees that φ satisfies $\varphi(x\alpha g) = \Omega(\alpha)\rho_\mathbf{k}(g^{-1})\varphi(x)$ for $\alpha \in \hat{\mathcal{O}}^\times$ and $g \in B_\infty^\times$. Taking such maps $f_0 \mapsto \varphi$ on a basis of $\pi_\mathbf{k}^\Omega$ gives an embedding of $\pi_\mathbf{k}^\Omega$ into the collection of $\varphi : B^\times \backslash B^\times(\mathbb{A}) \to V_\mathbf{k}$ satisfying this transformation property.

4.1. **Quaternionic modular forms.** Now suppose $K = \hat{\mathcal{O}}^\times$ with $\mathcal{O}$ a special order of B of level $\mathfrak{N} = \prod \mathfrak{p}_v^{r_v}$, by which we mean $\mathcal{O}_v$ is special order of B_v of level $\mathfrak{p}_v^{r_v}$ for each $v < \infty$. In Section 5.3, we will assume $\mathcal{O}_v$ is of "unramified quadratic type" when $v|\mathfrak{D}$ and r_v is odd, but we do not impose this restriction until then.

We want to construct spaces of modular forms on B which correspond to Hilbert modular forms of level $\mathfrak{N}$ and central character (i.e., nebentypus) ω. For such forms to exist, we need that $c(\omega_v) \leq r_v$ for each $v < \infty$ and further that $c(\omega_v) \leq \frac{r_v}{2}$ for $v|\mathfrak{D}$, so we assume this now. (All forms which are primitive at primes $v|\mathfrak{D}$ satisfy this by

[SW93, Theorem 6.8]). We extend $\omega|_{\hat{\mathfrak{o}}_F^\times}$ to a character Ω of $\hat{\mathcal{O}}^\times$ as follows. In fact, for later use, we will extend Ω to be a semigroup homomorphism from $\hat{\mathcal{O}}$ to $\mathbb{C}$.

Let $v < \infty$. On $\mathfrak{o}_v$, we may view ω_v as a character of $\mathfrak{o}_v^\times/\mathfrak{u}_v^{c(\omega_v)}$, which we may pullback to a multiplicative map $\mathfrak{o}_v/\mathfrak{p}^{c(\omega_v)}$ by extending it to be 1 on noninvertible elements if ω_v is unramified and 0 on $\mathfrak{p}_v$ if ω_v is ramified.

If B_v is split, we can realize $\mathcal{O}_v = R_0(\mathfrak{p}_v^n)$. Then set $\Omega_v(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \omega_v(d)$.

If B_v is division, we can realize $\mathcal{O}_v = \mathfrak{o}_{E,v} + \mathfrak{P}_v^{r_v-1}$, where E_v/F_v is a quadratic extension. Assume E_v/F_v is unramified if $r_v = 1$. We first extend ω_v to a semigroup homomorphism Ω_v of $\mathfrak{o}_{E,v}$ such that $c_{E_v}(\Omega_v) = c(\omega_v)$ if E_v/F_v is unramified and $c_{E_v}(\Omega_v) = \max\{2c(\omega_v) - 1, 0\}$ if E_v/F_v is ramified. (This is always possible by Frobenius reciprocity as $\mathfrak{o}_v^\times/\mathfrak{u}_v^m$ embeds in $\mathfrak{o}_{E,v}^\times/\mathfrak{u}_{E,v}^{m'}$ where $m' = m$ or $\max\{0, 2m-1\}$ according to whether E_v/F_v is unramified or ramified.) Assume $\Omega_v = 1$ if ω_v is unramified. If E_v/F_v is unramified, then also $\mathcal{O}_v = \mathfrak{o}_{E,v} + \mathfrak{P}_v^{r_v}$ and for $x \in \mathfrak{o}_{E,v}$, $\alpha \in \mathfrak{P}_v^{r_v}$ we define $\Omega_v(x + \alpha) = \Omega_v(x)$. If E_v/F_v is ramified, simply define $\Omega_v(x + \alpha) = \Omega_v(x)$ for $x \in \mathfrak{o}_{E,v}$, $\alpha \in \mathfrak{P}_v^{r_v-1}$. In either case, this is well defined since $c(\omega_v) \leq \frac{r_v}{2}$, and makes Ω_v a minimally ramified extension of $\omega_v|_{\mathfrak{o}_v^\times}$.

Note that Ω_v is trivial on $\mathcal{U}_v^{r_v}$, and in fact trivial on $\mathcal{U}_v^{r_v-1}$ when B_v is division. Further Ω_v is trivial on $\mathcal{O}_v^\times$ whenever ω_v is unramified. Note the above definition of Ω actually makes it a semigroup homomorphism $\Omega : \hat{\mathcal{O}} \to \mathbb{C}$, which is important for defining Hecke operators and Brandt matrices with character.

Let $\mathbf{k} = (k_{\nu_1}, \ldots, k_{\nu_d})$. We define the space of quaternionic modular forms of weight $\mathbf{k}$, level $\mathcal{O}$, and character Ω to be

$$M_{\mathbf{k}}(\mathcal{O}, \Omega) = \big\{ \varphi : B^\times \backslash B^\times(\mathbb{A}) \to V_{\mathbf{k}} \mid \varphi(x\alpha g) = \Omega(\alpha)\rho_{\mathbf{k}}(g^{-1})\varphi(x),$$
$$\text{for all } x \in B^\times(\mathbb{A}), \, \alpha \in \hat{\mathcal{O}}^\times, \, g \in B_\infty^\times \big\}.$$

When the class number h_F of F is 1, then $\mathbb{A}^\times = F^\times \hat{\mathfrak{o}}^\times F_\infty^\times \subset B^\times \hat{\mathcal{O}}^\times B_\infty^\times$, so all such forms must transform under the center by ω (assuming $\omega_v = \text{sgn}^{k_v}$ for all $v|\infty$—if not this space must be 0). In general, not all such forms will transform on the center by ω, but we define the subspace of those that do to be

$$M_{\mathbf{k}}(\mathcal{O}, \Omega, \omega) = \big\{ \varphi : B^\times \backslash B^\times(\mathbb{A}) \to V_{\mathbf{k}} \mid \varphi(zx\alpha g) = \omega(z)\Omega(\alpha)\rho_{\mathbf{k}}(g^{-1})\varphi(x),$$
$$\text{for all } z \in \mathbb{A}^\times, \, x \in B^\times(\mathbb{A}), \, \alpha \in \hat{\mathcal{O}}^\times, \, g \in B_\infty^\times \big\}.$$

Viewing $M_{\mathbf{k}}(\mathcal{O}, \Omega)$ as a representation space for $F^\times \backslash \mathbb{A}^\times$, we get a decomposition

$$M_{\mathbf{k}}(\mathcal{O}, \Omega) = \bigoplus_\omega M_{\mathbf{k}}(\mathcal{O}, \Omega, \omega),$$

where ω runs over all (necessarily finite order) idele class characters of F which agree with Ω on $\hat{\mathfrak{o}}^\times$ such that $\omega_v = \text{sgn}^{k_v}$ for all $v|\infty$. We note there are at most h_F such ω.

When $\mathbf{k} = \mathbf{0} = (0, 0, \cdots, 0)$, $V_{\mathbf{k}}$ is 1-dimensional, and φ may factor through the (reduced) norm, i.e., $\varphi = \mu \circ N$ for some character μ of $F^\times \backslash \mathbb{A}^\times$. Let $E_{\mathbf{0}}(\mathcal{O}, \Omega, \omega)$ be the linear span of such $\varphi = \mu \circ N$, which we think of as the Eisenstein subspace of $M_{\mathbf{k}}(\mathcal{O}, \Omega, \omega)$ (even though there are no cusps). More explicitly, a basis of $E_{\mathbf{0}}(\mathcal{O}, \Omega, \omega)$ is

given by the set of $\mu \circ N$ where μ ranges over idele class characters such that $\mu^2 = \omega$ and the local components of $\mu \circ N$ and Ω agree on each $\mathcal{O}_{\mathfrak{p}}^\times$. In particular, each such $\mu_{\mathfrak{p}}$ is unramified when $\mathfrak{p} \nmid \mathfrak{N}$. Moreover, if ω (and thus Ω) is trivial, μ runs over the set of quadratic idele class characters (including the trivial character) which are unramified at all finite places (note the number of such μ equals the 2-rank of the narrow Hilbert class group of F).

We can define an inner product on $M_{\mathbf{0}}(\mathcal{O}, \Omega, \omega)$ given by

$$(\varphi, \varphi') = \int_{\mathbb{A}^\times B^\times \backslash B^\times(\mathbb{A})} \varphi(x) \overline{\varphi'(x)} \, dx$$

for a suitable choice of Haar measure dx on $B^\times(\mathbb{A})$. Let $S_{\mathbf{0}}(\mathcal{O}, \Omega, \omega)$ be the orthogonal complement of $E_{\mathbf{0}}(\mathcal{O}, \Omega, \omega)$ in $M_{\mathbf{0}}(\mathcal{O}, \Omega, \omega)$ with respect to this inner product, which we call the space of cusp forms. If $\mathbf{k} \neq \mathbf{0}$, set $E_{\mathbf{k}}(\mathcal{O}, \Omega, \omega) = 0$ and $S_{\mathbf{k}}(\mathcal{O}, \Omega, \omega) = M_{\mathbf{k}}(\mathcal{O}, \Omega, \omega)$.

Using (4.1) to translate from automorphic forms to quaternionic modular forms gives vector-space isomorphisms

$$(4.2) \qquad M_{\mathbf{k}}(\mathcal{O}, \Omega, \omega) \simeq \bigoplus_\pi \pi_{\mathbf{k}}^\Omega,$$

and

$$(4.3) \qquad S_{\mathbf{k}}(\mathcal{O}, \Omega, \omega) \simeq \bigoplus_{\dim \pi > 1} \pi_{\mathbf{k}}^\Omega,$$

where π runs over equivalence classes of irreducible automorphic representations of $B^\times(\mathbb{A})$ with central character ω. We will often identify $\pi_{\mathbf{k}}^\Omega$ with the corresponding subspace of $M_{\mathbf{k}}(\mathcal{O}, \Omega, \omega)$ obtained from (4.1). The dimensions of the left-hand sides are finite (e.g., from finiteness of the class number and the description below), so there are only finitely many nonzero π^Ω's appearing on the right-hand side. In essence, the main goal of this paper is an explicit description of these decompositions for special orders $\mathcal{O}$.

Note one gets analogous decompositions of $M_{\mathbf{k}}(\mathcal{O}, \Omega)$ and $S_{\mathbf{k}}(\mathcal{O}, \Omega)$ by summing the decompositions in (4.2) and (4.3) over suitable ω.

We may identify the (invertible, i.e., locally principal) fractional right $\mathcal{O}$-ideal classes with the set $\mathrm{Cl}(\mathcal{O}) = B^\times \backslash \hat{B}^\times / \hat{\mathcal{O}}^\times$. Now fix a set of representatives $x_1, \ldots, x_h \in \hat{B}^\times$ for $\mathrm{Cl}(\mathcal{O})$. Since any $\varphi \in M_{\mathbf{k}}(\mathcal{O}, \Omega)$ is determined by its values on $x_1, \ldots, x_h$, we can view

$$\varphi : \{x_1, \ldots, x_h\} \to V_{\mathbf{k}},$$

which must satisfy the compatibility condition: if $\gamma x_i = x_i \alpha g$ for some $\gamma \in B^\times$, $\alpha \in \hat{\mathcal{O}}^\times$ and $g \in B_\infty^\times$ (necessarily $g = \gamma_\infty$), then

$$(4.4) \qquad \varphi(x_i) = \Omega(\alpha) \rho_{\mathbf{k}}(g^{-1}) \varphi(x_i).$$

Put $\kappa = \dim V_{\mathbf{k}} = \sum (k_i + 1)$. Then we can view $\varphi \in M_{\mathbf{k}}(\mathcal{O}, \Omega)$ as the element $(\varphi(x_1), \cdots, \varphi(x_h)) \in \mathbb{C}^{\kappa h}$.

Let $V_{\mathbf{k}}^{\Gamma_i, \Omega}$ denote the set of $v \in V_{\mathbf{k}}$ such that $\rho_{\mathbf{k}}(\gamma) v = \Omega(\alpha) v$ for all $\gamma \in \Gamma_i = x_i \hat{\mathcal{O}}^\times B_\infty^\times x_i^{-1} \cap B^\times$, where $\alpha = x_i^{-1} \gamma \gamma_\infty^{-1} x_i \in \hat{\mathcal{O}}^\times$ and $\rho_{\mathbf{k}}(\gamma)$ means $\rho_{\mathbf{k}}(\gamma_\infty)$. Then (4.4) implies each $\varphi(x_i) \in V_{\mathbf{k}}^{\Gamma_i, \Omega}$, which is just the set of Γ_i-invariant vectors in $V_{\mathbf{k}}$ if Ω is

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unramified. This implies that we have a vector-space isomorphism

$$M_{\mathbf{k}}(\mathcal{O}, \Omega) \simeq \bigoplus_{i=1}^{h} V_{\mathbf{k}}^{\Gamma_i, \Omega}.$$

Namely, for $1 \leq i \leq h$, we associate with an element $v_i \in V_{\mathbf{k}}^{\Gamma_i, \Omega}$ the unique element of $M_{\mathbf{k}}(\mathcal{O}, \Omega)$ such that $\varphi(x_i) = v_i$ and $\varphi(x_j) = 0$ for $j \neq i$.

4.2. Hecke operators.
Now we define Hecke operators, whose action will be given by Brandt matrices (with character) operating on $M(\mathcal{O}, \Omega) \simeq \bigoplus V_{\mathbf{k}}^{\Gamma_i, \Omega} \subset \mathbb{C}^{h\kappa}$.

For $v < \infty$, let $\mathcal{O}_v^{\bullet} = \mathcal{O}_v \cap B_v^{\times}$. We have chosen Ω so that it is a semigroup homomorphism from $\hat{\mathcal{O}}^{\bullet}$ to $\mathbb{C}$, where $\hat{\mathcal{O}}^{\bullet}$ is the set of $(\alpha_v) \in \hat{B}^{\times}$ such that $\alpha_v \in \mathcal{O}_v^{\bullet}$ for all $v < \infty$. (Our prescription of Ω is not as specific as that in [HPS89a], [HPS89b].)

For $\varphi \in M_{\mathbf{k}}(\mathcal{O}, \Omega)$ and $\alpha \in \hat{\mathcal{O}}^{\bullet}$, we define the double coset operator $\hat{\mathcal{O}}^{\times} \alpha \hat{\mathcal{O}}^{\times}$ by

$$(4.5) \qquad (\hat{\mathcal{O}}^{\times} \alpha \hat{\mathcal{O}}^{\times} \cdot \varphi)(x) = \sum_{\beta} \Omega^{-1}(\beta) \varphi(x\beta), \quad \text{where } \hat{\mathcal{O}}^{\times} \alpha \hat{\mathcal{O}}^{\times} = \bigsqcup_{\beta} \beta \hat{\mathcal{O}}^{\times}.$$

Here by $\Omega^{-1}(\beta)$ we mean $\frac{1}{\Omega(\beta)}$ if $\Omega(\beta) \neq 0$ and 0 if $\Omega(\beta) = 0$. Note that if $\beta' = \beta u$ where $u \in \hat{\mathcal{O}}^{\times}$, then $\Omega^{-1}(\beta')\varphi(x\beta') = \Omega^{-1}(\beta)\varphi(x\beta)$, so (4.5) does not depend upon the choice of decomposition $\hat{\mathcal{O}}^{\times} \alpha \hat{\mathcal{O}}^{\times} = \bigsqcup_{\beta} \beta \hat{\mathcal{O}}^{\times}$.

For a nonzero integral ideal $\mathfrak{n}$, let $T_{\mathfrak{n}} = T_{\mathfrak{n}}^{\mathcal{O}}$ denote the formal sum of (distinct) double cosets $\hat{\mathcal{O}}^{\times} \alpha \hat{\mathcal{O}}^{\times} \subset \hat{B}^{\times}$, where α runs over elements of $\hat{\mathcal{O}}^{\bullet}$ such that $N(\alpha) \in \hat{F}^{\times}$ corresponds to the integral ideal $\mathfrak{n}$, i.e., such that $N(\alpha)\hat{\mathfrak{o}} \cap F = \mathfrak{n}$. In the obvious way, we may view $T_{\mathfrak{n}}$ as a (Hecke) operator on $M_{\mathbf{k}}(\mathcal{O}, \Omega)$ via (4.5).

Note for $\alpha = (\alpha_{\mathfrak{p}}) \in \hat{\mathcal{O}}^{\bullet}$, $\mathcal{I} = \bigcap \alpha_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}$ is an integral (nonzero locally principal) right $\mathcal{O}$-ideal. Conversely, given any integral (nonzero locally principal) right $\mathcal{O}$-ideal $\mathcal{I}$, we can write each $\mathcal{I}_{\mathfrak{p}} = \alpha_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}$, where $\alpha_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}}^{\times}$. Thus the integral right $\mathcal{O}$-ideals are in one-to-one correspondence with $\hat{\mathcal{O}}^{\bullet}/\hat{\mathcal{O}}^{\times}$. Hence we have

$$(4.6) \quad (T_{\mathfrak{n}}\varphi)(x) = \sum_{\beta \in \hat{\mathcal{O}}(\mathfrak{n})/\hat{\mathcal{O}}^{\times}} \Omega^{-1}(\beta)\varphi(x\beta), \quad \text{where } \hat{\mathcal{O}}(\mathfrak{n}) = \{\alpha \in \hat{\mathcal{O}}^{\bullet} : N(\alpha)\hat{\mathfrak{o}} \cap F = \mathfrak{n}\}.$$

In particular, if ω is unramified so Ω is trivial on $\hat{\mathcal{O}}$, we can just interpret $T_{\mathfrak{n}}\varphi$ as the sum of right translates of φ by integral right $\mathcal{O}$-ideals of norm $\mathfrak{n}$. We always have that $T_{\mathfrak{o}}$ acts trivially on $M_{\mathbf{k}}(\mathcal{O}, \Omega)$.

The full Hecke algebra $\mathcal{H}(\mathcal{O}, \Omega)$ for $M_{\mathbf{k}}(\mathcal{O}, \Omega)$ is the algebra over $\mathbb{C}$ generated by all $T_{\mathfrak{n}}$'s. The unramified Hecke algebra $\mathcal{H}^S = \mathcal{H}^S(\mathcal{O}, \Omega)$ is the subalgebra generated by the $T_{\mathfrak{n}}$'s for $\mathfrak{n}$ coprime to all prime ideals $\mathfrak{p}$ such that $\mathcal{O}_{\mathfrak{p}} \not\simeq M_2(\mathfrak{o}_{\mathfrak{p}})$ (implicitly, S denotes the set of such primes $\mathfrak{p}$ together with the set of infinite places). Since $\mathcal{H}^S$ is a commutative algebra of normal operators, $M_{\mathbf{k}}(\mathcal{O}, \Omega)$ has a basis of eigenforms for $\mathcal{H}^S$. Moreover, if $\varphi \in M_{\mathbf{k}}(\mathcal{O}, \Omega)$ is an eigenform for $\mathcal{H}^S$, then $\varphi \in \pi_{\mathbf{k}}^{\Omega}$ for some irreducible π in the decomposition (4.2).

Since $M_{\mathbf{k}}(\mathcal{O}, \Omega)$ has a factorizable basis of eigenfunctions $\varphi = \bigotimes \varphi_v$, we can decompose our global Hecke operators into a product a local Hecke operators via

$$(4.7) \qquad\qquad T_{\mathfrak{n}}\varphi = \prod T_{\mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{n})}} \varphi_{\mathfrak{p}},$$

where we let $T_{\mathfrak{p}^m}$ act on the local component $\varphi_{\mathfrak{p}}$ by the local analogue of (4.6). (We can view each local operator $T_{\mathfrak{p}^m}$ as acting on local representation spaces $\pi_{\mathfrak{p}}$ and locally β runs over elements of $\mathcal{O}_{\mathfrak{p}}^{\bullet}/\mathcal{O}_{\mathfrak{p}}^{\times}$ such that $v_{B_{\mathfrak{p}}}(\beta) = m$.) When $\mathcal{O}_{\mathfrak{p}} \simeq M_2(\mathfrak{o}_{\mathfrak{p}})$, these local Hecke operators are the usual unramified local Hecke operators for $\mathrm{GL}_2(F_{\mathfrak{p}})$.

The following is a quaternionic analogue of the calculation of ramified Hecke eigenvalues for elliptic or Hilbert newforms (e.g., [AL70, Theorem 3]).

Proposition 4.1. *Let π an irreducible representation appearing in* (4.2). *Let $\mathfrak{p}|\mathfrak{D}$, $\mathfrak{p}^{n_{\mathfrak{p}}}$ be the level of the local order $\mathcal{O}_{\mathfrak{p}}$, and $m \geq 1$.*

(i) *If $\omega_{\mathfrak{p}}$ is ramified then $T_{\mathfrak{p}^m} = 0$ on $\pi_{\mathbf{k}}^{\Omega}$.*

(ii) *If $\omega_{\mathfrak{p}}$ is unramified, $n_{\mathfrak{p}} = 1$, and $\pi_{\mathfrak{p}} \simeq \mu_{\mathfrak{p}} \circ N_{B_{\mathfrak{p}}/F_{\mathfrak{p}}}$ for an unramified character $\mu_{\mathfrak{p}}$ of $F_{\mathfrak{p}}^{\times}$, then $T_{\mathfrak{p}^m}$ acts by $\mu_{\mathfrak{p}}(\varpi_{\mathfrak{p}})^m$ on $\pi_{\mathfrak{p}}$.*

(iii) *If $n_{\mathfrak{p}} = c(\pi_{\mathfrak{p}}) \geq 2$, then $T_{\mathfrak{p}^m} = 0$ on $\pi_{\mathbf{k}}^{\Omega}$.*

(iv) *If $c(\pi_{\mathfrak{p}}) \geq 3$ and $m \geq n_{\mathfrak{p}} - c(\pi_{\mathfrak{p}}) - 1$, then $T_{\mathfrak{p}^m} = 0$ on $\pi_{\mathbf{k}}^{\Omega}$.*

Proof. As explained above, the proposition boils down to a local calculation. So for simplicity, for the proof, we drop subscripts and revert to the notation of the local sections. E.g., F, B, $\mathcal{O}$, etc. now denote what were earlier denoted by $F_{\mathfrak{p}}$, $B_{\mathfrak{p}}$, $\mathcal{O}_{\mathfrak{p}}$, etc. In particular, $\mathcal{O}$ is a special order of level $\mathfrak{p}^n$, Ω is a minimally ramified extension of (the restriction to $\mathfrak{o}^{\times}$ of) $\omega = \omega_{\pi}$ to $\mathcal{O}^{\times}$, and Ω extends to a multiplicative function of $\mathcal{O}$ such that $\Omega = 1$ if ω is unramified and $\Omega = 0$ outside $\mathcal{O}^{\times}$ if ω is ramified. Consequently, $T_{\mathfrak{p}^m} = 0$ if ω is ramified, and we may assume ω is unramified and $\Omega = 1$.

Then the local Hecke operator is simply

$$(T_{\mathfrak{p}^m}\varphi)(x) = \sum_{\beta} \varphi(x\beta),$$

where β runs over the elements of $\mathcal{O}/\mathcal{O}^{\times}$ of norm q^m, i.e., β runs over the right $\mathcal{O}$-ideals of norm $\mathfrak{p}^m$.

Suppose $\mathcal{O} = \mathcal{O}_B$ is the maximal order. Then $\pi = \varphi = \mu \circ N$ for some unramified character μ of $F^{\times}$ whose square is ω. Hence $(T_{\mathfrak{p}^m}\varphi)(x) = \mu(N(x\varpi_B^m)) = \mu(\varpi^m)\varphi(x)$, which gives the second part.

Now suppose $\mathcal{O}$ has level $\mathfrak{p}^n$, where $n \geq 2$.

Consider an element $\beta \in \mathcal{O}$ with $v_B(\beta) = m$. Then $u\beta \in \beta + \mathfrak{P}^{m+m'}$ for any $u \in \mathcal{U}^{m'}$. In particular, if we take $m' \geq \max\{1, n-m-1\}$, then such $u\beta$ lies in $\mathcal{O}$ with valuation m. Suppose $c(\pi) \geq 3$, $m \geq n - c(\pi) - 1$, and take $m' = c(\pi) - 2$. Then the abelian group $\mathcal{U}^{m'}/\mathcal{U}^{n-1}$, which is nontrivial if $n \geq 3$, acts on the left on the right $\mathcal{O}$-ideals of norm $\mathfrak{p}^m$. Now π restricted to $\mathcal{U}^{m'}$ breaks up as a sum of characters, all of which are nontrivial. Assume φ is an eigenfunction of one of these characters, say ψ'. Then $(T_{\mathfrak{p}^m}\varphi)(x)$ breaks up as a sum of expressions of the form $\sum_u \psi'(u)\varphi(x\beta)$, all of which are 0. In particular, for all $n \geq 3$ and $m \geq 1$, $T_{\mathfrak{p}^m}$ annihilates all π with $c(\pi) = n$. This proves case (iv) and, in the situation that $n_{\mathfrak{p}} \geq 3$, case (iii).

Finally, suppose $n = 2$. Then $\mathcal{O} = \mathfrak{o}_E + \mathfrak{P}$, where E/F is a (in fact any) ramified quadratic extension. Since $\mathfrak{P} \subset \mathcal{O}$, $\mathcal{O}_B^{\times}$ acts (transitively) by left multiplication on the elements of $\mathcal{O}$ of a fixed valuation $m \geq 1$. Then, as above, if $c(\pi) = 2$, $(T_{\mathfrak{p}^m}\varphi)$ breaks up as a sum of sums of nontrivial characters on the abelian group $\mathcal{O}_B^{\times}/\mathcal{U}^1$, and thus kills π. $\square$

Remark 4.1. Continuing with the local notation in the proof, if $\dim \pi = 1$, again we can write $\pi = \varphi = \mu \circ N$. Suppose $\Omega = 1$. Then by Theorem 3.5 and Theorem 3.6, the only way $\pi^\Omega \neq 0$ is if μ is trivial on $N_{E/F}(\mathfrak{o}_E^\times)$. Now $(T_{\mathfrak{p}^m}\varphi)(x) = \sum \mu(N(\beta))\varphi(x)$, where β runs over the elements of $\mathcal{O}/\mathcal{O}^\times$ of norm q^m. This generalizes (ii) to $n_{\mathfrak{p}} \geq 1$. If μ is unramified, we see the local Hecke eigenvalue is just $\mu(q^m)$ times the number of $\mathcal{O}$-ideals of norm $\mathfrak{p}^m$.

We will also use the following calculation of ramified Hecke eigenvalues when $B_{\mathfrak{p}}$ is split. Note, even though $B_{\mathfrak{p}} \simeq M_2(F_{\mathfrak{p}})$, these local ramified Hecke operators are *not* the standard ones that will arise in the next section, however this calculation shows that they do agree on new forms of the appropriate level. (A general comparison of these two definitions of ramified Hecke operators seems not so simple.)

Proposition 4.2. *Suppose $\mathfrak{p} \nmid \mathfrak{D}$, $n_{\mathfrak{p}} \geq 1$ is the level of the local Eichler order $\mathcal{O}_{\mathfrak{p}}$, and $m \geq 1$. Let π an irreducible representation appearing in (4.2) such that $c(\pi_{\mathfrak{p}}) = n_{\mathfrak{p}}$. Then $T_{\mathfrak{p}^m}$ acts by $\hat{\mathcal{O}}^\times \begin{pmatrix} \varpi_{\mathfrak{p}}^m & \\ & 1 \end{pmatrix} \hat{\mathcal{O}}^\times$ on π^Ω.*

Proof. As in the previous proof, this is really a local calculation, so we will just use local notation and drop the $\mathfrak{p}$'s from our subscripts. However, now $B \simeq M_2(F)$ is the split local quaternion algebra, and we may take $\mathcal{O} = R_0(\mathfrak{p}^n)$. By assumption $\dim \pi^{\mathcal{O}^\times} = 1$.

Take a (nonzero) new vector $W \in \pi^{\mathcal{O}^\times}$ in the Whittaker model with respect to an additive character ψ of order 0. We know $W(1) = 1$, so it suffices to compute

$$T_{\mathfrak{p}^m} W(1) = \sum_g \omega^{-1}(d) W(g),$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ runs over the elements in $\mathcal{O}/\mathcal{O}^\times$ with $v(\det g) = m$. Note that we can break up the set of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathcal{O}$ with $v(\det g) = m$ into two disjoint subsets X and Y, where X consists of such g with $d \in \mathfrak{o}^\times$ and Y consists of such g with $d \in \mathfrak{p}$. For any such $g \in X$, we can write

$$g = \begin{pmatrix} d^{-1} & bd^{-1} \\ & 1 \end{pmatrix} \begin{pmatrix} \varpi^m & \\ & 1 \end{pmatrix} \begin{pmatrix} u & \\ c & d \end{pmatrix} \in \mathcal{O}^\times \begin{pmatrix} \varpi^m & \\ & 1 \end{pmatrix} \mathcal{O}^\times,$$

where $u\varpi^m = \det g$. Hence $X = \mathcal{O}^\times \begin{pmatrix} \varpi^m & \\ & 1 \end{pmatrix} \mathcal{O}^\times$, and it suffices to show that there is no contribution from $g \in Y/\mathcal{O}^\times$ to the above sum for $T_{\mathfrak{p}^m} W(1)$.

Let $U = \{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} : x \in \mathfrak{p}^{-1}/\mathfrak{o} \}$. Note that U acts by left multiplication on Y. Let s_g be the size of the stabilizer in U of $g\mathcal{O}^\times$, which only depends upon the U-orbit of g. Then the contribution of $g \in Y/\mathcal{O}^\times$ to $T_{\mathfrak{p}^m} W(1)$ is

$$\sum_{g \in Y/\mathcal{O}^\times} \omega^{-1}(d) W(g) = \sum_{g \in U \backslash Y/\mathcal{O}^\times} \frac{1}{s_g} \omega^{-1}(d) \sum_{x \in \mathfrak{p}^{-1}/\mathfrak{o}} W(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g).$$

The inner sum is $\sum \psi(x) W(g)$, which vanishes since ψ is nontrivial on $\mathfrak{p}^{-1}/\mathfrak{o}$. $\qquad \square$

4.3. **Brandt matrices.** Here we explain how to describe $T_\mathfrak{n}$ in terms of matrices. Recall we have a fixed set of representatives $x_1, \ldots, x_h$ for $\mathrm{Cl}(\mathcal{O})$. We also fix a basis $v_1, \ldots, v_\kappa$ of $V_\mathbf{k}$, and use this to realize $\rho_\mathbf{k}(\gamma) \in \mathrm{GL}_\kappa(\mathbb{C})$ for $\gamma \in B^\times$. Regard any $\varphi \in M_\mathbf{k}(\mathcal{O}, \Omega)$ as a column vector ${}^t(\varphi(x_1) \cdots \varphi(x_h)) \in \mathbb{C}^{h\kappa}$. To simplify notation, we extend Ω trivially from $\hat{\mathcal{O}}$ to $\hat{\mathcal{O}} \times B_\infty^\times$, i.e., $\Omega(\beta, g) = \Omega(\beta)$ for $\beta \in \hat{\mathcal{O}}$, $g \in B_\infty^\times$.

For $1 \le i \le h$, let $\bar{\Gamma}_i = \Gamma_i / \mathfrak{o}^1$, where $\mathfrak{o}^1$ denotes the group of (absolute) norm one units of $\mathfrak{o}$. Since $\rho_\mathbf{k}$ is trivial on $\mathfrak{o}^1$, the finite group $\bar{\Gamma}_i$ acts on $V_\mathbf{k}$ via $\rho_\mathbf{k}$. Then $V_\mathbf{k}$ has a canonical decomposition into isotypic subspaces with respect to this action, with $V_\mathbf{k}^{\Gamma_i, \Omega}$ being one of these subspaces. Put $e_i = |\bar{\Gamma}_i|$. The map

$$\Xi_i(v) = \frac{1}{e_i} \sum_{\gamma \in \bar{\Gamma}_i} \Omega(x_i^{-1} \gamma x_i) \rho_\mathbf{k}(\gamma^{-1}) v$$

defines the orthogonal projection $\Xi_i : V_\mathbf{k} \to V_\mathbf{k}^{\Gamma_i, \Omega}$ with respect to the canonical decomposition. Let Ξ be the block diagonal matrix with diagonal blocks $\Xi_1, \ldots, \Xi_h$. Then Ξ is a linear projection from $\mathbb{C}^{h\kappa}$ to $M_\mathbf{k}(\mathcal{O}, \Omega)$, and for a nonzero integral ideal $\mathfrak{n}$, we define the Brandt matrix $A_\mathfrak{n}$ to be the $h\kappa \times h\kappa$ matrix such that

$$(4.8) \qquad\qquad\qquad\qquad A_\mathfrak{n} = T_\mathfrak{n} \circ \Xi,$$

as a linear operator from $\mathbb{C}^{h\kappa}$ to $M_\mathbf{k}(\mathcal{O}, \Omega)$.

We now describe the entries of $A_\mathfrak{n}$ in classical terms. For $1 \le i \le h$, let $\mathcal{I}_i = (x_i \hat{\mathcal{O}} \times B_\infty) \cap B$ denote the right $\mathcal{O}$-ideal associated to x_i. Then Γ_i is the unit group of the left order $\mathcal{O}_l(\mathcal{I}_i) = (x_i \hat{\mathcal{O}} x_i^{-1} \times B_\infty) \cap B$ of $\mathcal{I}_i$.

Proposition 4.3. *Let a_{ij} denote the (i,j)-th $\kappa \times \kappa$ block of the Brandt matrix $A_\mathfrak{n}$. Then*

$$(4.9) \qquad\qquad\qquad a_{ij} = \frac{1}{e_j} \sum_\gamma \Omega^{-1}(x_i^{-1} \gamma x_j) \rho_\mathbf{k}(\gamma),$$

where $e_j = [\mathcal{O}_l(\mathcal{I}_j)^\times : \mathfrak{o}^1]$ and γ runs over a set of representatives for

$$\{\gamma \in \mathcal{I}_i \mathcal{I}_j^{-1} : N(\gamma)\mathfrak{o} = \mathfrak{n} N(\mathcal{I}_i \mathcal{I}_j^{-1})\}/\mathfrak{o}^1.$$

Proof. For $1 \le i, j \le h$ and $\beta \in \hat{\mathcal{O}}^\bullet$, put

$$\Gamma^{i,j}(\beta) = x_i \beta \hat{\mathcal{O}}^\times B_\infty^\times x_j^{-1} \cap B^\times.$$

Note $\Gamma^{i,j}(\beta)$ only depends upon the class of β in $\hat{\mathcal{O}}^\bullet / \hat{\mathcal{O}}^\times$. Conversely, $\Gamma^{i,j}(\beta) \cap \Gamma^{i,j}(\beta') \ne \emptyset$ implies $\beta' \in \beta \hat{\mathcal{O}}^\times$. Moreover if $\Gamma^{i,j}(\beta) \ne \emptyset$, then $\Gamma^{i,j}(\beta) = \gamma_\beta \Gamma_j$ for any choice of $\gamma_\beta \in \Gamma^{i,j}(\beta)$. Hence there is a bijection between the classes of $\beta \in \hat{\mathcal{O}}^\bullet / \hat{\mathcal{O}}^\times$ such that $\Gamma^{i,j}(\beta) \ne \emptyset$ and the classes of $\gamma_\beta \in (x_i \hat{\mathcal{O}}^\bullet B_\infty^\times x_j^{-1} \cap B^\times)/\Gamma_j$ via $\Gamma^{i,j}(\beta) = \gamma_\beta \Gamma_j$.

Since $(T_\mathfrak{n}\varphi)(x_i) = \sum_\beta \Omega^{-1}(\beta)\varphi(x_i \beta)$ where β runs over $\hat{\mathcal{O}}(\mathfrak{n})/\hat{\mathcal{O}}^\times$ as in (4.6), for $v \in V_\mathbf{k}^{\Gamma_j, \Omega}$ we have

$$a_{ij}v = \sum_\beta \Omega^{-1}(\beta u_\beta^{-1})\rho_\mathbf{k}(\gamma_\beta)v = \sum_\beta \Omega^{-1}(x_i^{-1}\gamma_\beta x_j)\rho_\mathbf{k}(\gamma_\beta)v,$$

where now β runs over a set of representatives of $\hat{\mathcal{O}}(\mathfrak{n})/\hat{\mathcal{O}}^\times$ such that $\Gamma^{i,j}(\beta) \ne \emptyset$, and for each such β we choose a $u_\beta \in \hat{\mathcal{O}}^\times$ and $\gamma_\beta \in \Gamma^{i,j}(\beta)$ such that $x_i \beta = \gamma_\beta x_j u_\beta \gamma_{\beta,\infty}^{-1}$.

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Then the matrix a_{ij} can be computed as the above operator (which depends upon the choices of γ_β's) times Ξ_j. Multiplying these yields

$$a_{ij} = \frac{1}{e_j} \sum_\beta \sum_{\gamma \in \bar{\Gamma}_j} \Omega^{-1}(x_i^{-1}\gamma_\beta\gamma^{-1}x_j)\rho_{\mathbf{k}}(\gamma_\beta\gamma^{-1}).$$

Now applying the change of variable $\gamma_\beta\gamma^{-1} \mapsto \gamma$ gives (4.8) with γ running over a set of representatives of $(x_i\hat{\mathcal{O}}(\mathfrak{n})B_\infty^\times x_j^{-1} \cap B^\times)/\mathfrak{o}^1$. Since $(x_i\hat{\mathcal{O}}x_j^{-1} \times B_\infty) \cap B = \mathcal{I}_i\mathcal{I}_j^{-1}$, the proposition follows. $\square$

Note that in the special case of trivial weight and character ($\mathbf{k} = \mathbf{0}$, $\Omega = 1$), we simply get

$$a_{ij} = |\{\gamma \in \mathcal{I}_i\mathcal{I}_j^{-1} : N(\gamma)\mathfrak{o} = \mathfrak{n}N(\mathcal{I}_i\mathcal{I}_j^{-1})\}/\mathcal{O}_l(\mathcal{I}_j)^\times|.$$

We also define a Brandt matrix A_0 associated to the zero ideal. If $\mathbf{k} = \mathbf{0}$ and $\Omega = 1$, let A_0 be the $h \times h$ diagonal matrix whose i-th diagonal entry is $\frac{1}{e_i}$. Otherwise, set $A_0 = 0$.

The above approach to defining Brandt matrices is similar in spirit to that in [HPS89b], which treats special orders with character over $\mathbb{Q}$. Brandt matrices without character were also defined for maximal orders over totally real fields by Eichler in [Eic77]. Our definition coincides with Eichler's (up to a formal difference of working with right ideals rather than left ideals).

4.4. **Quaternionic new and old forms.** For a special order $\mathcal{O}' \supset \mathcal{O}$, we say Ω' is an admissible extension of Ω to $\hat{\mathcal{O}}'$ if Ω' is an extension of ω to $\hat{\mathcal{O}}'$ constructed as in Section 4.1 which agrees with Ω on $\hat{\mathcal{O}}$. We define the subspace $M_{\mathbf{k}}^{\mathrm{old}}(\mathcal{O}, \Omega, \omega)$ of *oldforms* in $M_{\mathbf{k}}(\mathcal{O}, \Omega, \omega)$ to be the space generated by $\varphi \in M_{\mathbf{k}}(\mathcal{O}, \Omega, \omega)$ such that $\varphi \in M_{\mathbf{k}}(\mathcal{O}', \Omega', \omega)$, where $\mathcal{O}'$ is a special order properly containing $\mathcal{O}$ and Ω' is an admissible extension of Ω to $\hat{\mathcal{O}}'$. Define the subspace $M_{\mathbf{k}}^{\mathrm{new}}(\mathcal{O}, \Omega, \omega)$ of *newforms* to be the orthogonal complement of $M_{\mathbf{k}}^{\mathrm{old}}(\mathcal{O}, \Omega, \omega)$ in $M_{\mathbf{k}}(\mathcal{O}, \Omega, \omega)$ with respect to (4.2).

We define the new and old spaces of cusp forms and Eisenstein forms in $M_{\mathbf{k}}(\mathcal{O}, \Omega, \omega)$ to be the intersection of the new and old spaces with $S_{\mathbf{k}}(\mathcal{O}, \Omega, \omega)$ and $E_{\mathbf{0}}(\mathcal{O}, \Omega, \omega)$, and denote them in a similar way.

Proposition 4.4. *(i) We have the decomposition*

$$S_{\mathbf{k}}^{\mathrm{new}}(\mathcal{O}, \Omega, \omega) \simeq \bigoplus \pi_{\mathbf{k}}^\Omega,$$

where π runs over the representations in (4.3) such that $c(\pi) := \prod \mathfrak{p}^{c(\pi_\mathfrak{p})} = \mathfrak{N}$.
(ii) Moreover, as $\mathcal{H}^S(\mathcal{O}, \Omega)$-modules, we have

$$S_{\mathbf{k}}^{\mathrm{old}}(\mathcal{O}, \Omega, \omega) \simeq \bigoplus S_{\mathbf{k}}^{\mathrm{new}}(\mathcal{O}', \Omega', \omega),$$

where $\mathcal{O}'$ runs over all special orders of B properly containing $\mathcal{O}$, and Ω' runs over all admissible semigroup homomorphisms of $\hat{\mathcal{O}}'$ extending Ω.

Note that Ω' admissible implies that when $B_\mathfrak{p}$ is split (resp. ramified) $\Omega'_\mathfrak{p}$ is trivial on $\mathcal{U}_\mathfrak{p}^{r'_\mathfrak{p}}$ (resp. $\mathcal{U}_\mathfrak{p}^{r'_\mathfrak{p}-1}$), where $\mathfrak{p}^{r'_\mathfrak{p}}$ is the level of $\mathcal{O}'_\mathfrak{p}$. It will follow from the proof below that, for each such $\mathcal{O}'$, there is always at most one such admissible Ω'.

Proof. Consider π occurring in (4.3), and write $n_{\mathfrak{p}} = c(\pi_{\mathfrak{p}})$. By Lemma 3.1 and its well-known analogue for places where B splits, if $c(\pi) = \mathfrak{N}$, then $\pi_{\mathbf{k}}^{\Omega} \subset S_{\mathbf{k}}^{\text{new}}(\mathcal{O}, \Omega, \omega)$. This proves one direction of (i).

Now suppose $c(\pi) := \prod \mathfrak{p}^{n_{\mathfrak{p}}} \neq \mathfrak{N} := \prod \mathfrak{p}^{r_{\mathfrak{p}}}$, so $n_{\mathfrak{p}} \leq r_{\mathfrak{p}}$ for all $\mathfrak{p}$ and this equality is strict for at least one $\mathfrak{p}$. Suppose $\mathfrak{p}$ is such that $n_{\mathfrak{p}} < r_{\mathfrak{p}}$.

If $B_{\mathfrak{p}}$ is split, we may take $\mathcal{O}_{\mathfrak{p}} = R_0(\mathfrak{p}^{r_{\mathfrak{p}}})$. By local newform theory (see [Cas73]) we have that $\pi_{\mathfrak{p}}^{\Omega_{\mathfrak{p}}'}$ is generated by the lines which are $\Omega_{\mathfrak{p}}'$-equivariant under the orders $\begin{pmatrix} \mathfrak{o}_{\mathfrak{p}} & \mathfrak{p}_{\mathfrak{p}}^{-j} \\ \mathfrak{p}_{\mathfrak{p}}^{n_v+j} & \mathfrak{o}_{\mathfrak{p}} \end{pmatrix}$ where $0 \leq j \leq r_{\mathfrak{p}} - n_{\mathfrak{p}}$. Here there is a unique admissible extension $\Omega_{\mathfrak{p}}'$ of $\Omega_{\mathfrak{p}}$ to each such superorder, namely $\Omega_{\mathfrak{p}}' \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \omega_{\mathfrak{p}}(d)$. These superorders are precisely the local Eichler orders of level $\mathfrak{p}^{n_{\mathfrak{p}}}$ containing $\mathcal{O}_{\mathfrak{p}}$.

If $B_{\mathfrak{p}}$ is a division algebra, then we can write $\mathcal{O}_{\mathfrak{p}}^{\times} = \mathcal{O}_{r_{\mathfrak{p}}}(E_{\mathfrak{p}})^{\times} = \mathfrak{o}_{E,\mathfrak{p}}^{\times} \mathcal{U}_{\mathfrak{p}}^{r_{\mathfrak{p}}-1}$ where $E_{\mathfrak{p}}/F_{\mathfrak{p}}$ is a quadratic field extension. Since $\pi_{\mathfrak{p}}$ is trivial on $\mathcal{U}_{\mathfrak{p}}^{n_{\mathfrak{p}}-1}$, $\Omega_{\mathfrak{p}}$ must be trivial on $\mathcal{U}_{\mathfrak{p}}^{n_{\mathfrak{p}}-1} \cap \mathcal{O}_{\mathfrak{p}}^{\times}$. Note that since $\mathcal{O}_{n_{\mathfrak{p}}}(E_{\mathfrak{p}}) = \mathcal{O}_{r_{\mathfrak{p}}}(E_{\mathfrak{p}})\mathcal{U}_{\mathfrak{p}}^{n_{\mathfrak{p}}-1}$, there is a unique extension $\Omega_{\mathfrak{p}}'$ of $\Omega_{\mathfrak{p}}$ to $\mathcal{O}_{n_{\mathfrak{p}}}(E_{\mathfrak{p}})$ which is trivial on $\mathcal{U}_{\mathfrak{p}}^{n_{\mathfrak{p}}-1}$ Then $\pi_{\mathfrak{p}}^{\Omega_{\mathfrak{p}}'}$ is the also set of $\Omega_{\mathfrak{p}}'$-equivariant vectors for $\mathcal{O}_{n_{\mathfrak{p}}}(E_{\mathfrak{p}})^{\times}$ in π. Recall that any local order in $B_{\mathfrak{p}}$ containing $\mathfrak{o}_{E,\mathfrak{p}}$ is of the form $\mathcal{O}_j(E_{\mathfrak{p}})$ for some j, so $\mathcal{O}_{n_{\mathfrak{p}}}(E_{\mathfrak{p}})$ is the unique order of level $\mathfrak{p}^{n_{\mathfrak{p}}}$ containing $\mathcal{O}_{\mathfrak{p}}$.

Consequently, using the proven direction of (i), we see

$$\pi_{\mathbf{k}}^{\Omega} = \bigoplus S_{\mathbf{k}}^{\text{new}}(\mathcal{O}', \Omega', \omega),$$

where $\mathcal{O}'$ runs over special orders of B of level $c(\pi)$ containing $\mathcal{O}$ and Ω' is the unique extension of Ω to $\hat{\mathcal{O}}'$ where $\Omega_{\mathfrak{p}}'$ is as described above when $\mathcal{O}_{\mathfrak{p}}' \not\simeq \mathcal{O}_{\mathfrak{p}}$ and $\Omega_{\mathfrak{p}}' = \Omega_{\mathfrak{p}}$ otherwise. This implies both the other direction of (i) and (ii). $\qquad\square$

This yields the following Atkin–Lehner type decomposition.

Corollary 4.5. *As $\mathcal{H}^S(\mathcal{O}, \Omega)$-modules, we have*

$$S_{\mathbf{k}}(\mathcal{O}, \Omega, \omega) \simeq \bigoplus S_{\mathbf{k}}^{\text{new}}(\mathcal{O}', \Omega', \omega),$$

where $\mathcal{O}'$ runs over all special orders of B containing $\mathcal{O}$, and Ω' runs over all admissible semigroup homomorphisms of $\hat{\mathcal{O}}'$ extending Ω.

For a special order $\mathcal{O}$, we say the local order $\mathcal{O}_{\mathfrak{p}}$ is of *unramified quadratic type* if $\mathcal{O}_{\mathfrak{p}}$ is an Eichler order or is isomorphic to $\mathcal{O}_r(E_{\mathfrak{p}})$ for some $r \geq 1$ where $E_{\mathfrak{p}}/F_{\mathfrak{p}}$ is the unramified quadratic extension. Otherwise, we say $\mathcal{O}_{\mathfrak{p}}$ is of *ramified quadratic type*. We say $\mathcal{O}$ is globally of *unramified quadratic type* if it is everywhere locally.

Write the level $\mathfrak{N} = \prod \mathfrak{p}^{r_{\mathfrak{p}}}$ of $\mathcal{O}$ as $\mathfrak{N} = \mathfrak{N}_1 \mathfrak{N}_2 \mathfrak{M}$ where $\mathfrak{N}_1$, $\mathfrak{N}_2$ and $\mathfrak{M}$ are the unique pairwise coprime ideals such that $\mathfrak{M}$ is coprime to $\mathfrak{D}$ and all primes $\mathfrak{p}|\mathfrak{D}$ at which $\mathcal{O}$ is of unramified (resp. ramified) quadratic type divide $\mathfrak{N}_1$ (resp. $\mathfrak{N}_2$). Necessarily, $r_{\mathfrak{p}}$ is odd for $\mathfrak{p}|\mathfrak{N}_1$ and $r_{\mathfrak{p}} \geq 2$ for $\mathfrak{p}|\mathfrak{N}_2$.

Consider a level $\mathfrak{N}'|\mathfrak{N}$, which we write as $\mathfrak{N}' = \mathfrak{N}_1' \mathfrak{N}_2' \mathfrak{M}'$, with $\mathfrak{N}_1'|\mathfrak{N}_1$, $\mathfrak{N}_2'|\mathfrak{N}_2$, and $\mathfrak{M}'|\mathfrak{M}$. Write $\mathfrak{N}' = \prod \mathfrak{p}^{r_{\mathfrak{p}}'}$. For $\mathfrak{N}'$ to be the level of a special order $\mathcal{O}'$ in B containing $\mathcal{O}$, we need $\mathcal{O}_{\mathfrak{p}}'$ to be of unramified quadratic type for $\mathfrak{p}|\mathfrak{N}_1'$ (so $r_{\mathfrak{p}}'$ is necessarily odd), which we assume. (For $\mathfrak{p}|\mathfrak{N}_2'$, $\mathcal{O}_{\mathfrak{p}}'$ is unramified quadratic type if $r_{\mathfrak{p}}' = 1$ and of ramified

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quadratic type if $r'_{\mathfrak{p}} \geq 2$.) From the proof of the above proposition, we find that the number of special orders $\mathcal{O}'$ of level $\mathfrak{N}'$ containing $\mathcal{O}$ is

$$m_{\mathcal{O}}(\mathfrak{N}') = \prod_{\mathfrak{p}|\mathfrak{M}} (r_{\mathfrak{p}} - r'_{\mathfrak{p}} + 1).$$

Since, for fixed $\mathfrak{N}'$, all such $\mathcal{O}'$ are conjugate, the isomorphism type (as a Hecke module) of the subspace $M_{\mathbf{k}}(\mathcal{O}', \Omega', \omega)$ in $M_{\mathbf{k}}(\mathcal{O}, \Omega, \omega)$ only depends on $\mathfrak{N}'$.

For each $\mathfrak{N}'|\mathfrak{N}$ as above, we fix a special order $\mathcal{O}'(\mathfrak{N}')$ of level $\mathfrak{N}'$ containing $\mathcal{O}$. Then Ω has a (unique) admissible extension to $\hat{\mathcal{O}}'$ if and only if $c(\omega_{\mathfrak{p}}) \leq \frac{r'_{\mathfrak{p}}}{2}$ for each $\mathfrak{p}|\mathfrak{N}_1\mathfrak{N}_2$. Consequently, the above corollary asserts a $\mathcal{H}^S(\mathcal{O}, \Omega)$-module isomorphism

$$(4.10) \qquad S_{\mathbf{k}}(\mathcal{O}, \Omega, \omega) \simeq \bigoplus_{\mathfrak{N}'} m_{\mathcal{O}}(\mathfrak{N}') S_{\mathbf{k}}^{\text{new}}(\mathcal{O}'(\mathfrak{N}'), \Omega', \omega),$$

where $\mathfrak{N}' = \prod \mathfrak{p}^{r'_{\mathfrak{p}}}$ runs over divisors of $\mathfrak{N}$ such that (i) $r'_{\mathfrak{p}} \geq \max\{1, 2c(\omega_{\mathfrak{p}})\}$ for all $\mathfrak{p}|\mathfrak{N}_1\mathfrak{N}_2$ and (ii) $r'_{\mathfrak{p}}$ is odd for all $\mathfrak{p}|\mathfrak{N}_1$.

When $\mathbf{k} = \mathbf{0}$, the above decompositions do not apply verbatim to the Eisenstein spaces because the local representations are all 1-dimensional. Instead, we simply have

$$(4.11) \qquad E_{\mathbf{0}}(\mathcal{O}, \Omega, \omega) \simeq \bigoplus_{\mathfrak{N}'} E_{\mathbf{0}}^{\text{new}}(\mathcal{O}'(\mathfrak{N}'), \Omega', \omega),$$

with the sum as in (4.10).

Lemma 4.6. *We have $E_{\mathbf{0}}(\mathcal{O}, \Omega, \omega) = 0$ unless $\omega_{\mathfrak{p}}$ is unramified for all $\mathfrak{p} \nmid \mathfrak{D}$.*

Proof. Consider $\mu \circ N \in E_{\mathbf{0}}(\mathcal{O}, \Omega, \omega)$. If $B_{\mathfrak{p}}$ is split, for $\Omega_{\mathfrak{p}}$ to agree with $\mu_{\mathfrak{p}} \circ N$ on $\begin{pmatrix} \mathfrak{o}_{\mathfrak{p}}^{\times} & \\ & 1 \end{pmatrix} \subset \mathcal{O}_{\mathfrak{p}}^{\times}$, we need $\mu_{\mathfrak{p}}$, and thus $\omega_{\mathfrak{p}} = \mu_{\mathfrak{p}}^2$, to be unramified. $\qquad\square$

The next proposition says that typically only "small levels" contribute to the Eisenstein subspaces.

Proposition 4.7. *Suppose $\mathcal{O}$ has level $\mathfrak{N} = \prod \mathfrak{p}^{r_{\mathfrak{p}}}$. Then $E_{\mathbf{0}}^{\text{new}}(\mathcal{O}, \Omega, \omega) = 0$ if one of the following holds:*

(i) $\mathfrak{M} \neq \mathfrak{o}$;
(ii) $c(\omega_{\mathfrak{p}}) < \frac{r_{\mathfrak{p}}}{2}$ for some $\mathfrak{p}^2|\mathfrak{N}_1$ or nondyadic $\mathfrak{p}^3|\mathfrak{N}_2$; or
(iii) $c(\omega_{\mathfrak{p}}) < \frac{r_{\mathfrak{p}}}{2}$ for some $\mathfrak{p}^3|\mathfrak{N}_2$ with $v_{\mathfrak{p}}(\mathfrak{N}_2) \geq 2t(E_{\mathfrak{p}}/F_{\mathfrak{p}}) + 2$, where $\mathcal{O}_{\mathfrak{p}} \simeq \mathcal{O}_{r_{\mathfrak{p}}}(E_{\mathfrak{p}})$.

In particular, if $\mathcal{O}$ is of unramified quadratic type and ω is unramified, then $E_{\mathbf{0}}^{\text{new}}(\mathcal{O}, \Omega, \omega) = 0$ unless $\mathcal{O}$ is maximal, in which case $E_{\mathbf{0}}^{\text{new}}(\mathcal{O}, \Omega, \omega) = E_{\mathbf{0}}(\mathcal{O}, \Omega, \omega)$.

Proof. Recall that $E_{\mathbf{0}}(\mathcal{O}, \Omega, \omega)$ is generated by the characters $\mu \circ N$ of $B^{\times}(\mathbb{A})$, where μ ranges over idele class characters such that $\mu^2 = \omega$ and the local components of $\mu \circ N$ and Ω agree on each $\mathcal{O}_{\mathfrak{p}}^{\times}$. Thus, for a special order $\mathcal{O}' \supset \mathcal{O}$ of level $\mathfrak{N}'$, we have $E_{\mathbf{0}}(\mathcal{O}, \Omega, \omega) = E_{\mathbf{0}}(\mathcal{O}', \Omega', \omega)$ (with Ω' as before) if $N((\mathcal{O}'_v)^{\times}) = N(\mathcal{O}_v^{\times})$ for all finite v.

Note that for the existence of a proper superorder $\mathcal{O}' \supset \mathcal{O}$ with an admissible extension Ω' of Ω, we need that $c(\omega_{\mathfrak{p}}) < r_{\mathfrak{p}}$ for some split $B_{\mathfrak{p}}$ or $c(\omega_{\mathfrak{p}}) < \frac{r_{\mathfrak{p}}}{2}$ for some ramified $B_{\mathfrak{p}}$.

If a local order $\mathcal{O}'_v$ is of quadratic unramified type, then $N((\mathcal{O}'_v)^{\times}) = \mathfrak{o}_v^{\times}$. This proves $E_{\mathbf{0}}^{\text{new}}(\mathcal{O}, \Omega, \omega) = 0$ if $c(\omega_{\mathfrak{p}}) < r_{\mathfrak{p}}$ for some $\mathfrak{p}|\mathfrak{M}$, as well as the $\mathfrak{p}^2|\mathfrak{N}_1$ part of (ii). Then (i) follows from the previous lemma.

Since (iii) implies the rest of (ii), it remains to prove (iii). Suppose $v|\mathfrak{N}_2$ and $\mathcal{O}_v \simeq \mathcal{O}_{E_v}(r_v)$ where E_v/F_v is some ramified extension. Write $t_v = t(E_v/F_v)$, and assume $r_v \geq 2t_v + 3$ (i.e., $r_v \geq 3$ if v is non-dyadic). Take $\mathcal{O}'_v \simeq \mathcal{O}_{E_v}(r_v - 1)$. Now $N(\mathcal{O}_v^\times) = N(\mathfrak{o}_{E_v}^\times)\mathfrak{u}_v^{[(r_v-1)/2]}$ and $N((\mathcal{O}'_v)^\times) = N(\mathfrak{o}_{E_v}^\times)\mathfrak{u}_v^{[(r_v-2)/2]}$. Thus these norm subgroups are equal if $N(\mathfrak{o}_{E_v}^\times) \supset \mathfrak{u}_v^{[(r_v-2)/2]}$. By assumption on r_v, this is the case since $N(\mathfrak{o}_{E_v}^\times) = N(\mathfrak{u}_{E_v}^{t_v+1}) = \mathfrak{u}_v^{t_v+1}$. $\qquad\square$

5. A CLASSICAL JACQUET–LANGLANDS CORRESPONDENCE

Here we reinterpret the representation-theoretic Jacquet–Langlands correspondence from [JL70] in more classical language, namely as a Hecke-module homomorphism from a space of quaternionic modular forms to Hilbert modular forms. (Note: analytic details of the proof of the Jacquet–Langlands correspondence were not actually completed in [JL70], but for instance in [DL71]—see also [Gel75] for an exposition.) This interpretation is essentially a generalization of [Shi72] (see also [Gel75]) from Eichler orders to special orders.

5.1. **Hilbert modular forms.** First we recall some facts and set notation about Hilbert modular forms. See [Shi78] for more details. We continue the notation from the previous section. In particular, F is a totally real number field of degree d with adele ring $\mathbb{A}$.

Let $\mathfrak{o} = \mathfrak{o}_F$ denote the integer ring of F, $\mathfrak{d}_F$ the absolute different, and $\mathfrak{N}$ a nonzero integral ideal of $\mathfrak{o}$. Let $W = W(\mathfrak{N})$ (resp. $Y = Y(\mathfrak{N})$) be image under the canonical involution ι of the level $\mathfrak{N}$ subgroup of (resp. semisubgroup) of $\mathrm{GL}_2(\mathbb{A})$ denoted by the same letter in [Shi78, Sec 2]. (These involuted subsets make the subsequent notation more straightforward.) Namely, these are the subsets of $\mathrm{GL}_2(\mathbb{A})$ with finite local components given by

$$\begin{cases} Y_{\mathfrak{p}} = \begin{pmatrix} \mathfrak{o}_{\mathfrak{p}} & \mathfrak{d}_F^{-1}\mathfrak{o}_{\mathfrak{p}} \\ \mathfrak{d}_F\mathfrak{o}_{\mathfrak{p}} & \mathfrak{o}_{\mathfrak{p}} \end{pmatrix}, & W_{\mathfrak{p}} = Y_{\mathfrak{p}}^\times & \text{if } \mathfrak{p} \nmid \mathfrak{N} \\[2ex] Y_{\mathfrak{p}} = \begin{pmatrix} \mathfrak{o}_{\mathfrak{p}} & \mathfrak{d}_F^{-1}\mathfrak{o}_{\mathfrak{p}} \\ \mathfrak{N}\mathfrak{d}_F\mathfrak{o}_{\mathfrak{p}} & \mathfrak{o}_{\mathfrak{p}}^\times \end{pmatrix}, & W_{\mathfrak{p}} = \begin{pmatrix} \mathfrak{o}_{\mathfrak{p}}^\times & \mathfrak{d}_F^{-1}\mathfrak{o}_{\mathfrak{p}} \\ \mathfrak{N}\mathfrak{d}_F\mathfrak{o}_{\mathfrak{p}} & \mathfrak{o}_{\mathfrak{p}}^\times \end{pmatrix} = Y_{\mathfrak{p}}^\times & \text{if } \mathfrak{p}|\mathfrak{N}, \end{cases}$$

and infinite components $\mathrm{GL}_2^+(\mathbb{R})^d$. Let ψ be a finite-order Hecke character of F of conductor dividing $\mathfrak{N}$. We extend ψ to $Y(\mathfrak{N})$ by $\psi_Y \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \psi(d_{\mathfrak{N}} \bmod \mathfrak{N})$, where $d_{\mathfrak{N}} = \prod_{\mathfrak{p}|\mathfrak{N}} d_{\mathfrak{p}}$ denotes the $\mathfrak{N}$-part of d.

Let $\mathbf{k} = (k_1, \ldots, k_d)$. We denote by $M_{\mathbf{k}}(\mathfrak{N}, \psi)$ the space of adelic holomorphic Hilbert modular forms of level $\mathfrak{N}$ and character ψ, which can be viewed the space of functions f on $\mathrm{GL}_2(\mathbb{A}_F)$ satisfying

$$f(z\gamma g w) = \psi(z)\psi_Y(w)f(g), \quad z \in \mathbb{A}_F^\times, \gamma \in \mathrm{GL}_2(F), w \in W(\mathfrak{N}), w_\infty = 1,$$

together with the usual holomorphy conditions and weight $\mathbf{k}$ transformation law at infinity. Specifically, let $t_1, \ldots, t_{h_F}$ denote a set of ideal class representatives for F. Then for $1 \leq j \leq h_F$, we require

$$f_j(y \cdot (i, \ldots, i)) = \det y^{k/2}(c(i, \ldots, i) + d)^k f(\begin{pmatrix} t_j^{-1} & \\ & 1 \end{pmatrix} y), \quad y \in \mathrm{GL}_2(F_\infty)^+$$

is a classical Hilbert modular form of weight $\mathbf{k}$. Denote by $S_{\mathbf{k}}(\mathfrak{N}, \psi)$ the subspace of cusp forms of $M_{\mathbf{k}}(\mathfrak{N}, \psi)$.

For $f \in M_{\mathbf{k}}(\mathfrak{N}, \psi)$, $x \in \mathbb{A}_F$ and $y \in \mathbb{A}_F^{\times}$ such that $y_{\infty} \gg 0$, we have a Fourier expansion of the form

$$f \begin{pmatrix} y & x \\ & 1 \end{pmatrix} = c_0(y\mathfrak{o})|y|^{\mathbf{k}/2} + \sum_{0 \ll \zeta \in F} c(\zeta y \mathfrak{o}, f)(\zeta y_{\infty})^{\mathbf{k}/2} e_F(\zeta i y_{\infty}) \chi_F(\zeta x),$$

where $e_F(z_1, \ldots, z_d) = \exp(2\pi i \sum z_j)$ and χ_F is the character of $\mathbb{A}_F/F$ agreeing with e_F at infinity. The coefficients $c(\mathfrak{n}, f)$ are 0 unless $\mathfrak{n}$ is integral (i.e., we may take ζ to run over totally positive elements of $y^{-1}\mathfrak{o}$ in the above sum) and $c_0(y\mathfrak{o})$ is a function of strict ideal classes that is 0 unless $\mathbf{k}$ is a parallel weight. If each f_j is a classical Hilbert modular form for the subgroup denoted $\Gamma(t_j \mathfrak{d}_F, \mathfrak{N})$ in [Shi78], then the adelic Fourier expansion corresponds to the classical Fourier expansions

$$f_j(z) = \sum a_j(\xi) e_F(\xi z), \quad \xi \in t_j \text{ such that } \xi = 0 \text{ or } \xi \gg 0,$$

where $a_j(\xi) = \xi^{\mathbf{k}/2} c(\xi t_j^{-1}, f)$ for $\xi \gg 0$ in t_j and, in the case of parallel weight, $a_j(0) = N(t_j^{\mathbf{k}/2}) c_0(\eta t_j^{-1})$ for any $\eta \gg 0$.

For an integral ideal $\mathfrak{n}$ of F, one defines the Hecke operator $T_{\mathfrak{n}}$ to be the sum over distinct $W(\mathfrak{N}) y W(\mathfrak{N})$ where $y \in Y(\mathfrak{N})$ such that $(\det y)\mathfrak{o} = \mathfrak{n}$. Here, if $W(\mathfrak{N}) y W(\mathfrak{N}) = \bigsqcup y_j W(\mathfrak{N})$, the action on f is given by $\sum_j \psi_Y(y_j)^{-1} \pi(y_j) f$. Then $T_{\mathfrak{n}}$ factors as a product of local Hecke operators $T_{\mathfrak{n}_{\mathfrak{p}}}$ as in (4.7), and one readily sees the definition of the local Hecke operators $T_{\mathfrak{n}_{\mathfrak{p}}}$ on $\mathrm{GL}_2(F_v)$ matches the definition on the local quaternionic groups $B_v^{\times}$ when the local levels $\mathfrak{N}_v = 1$ (so necessarily $B_v^{\times} \simeq \mathrm{GL}_2(F_v)$). Denote by $\mathcal{H}^S$ the Hecke algebra generated by $T_{\mathfrak{n}}$ for $\mathfrak{n}$ coprime to $\mathfrak{N}$.

Define the normalized Fourier coefficient $C(\mathfrak{n}, f) = N(\mathfrak{n}) c(\mathfrak{n}, f)$. (Our normalization is different from that in [Shi78] when $\mathbf{k} \neq (2, 2, \ldots, 2)$.) If f is a common eigenfunction of the Hecke operators $T_{\mathfrak{n}}$, and f is normalized so that $C(\mathfrak{o}, f) = 1$, then the eigenvalue of $T_{\mathfrak{n}}$ is $\lambda_f(\mathfrak{n}) = C(\mathfrak{n}, f)$. If further f is a cusp form, it generates an irreducible cuspidal automorphic representation $\pi = \pi_f$ of weight $\mathbf{k}$ (each archimedean component π_{ν_j} is the discrete series of weight k_j) with central character $\omega_{\pi} = \psi$ such that $c(\pi) = \prod_{\mathfrak{p}} \mathfrak{p}^{c(\pi_{\mathfrak{p}})}$ divides $\mathfrak{N}$. We say such an f is a newform if $c(\pi) = \mathfrak{N}$, and denote the span of newforms in $S_{\mathbf{k}}(\mathfrak{N}, \psi)$ by $S_{\mathbf{k}}^{\mathrm{new}}(\mathfrak{N}, \psi)$.

We have an Atkin–Lehner type decomposition in terms of newforms:

$$S_{\mathbf{k}}(\mathfrak{N}, \psi) = \bigoplus_{\mathfrak{M} | \mathfrak{N}} \bigoplus_{\mathfrak{d} | \mathfrak{N}\mathfrak{M}^{-1}} \iota_{\mathfrak{d}} \left(S_{\mathbf{k}}^{\mathrm{new}}(\mathfrak{M}, \psi) \right),$$

where the $\iota_{\mathfrak{d}}$ are the embeddings of $S_{\mathbf{k}}^{\mathrm{new}}(\mathfrak{M}, \psi)$ into $S_{\mathbf{k}}(\mathfrak{d}\mathfrak{M}, \psi)$ defined by $C(\mathfrak{n}, \iota_{\mathfrak{d}} f) = C(\mathfrak{n}\mathfrak{d}^{-1}, f)$ (see [SW93, Section 3]).

The usual dictionary between modular forms and automorphic representations defines an isomorphism (initially as vector spaces, but also as Hecke modules with Hecke operators appropriately normalized):

$$(5.1) \qquad\qquad S_{\mathbf{k}}(\mathfrak{N}, \psi) \simeq \bigoplus \pi_{\mathbf{k}}^{K_1(\mathfrak{N})},$$

where π runs over irreducible holomorphic weight $\mathbf{k}$ cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ with central character ψ, $K_1(\mathfrak{N})$ is the level $\mathfrak{N}$ compact open subgroup of $\mathrm{GL}_2(\hat{F})$ of "type Γ_1," and $\pi_{\mathbf{k}}$ denotes the subspace of π of weight $\mathbf{k}$ vectors.

5.2. **Eisenstein series.** The Jacquet–Langlands correspondence is a only statement about *cuspidal* representations. We also want to know that the 1-dimensional representations of quaternion algebras correspond to weight $\mathbf{2} = (2, 2, \ldots, 2)$ Eisenstein series, which we explain here.

Let B, $\mathcal{O}$ (a special order of level $\mathfrak{N}$), ω and Ω be as in Section 4. Let μ be a character of $F^\times \backslash \mathbb{A}^\times$ and suppose $\varphi = \mu \circ N \in M_{\mathbf{0}}(\mathcal{O}, \Omega)$, which means $\mu^2 = \omega$ and Ω agrees with $\mu \circ N$ on $\hat{\mathcal{O}}^\times$. Then it is immediate from (4.6) that φ is an eigenform for each Hecke operator $T_{\mathfrak{n}}$ with eigenvalue $\lambda_{\mathfrak{n}}(\varphi) = \sum \Omega^{-1}(\beta) \mu(N(\beta))$, where $\beta \in \hat{\mathcal{O}}(\mathfrak{n})/\mathcal{O}^\times$. By Remark 4.1, $\lambda_{\mathfrak{p}^m}(\varphi)$ is just the number of right $\mathcal{O}$-ideals of norm $\mathfrak{p}^m$ if μ is unramified at $\mathfrak{p}$.

Given an idele class character μ of F, we view this as a character on ideals of F by setting $\mu(\mathfrak{p})$ to be $\mu_{\mathfrak{p}}(\varpi_{\mathfrak{p}})$ if $\mu_{\mathfrak{p}}$ is unramified and 0 otherwise.

Lemma 5.1. *Let $\mathfrak{a}$, $\mathfrak{b}$ be coprime integral ideals of F. If $F = \mathbb{Q}$, assume $\mathfrak{a} \neq \mathbb{Z}$.*

(i) There is an Eisenstein series $E_{\mathbf{2}, \mathfrak{a}, \mathfrak{b}} \in M_{\mathbf{2}}(\mathfrak{a}\mathfrak{b}, 1)$ such that $C(\mathfrak{n}, E_{\mathbf{2}, \mathfrak{a}, \mathfrak{b}}) = \sum_{\mathfrak{d}} N(\mathfrak{d})$ where $\mathfrak{d}$ runs over integral ideals dividing $\mathfrak{n}$ such that $\mathfrak{d}$ is coprime to $\mathfrak{a}$ and $\mathfrak{d}^{-1}\mathfrak{n}$ is coprime to $\mathfrak{b}$. Moreover, $c_0(y\mathfrak{o}, E_{\mathbf{2}, \mathfrak{a}, \mathfrak{b}}) = 2^{-d} \zeta_F(-1) \prod_{\mathfrak{p} | \mathfrak{a}\mathfrak{b}} (1 - N(\mathfrak{p})^{-1})$ for $y \in \mathbb{A}_F^\times$ with $y_\infty \gg 0$.

(ii) Let μ be a nontrivial finite-order idele class character of F, with conductor $\mathfrak{c} | \mathfrak{a}\mathfrak{b}$. The twisted Eisenstein series $E_{\mathbf{2}, \mathfrak{a}, \mathfrak{b}}(\mu) := E_{\mathbf{2}, \mathfrak{a}, \mathfrak{b}} \otimes \mu \in M_{\mathbf{2}}(\mathfrak{a}\mathfrak{b}\mathfrak{c}, \mu^2)$ satisfies $C(\mathfrak{n}, E_{\mathbf{2}, \mathfrak{a}, \mathfrak{b}}(\mu)) = \mu(\mathfrak{n}) C(\mathfrak{n}, E_{\mathbf{2}, \mathfrak{a}, \mathfrak{b}})$ and $c_0(y\mathfrak{o}, E_{\mathbf{2}, \mathfrak{a}, \mathfrak{b}}(\mu)) = 0$.

Proof. Let $\mu_{\mathfrak{M}}$ denote the character of ideals obtained by viewing μ as an (imprimitive if $\mathfrak{c} \neq \mathfrak{M}$) character of conductor $\mathfrak{M}$ (i.e., it is 0 on integral ideals not prime to $\mathfrak{M}$). The Eisenstein series $E_{\mathbf{2}, \mathfrak{a}, \mathfrak{b}}(\mu)$ is the Eisenstein series associated to the characters on ideal groups $\eta = \mu_{\mathfrak{a}}$ and $\chi = \mu_{\mathfrak{b}}$ in [Shi78, Proposition 3.4], which gives the level, central character and a formula for $C(\mathfrak{n}, E_{\mathbf{2}, \mathfrak{a}, \mathfrak{b}}(\mu))$ as in (i). (Technically, [Shi78, Proposition 3.4] assumes $F \neq \mathbb{Q}$, but the elliptic case is classical.) The constant term is calculated in [DDP11, Proposition 2.1]. The explicit relation for twists of Fourier coefficients in (ii) is given in [Shi80, Proposition 9.7]. $\qquad \square$

Note that the Fourier coefficients of $E_{\mathbf{2}, \mathfrak{a}, \mathfrak{b}}(\mu)$ are multiplicative, which means that $E_{\mathbf{2}, \mathfrak{a}, \mathfrak{b}}$ is an eigenfunction at least for the unramified Hecke algebra. We also note that $E_{\mathbf{2}, \mathfrak{a}, \mathfrak{b}}$ only depends on the squarefree parts $\mathfrak{a}_0$ and $\mathfrak{b}_0$ of $\mathfrak{a}$ and $\mathfrak{b}$, so in fact we have $E_{\mathbf{2}, \mathfrak{a}, \mathfrak{b}}(\mu) \in M_{\mathbf{2}}(\mathfrak{a}_0 \mathfrak{b}_0 \mathfrak{c}, \mu^2)$, where $\mathfrak{c}$ is the conductor of μ.

Proposition 5.2. *Let μ be a finite-order idele class character of F such that $\varphi_\mu = \mu \circ N \in M_{\mathbf{0}}(\mathcal{O}, \Omega)$. Write $\mathfrak{N} = \mathfrak{N}'\mathfrak{M}$ where $\mathfrak{N}'$ (resp. $\mathfrak{M}$) is of the form $\prod \mathfrak{p}^{r_{\mathfrak{p}}}$ where $\mathfrak{p}$ runs over all finite primes at which B is ramified (resp. split). Then φ_μ is an eigenform whose Hecke eigenvalue $\lambda_{\mathfrak{n}}(\varphi)$ for $T_{\mathfrak{n}}$ is $C(\mathfrak{n}, E_{\mathbf{2}, \mathfrak{N}', \mathfrak{M}}(\mu))$ for all $\mathfrak{n} \nmid \mathfrak{M}$. Moreover, $E_{\mathbf{2}, \mathfrak{N}', \mathfrak{M}}(\mu) \in M_{\mathbf{2}}(\mathfrak{N}, \mu^2)$.*

Proof. Since the Hecke eigenvalues are multiplicative, for the first part it suffices to consider prime power eigenvalues $\lambda_{\mathfrak{p}^m}(\varphi_\mu)$. Suppose $\mathfrak{p} \nmid \mathfrak{N}$. Then $\Omega_{\mathfrak{p}} = 1$ and $\mu_{\mathfrak{p}} = 1$.

By [Remark 4.1], $\lambda_{\mathfrak{p}^m}(\varphi_\mu)$ is $\mu(\mathfrak{p}^m)$ times the number of $\mathcal{O}_\mathfrak{p}$-ideals of norm $\mathfrak{p}^m$. It is well known that this is $\frac{q_\mathfrak{p}^{m+1}-1}{q_\mathfrak{p}-1} = C(\mathfrak{p}^m, E_{\mathbf{2},\mathfrak{N}',\mathfrak{M}})$. This proves the first assertion.

To see $E_{\mathbf{2},\mathfrak{N}',\mathfrak{M}}(\mu) \in M_\mathbf{2}(\mathfrak{N}, \mu^2)$, recall $c(\mu_\mathfrak{p} \circ N) = \max\{2c(\mu_\mathfrak{p}), 1\}$ for $\mathfrak{p}|\mathfrak{N}'$ and, by the proof of [Lemma 4.6], $c(\mu_\mathfrak{p}) = 0$ for $\mathfrak{p}|\mathfrak{M}$. Since $c(\mu \circ N)|\mathfrak{N}$, we see $c(\mu_\mathfrak{p}) = 0$ when $v_\mathfrak{p}(\mathfrak{N}) \leq 1$ and $c(\mu_\mathfrak{p}) \leq \frac{v_\mathfrak{p}(\mathfrak{N})}{2}$ otherwise. Hence $c(\mu)\mathfrak{N}_0|\mathfrak{N}$, where $\mathfrak{N}_0$ denotes the squarefree part of $\mathfrak{N}$. $\qquad\square$

Remark 5.1. Since the above proposition only considers Hecke eigenvalues away from the level, there are other Eisenstein series we could have used as well. However, we chose $E_{\mathbf{2},\mathfrak{N}',\mathfrak{M}}$ to correspond to the constant function $\varphi_1 = 1$ on $B^\times(\mathbb{A})$ because, at least in the case $\mathfrak{N}$ is squarefree, the appropriate definition of ramified Hecke operators makes the ramified Hecke eigenvalues of φ_1 match with the ramified Hecke eigenvalues of $E_{\mathbf{2},\mathfrak{N}',\mathfrak{M}}$. Here, by appropriate definition of ramified Hecke operators, we mean that one should use the same definition of local ramified Hecke operators for $\mathfrak{p}|\mathfrak{M}$ on $M_\mathbf{0}(\mathcal{O}, \Omega, \omega)$ as on $M_\mathbf{2}(\mathfrak{N}, \omega)$. See [Mar17] or [Mar18] for $\mathfrak{p}|\mathfrak{N}'$.

We also note that for a suitably normalized inner product (φ_1, φ_1) is the mass $m(\mathcal{O})$ of $\mathcal{O}$, and $(\varphi_\mu, \varphi_\mu) = 0$ for $\mu \neq 0$. These inner products correspond to the constant terms $c_0(y\mathfrak{o}, E_{\mathbf{2},\mathfrak{N}',\mathfrak{M}}(\mu))$, up to simple factors. For instance, if $\mathcal{O}$ is maximal, then $m(\mathcal{O}) = 2h_F N(\mathfrak{N})|c_0(\mathfrak{o}, E_{\mathbf{2},\mathfrak{N}',\mathfrak{M}}(\mu))|$ (see [Mar17, (1.6)]).

5.3. **Correspondence of Hecke modules.** Here we come to the main results of this section.

We summarize our notation from above: B is a totally definite quaternion algebra over F with discriminant $\mathfrak{D}$; $\mathfrak{N}$ is a nonzero integral ideal in $\mathfrak{o} = \mathfrak{o}_F$ such that $\mathfrak{p}|\mathfrak{N}$ for all finite $\mathfrak{p}$ where $B_\mathfrak{p}$ is division; $\mathcal{O}$ is a special order in B of level $\mathfrak{N} = \mathfrak{N}_1\mathfrak{N}_2\mathfrak{M} = \prod \mathfrak{p}^{r_\mathfrak{p}}$ as in [Section 4.4]; and Ω is a semigroup homomorphism of $\hat{\mathcal{O}}^\bullet$ extending $\omega|_{\hat{\mathfrak{o}}^\times}$ as in [Section 4.1]. Recall also that $c(\omega_\mathfrak{p}) \leq \frac{r_\mathfrak{p}}{2}$ for all $\mathfrak{p}|\mathfrak{N}_1\mathfrak{N}_2$. We will further assume $\mathcal{O}$ is chosen so that $r_\mathfrak{p}$ is even for all $\mathfrak{p}|\mathfrak{N}_2$; i.e., we choose $\mathcal{O}$ so that it is of unramified quadratic type at as many places as possible (given $\mathfrak{N}$).

We call an eigenform $\mathfrak{p}$-*primitive* if the associated local representation $\pi_\mathfrak{p}$ is minimal. In classical language, this means that there is no $\mathfrak{p}$-power conductor character χ such that twisting by χ lowers the level at $\mathfrak{p}$. For an ideal $\mathfrak{a}$ in $\mathfrak{o}$, we say a form is $\mathfrak{a}$-primitive if it is $\mathfrak{p}$-primitive for all $\mathfrak{p}|\mathfrak{a}$.

The following is the first, unrefined version of our "classical" Jacquet–Langlands correspondence.

Proposition 5.3. *There is a (non-canonical) homomorphism of $\mathcal{H}^S$-modules*

$$\mathrm{JL} : S_\mathbf{k}^{\mathrm{new}}(\mathcal{O}, \Omega, \omega) \to S_{\mathbf{k+2}}^{\mathrm{new}}(\mathfrak{N}, \omega)$$

such that any $\mathfrak{D}$-primitive newform $f \in S_{\mathbf{k+2}}^{\mathrm{new}}(\mathfrak{N}, \omega)$ lies in the image of this map.

Proof. Consider any π appearing in the decomposition of $S_\mathbf{k}^{\mathrm{new}}(\mathcal{O}, \Omega, \omega)$ from [Proposition 4.4](i). By the same proposition, the global Jacquet–Langlands correspondence associates to π an irreducible cuspidal automorphic representation π' of $\mathrm{GL}_2(\mathbb{A})$ with central character ω such that $\pi'_v \simeq \pi_v$ for all $v \nmid \mathfrak{D}$ and $c(\pi') := \prod \mathfrak{p}^{c(\pi'_\mathfrak{p})} = \mathfrak{N}$. Moreover, for each $\mathfrak{p}|\mathfrak{D}$, $\pi'_\mathfrak{p}$ is a discrete series representation, i.e., special or supercuspidal, and π'_{ν_i} is the discrete series of weight $k_i + 2$ for each infinite place ν_i. Further, all π' with

central character ω, conductor $\mathfrak{N}$ and holomorphic weight $\mathbf{k} + \mathbf{2}$ such that $\pi'_{\mathfrak{p}}$ is discrete series for $\mathfrak{p}|\mathfrak{D}$ appear in the image of the representation-theoretic Jacquet–Langlands correspondence. Such π' will correspond to a π appearing in the decomposition (4.3) if $\pi_{\mathfrak{p}}^{\Omega_{\mathfrak{p}}} \neq 0$ for $\mathfrak{p}|\mathfrak{D}$.

Since, for $\mathfrak{p}|\mathfrak{D}$, $\pi'_{\mathfrak{p}}$ being minimal implies $\pi'_{\mathfrak{p}}$ is discrete series (in fact supercuspidal if $c(\pi'_{\mathfrak{p}}) > 1$) whenever $c(\pi'_{\mathfrak{p}}) > 0$, Theorem 3.5 and Theorem 3.6 imply that all $\mathfrak{D}$-primitive representations appearing in the spectral decomposition of $S_{\mathbf{k}}^{\text{new}}(\mathfrak{N}, \omega)$, i.e., the new part of the decomposition (5.1), lie in the image of this correspondence.

Consequently, we can define a map at the level of modular forms as follows. For each π as above, fix a basis $\varphi_1, \ldots, \varphi_m$ of $\pi_{\mathbf{k}}^{\Omega}$. Map each φ_i to the unique (normalized) newform $f_{\pi'} \in S_{\mathbf{k}}^{\text{new}}(\mathfrak{N}, \omega)$ associated to π'. Extending this by linearity gives an $\mathcal{H}^S$-module homomorphism as $\pi'_v \simeq \pi_v$ for $v \nmid \mathfrak{D}$. $\qquad\square$

Remark 5.2. In fact, it follows from Proposition 4.1 and Proposition 4.2 that the Hecke eigenvalues for all $T_{\mathfrak{p}}$'s are also preserved under the above correspondence of newforms. See also [Mar18] for the case of squarefree conductor, where this was used to produce congruence of eigenforms mod 2. This preservation of eigenvalues will no longer be true for $T_{\mathfrak{p}}$'s with $\mathfrak{p}|\mathfrak{M}$ when we extend the correspondence to include oldforms.

The above result is sufficient to tell us that the basis problem has a solution, but we want to know more precise information about this map, namely what can we say about its kernel and its image, as well as understanding how it can be extended to $M_{\mathbf{k}}(\mathcal{O}, \Omega, \omega)$. It is clear from the above proof that understanding the kernel amounts to understanding $\dim \pi_{\mathbf{k}}^{\Omega}$, which was the main goal of our local calculations.

To get a more precise description of this map, and its extension to $S_{\mathbf{k}}(\mathcal{O}, \Omega, \omega)$, it will be convenient to define certain refinements of $S_{\mathbf{k}}(\mathfrak{N}, \psi)$ and $S_{\mathbf{k}}^{\text{new}}(\mathfrak{N}, \psi)$.

Let $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ and $\mathfrak{d}$ be nonzero pairwise coprime ideals in $\mathfrak{o}$ dividing $\mathfrak{N}$ such that $\mathfrak{p}^2|\mathfrak{b}$ whenever $\mathfrak{p}|\mathfrak{b}$. We define the subspace $S_{\mathbf{k}}^{[\mathfrak{a};\mathfrak{b},\mathfrak{c};\mathfrak{d}]}(\mathfrak{N}, \psi)$ of $S_{\mathbf{k}}(\mathfrak{N}, \psi)$ by

$$S_{\mathbf{k}}^{[\mathfrak{a};\mathfrak{b},\mathfrak{c};\mathfrak{d}]}(\mathfrak{N}, \psi) :\simeq \bigoplus \pi^{K_1(\mathfrak{N})},$$

where π runs over representations as in (5.1) satisfying: (i) $c(\pi_{\mathfrak{p}}) = v_{\mathfrak{p}}(\mathfrak{N})$ for $\mathfrak{p}|\mathfrak{a}\mathfrak{b}\mathfrak{c}\mathfrak{d}$; (ii) $\pi_{\mathfrak{p}}$ is discrete series for $\mathfrak{p}|\mathfrak{a}$; (iii) $\pi_{\mathfrak{p}}$ is minimal supercuspidal for $\mathfrak{p}|\mathfrak{b}$; and (iv) $\pi_{\mathfrak{p}}$ is special for $\mathfrak{p}|\mathfrak{c}$. If $f \in S_{\mathbf{k}}(\mathfrak{N}, \psi)$ is the newform (not necessarily of level $\mathfrak{N}$) associated to π, (i) means that f is $\mathfrak{p}$-new for each $\mathfrak{p}|\mathfrak{a}\mathfrak{b}\mathfrak{c}\mathfrak{d}$, i.e., f is in the orthogonal complement of forms coming from level $\mathfrak{p}^{-1}\mathfrak{N}$ for such $\mathfrak{p}$; (ii) means that the minimum $\mathfrak{p}$-part of the level among $\mathfrak{p}$-power twists $f \otimes \chi$ of f is strictly greater than the $\mathfrak{p}$-power conductor of $\psi\chi^2$; (iii) means that the $\mathfrak{p}$-power of level of f is minimal among twists, at least 2, and this power is strictly greater than (in fact at least twice) the $\mathfrak{p}$-power conductor of the nebentypus; and (iv) means that one can twist f such that the $\mathfrak{p}$-part of the level is $\mathfrak{p}$ and the nebentypus is prime to $\mathfrak{p}$. In particular, if the nebentypus conductor is prime to $\mathfrak{b}\mathfrak{c}$, (iii) just means f is $\mathfrak{b}$-primitive and (iv) just means $\mathfrak{p}$ sharply divides the exact level of some twist of f. We also write $S_{\mathbf{k}}^{[\mathfrak{a};\mathfrak{b};\mathfrak{d}]}(\mathfrak{N}, \psi)$ for $S_{\mathbf{k}}^{[\mathfrak{a};\mathfrak{b},\mathfrak{o};\mathfrak{d}]}(\mathfrak{N}, \psi)$ and $S_{\mathbf{k}}^{\mathfrak{d}\text{-new}}(\mathfrak{N}, \psi)$ for $S_{\mathbf{k}}^{[\mathfrak{o},\mathfrak{o},\mathfrak{o};\mathfrak{d}]}(\mathfrak{N}, \psi)$.

All isomorphisms below are as $\mathcal{H}^S$-modules.

Theorem 5.4. *(i) As $\mathcal{H}^S$-modules,*

$$S_{\mathbf{k}}^{\mathrm{new}}(\mathcal{O}, \Omega, \omega) \simeq 2^{\#\{\mathfrak{p}|\mathfrak{N}_2\}} S_{\mathbf{k+2}}^{[\mathfrak{N}_1;\mathfrak{N}_2;\mathfrak{M}]}(\mathfrak{N}, \omega) \oplus R_{\mathbf{k}}^{\mathrm{new}}(\mathcal{O}, \Omega, \omega),$$

where the "remainder" space $R_{\mathbf{k}}^{\mathrm{new}}(\mathcal{O}, \Omega, \omega)$ is the subspace of $S_{\mathbf{k}}^{\mathrm{new}}(\mathcal{O}, \Omega, \omega)$ given by $R_{\mathbf{k}}^{\mathrm{new}}(\mathcal{O}, \Omega, \omega) = \bigoplus \pi_{\mathbf{k}}^{\Omega}$ where π runs over representations as in (4.3) such that $\pi_{\mathfrak{p}}$ is non-minimal for some $\mathfrak{p}|\mathfrak{N}_2$.

(ii) If $\omega_{\mathfrak{p}}$ is unramified for each $\mathfrak{p}|\mathfrak{D}$, and each $\mathfrak{p}|\mathfrak{N}_2$ such that $v_{\mathfrak{p}}(\mathfrak{N}_2) \geq 4$ is non-dyadic, then

$$S_{\mathbf{k}}^{\mathrm{new}}(\mathcal{O}, \Omega, \omega) \simeq \bigoplus 2^{\#\{\mathfrak{p}|\mathfrak{N}_2'\}} S_{\mathbf{k+2}}^{[\mathfrak{N}_1;\mathfrak{N}_2',\mathfrak{N}_2'';\mathfrak{M}]}(\mathfrak{N}, \omega),$$

where $(\mathfrak{N}_2', \mathfrak{N}_2'')$ runs over all pairs of coprime divisors of $\mathfrak{N}_2$ such that $\mathfrak{N}_2'\mathfrak{N}_2'' = \mathfrak{N}_2$ and $v_{\mathfrak{p}}(\mathfrak{N}_2'') = 2$ for all $\mathfrak{p}|\mathfrak{N}_2''$.

Note $S_{\mathbf{k+2}}^{[\mathfrak{N}_1;\mathfrak{N}_2;\mathfrak{M}]}(\mathfrak{N}, \omega)$ is just the $\mathfrak{N}_1\mathfrak{N}_2$-primitive subspace of $S_{\mathbf{k+2}}^{\mathrm{new}}(\mathfrak{N}, \omega)$ by our assumptions on $\mathfrak{N}_1$ and $\mathfrak{N}_2$.

Proof. Consider an arbitrary π appearing in the decomposition of $S_{\mathbf{k}}^{\mathrm{new}}(\mathcal{O}, \Omega, \omega)$ from Proposition 4.4(i), and let π' be its Jacquet–Langlands transfer to $\mathrm{GL}_2(\mathbb{A})$.

First consider $\mathfrak{p}|\mathfrak{N}_1$. If $\mathfrak{p}\|\mathfrak{N}_1$, then $\omega_{\mathfrak{p}}$ is unramified and $\dim \pi_{\mathfrak{p}} = 1$, i.e., $\pi_{\mathfrak{p}}'$ is an unramified twist of Steinberg. If $\mathfrak{p}^3|\mathfrak{N}_1$, then $\dim \pi_{\mathfrak{p}} > 1$ and as $\pi_{\mathfrak{p}}$ has odd conductor it must be minimal. Hence $\pi_{\mathfrak{p}}'$ is a minimal supercuspidal. In either case, by Theorem 3.5, we have $\dim \pi_{\mathfrak{p}}^{\Omega_{\mathfrak{p}}} = 1$.

Now suppose $\mathfrak{p}|\mathfrak{N}_2$. Since $\mathfrak{p}^2|\mathfrak{N}_2$, $\dim \pi_{\mathfrak{p}} = 1$ implies $\pi_{\mathfrak{p}}$ is non-minimal. So assume $\dim \pi_{\mathfrak{p}} > 1$. By Theorem 3.6, if $\pi_{\mathfrak{p}}$ is minimal, then $\dim \pi_{\mathfrak{p}}^{\Omega_{\mathfrak{p}}} = 2$ and $\pi_{\mathfrak{p}}'$ is a minimal supercuspidal. Recall that, by Proposition 3.4, $\pi_{\mathfrak{p}}$ is necessarily minimal if $\mathfrak{p}$ is non-dyadic and $c(\omega_{\mathfrak{p}}) < \frac{v_{\mathfrak{p}}(\mathfrak{N}_2)}{2}$. On the other hand, if $c(\pi_{\mathfrak{p}}) = 2$ ($\mathfrak{p}$ dyadic or not), $\pi_{\mathfrak{p}}$ is necessarily minimal.

From the characterization of the image of the Jacquet–Langlands correspondence, we get all π''s appearing in the new part of (5.1) such that $\pi_{\mathfrak{p}}'$ is of the type specified above for $\mathfrak{p}|\mathfrak{N}_1\mathfrak{N}_2$. This proves (i).

Under the assumptions of (ii), the above argument implies the only contribution to $R_{\mathbf{k}}^{\mathrm{new}}(\mathcal{O}, \Omega, \omega)$ occurs for π such that $\dim \pi_{\mathfrak{p}} = 1$ for some $\mathfrak{p}|\mathfrak{N}_2$. Consider such π and $\mathfrak{p}$. By our assumption, $\Omega_{\mathfrak{p}} = 1$, which by Lemma 3.3 implies $c(\pi_{\mathfrak{p}}) = 2$, i.e., $\pi_{\mathfrak{p}}$ corresponds to a ramified quadratic twist of Steinberg. From the above calculations, one sees $\dim \pi_{\mathbf{k}}^{\Omega} = 2^{\#\{\mathfrak{p}|\mathfrak{N}_2:\dim \pi_{\mathfrak{p}}>1\}}$. $\qquad\square$

Remark 5.3. One can also extend (ii) to allow for dyadic primes dividing $\mathfrak{N}_2$ to a sufficiently large power, using Proposition 3.4(iii). For instance if $F = \mathbb{Q}$, we can allow $\mathfrak{N}_2$ to be even if $2^8|\mathfrak{N}_2$, or just $2^6|\mathfrak{N}_2$ if we take $\mathcal{O}_2 \simeq \mathcal{O}_{r_2}(E_2)$ where E_2 is $\mathbb{Q}_2(\sqrt{3})$ or $\mathbb{Q}_2(\sqrt{7})$. We note the same issue arises in [HPS89b], e.g., see Theorem 7.30 and Example 10.7 of *op. cit.*

It is also easy to see that if we replace $S_{\mathbf{k}}^{\mathrm{new}}(\mathcal{O}, \Omega, \omega)$ with the $\mathfrak{N}_1\mathfrak{N}_2$-new subspace, we get the analogous result by working with $\mathfrak{N}_1\mathfrak{N}_2$-new spaces on the Hilbert modular form side. Via the decomposition from Corollary 4.5 or (4.10) of $S_{\mathbf{k}}(\mathcal{O}, \Omega, \omega)$ into newspaces of smaller levels, we can use this to describe the Hecke module structure of $S_{\mathbf{k}}(\mathcal{O}, \Omega, \omega)$ in terms of spaces of Hilbert modular forms, at least in the case that $\mathfrak{N}_2$ is cube-free

(recall we have not described $S_{\mathbf{k}}^{\mathrm{new}}(\mathcal{O}, \Omega, \omega)$ when $v_{\mathfrak{p}}(\mathfrak{N}_2)$ is odd for some $\mathfrak{p}|\mathfrak{N}_2$). This can be regarded as a classical Jacquet–Langlands correspondence for the full cuspidal space. We just write things down in the simplest case of nebentypus conductor prime to $\mathfrak{D}$, so that the "remainder" spaces do not appear.

Corollary 5.5. *Suppose $\mathfrak{N}_2$ is cube-free, and $\omega_{\mathfrak{p}}$ is unramified for each $\mathfrak{p}|\mathfrak{D}$. Then*

$$S_{\mathbf{k}}(\mathcal{O}, \Omega, \omega) \simeq \bigoplus 2^{\#\{\mathfrak{p}|\mathfrak{b}\}} S_{\mathbf{k+2}}^{[\mathfrak{a};\mathfrak{b};\mathfrak{c};\mathfrak{o}]}(\mathfrak{abc}\mathfrak{M}, \omega),$$

where $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ run over divisors of $\mathfrak{N}_1 \mathfrak{N}_2$ such that (i) $\mathfrak{D}|\mathfrak{abc}$; (ii) $\mathfrak{a}|\mathfrak{N}_1$ and $v_{\mathfrak{p}}(\mathfrak{a})$ is odd for $\mathfrak{p}|\mathfrak{N}_1$; (iii) $\mathfrak{bc}|\mathfrak{N}_2$ with $\mathfrak{b}$, $\mathfrak{c}$ coprime; and (iv) $\mathfrak{b}$ is a square.

In particular, if $\mathcal{O}$ is of unramified quadratic type, i.e. $\mathfrak{N}_2 = 1$, we have

$$S_{\mathbf{k}}(\mathcal{O}, \Omega, \omega) \simeq \bigoplus_{\mathfrak{d}} S_{\mathbf{k+2}}^{\mathfrak{d}\text{-new}}(\mathfrak{d}\mathfrak{M}, \omega),$$

where $\mathfrak{d}$ runs over all divisors of $\mathfrak{N}_1$ such that $v_{\mathfrak{p}}(\mathfrak{d})$ is odd for all $\mathfrak{p}|\mathfrak{N}_1$. In this case,

$$S_{\mathbf{k}}^{\mathrm{new}}(\mathcal{O}, \Omega, \omega) \simeq S_{\mathbf{k+2}}^{\mathfrak{N}_1\text{-new}}(\mathfrak{N}_1 \mathfrak{M}, \omega).$$

Proof. From (4.10), we deduce that $S_{\mathbf{k}}(\mathcal{O}, \Omega, \omega) \simeq \bigoplus S_{\mathbf{k}}^{\mathfrak{ad}\text{-new}}(\mathcal{O}'(\mathfrak{ad}\mathfrak{M}), \Omega', \omega)$, where $\mathfrak{a}$ runs over divisors of $\mathfrak{N}_1$ satisfying (ii), $\mathfrak{d}$ runs over divisors of $\mathfrak{N}_2$ such that $\mathfrak{D}|\mathfrak{ad}$, and $\mathcal{O}'(\mathfrak{N}')$ denotes a special order in B of level $\mathfrak{N}'$ with an admissible extension Ω' of Ω.

For fixed $\mathfrak{a}$, $\mathfrak{d}$, we can decompose

$$S_{\mathbf{k}}^{\mathfrak{ad}\text{-new}}(\mathcal{O}'(\mathfrak{ad}\mathfrak{M}), \Omega', \omega) = \bigoplus S_{\mathbf{k}}^{[\mathfrak{a};\mathfrak{b};\mathfrak{c};\mathfrak{o}]}(\mathcal{O}'(\mathfrak{ad}\mathfrak{M}), \Omega', \omega),$$

where $\mathfrak{b}, \mathfrak{c}$ run over relatively prime divisors of $\mathfrak{d}$ such that $\mathfrak{bc} = \mathfrak{d}$, and each space on the right denotes the $\mathfrak{a}$-new subspace of $S_{\mathbf{k}}(\mathcal{O}'(\mathfrak{ad}\mathfrak{M}), \Omega', \omega)$ consisting of π such that $\pi_{\mathfrak{p}}$ is 1-dimensional for $\mathfrak{p}|\mathfrak{c}$ and higher dimensional for $\mathfrak{p}|\mathfrak{b}$. Since $\mathfrak{b}$ is cube-free, this means that $\pi_{\mathfrak{p}}$ is also minimal for $\mathfrak{p}|\mathfrak{b}$. Also note this space is can only be nonzero if $\mathfrak{b}$ is a square.

The (proof of the) above theorem gives $S_{\mathbf{k}}^{[\mathfrak{a};\mathfrak{b};\mathfrak{c};\mathfrak{o}]}(\mathcal{O}'(\mathfrak{ad}\mathfrak{M}), \Omega', \omega) \simeq 2^{\#\{\mathfrak{p}|\mathfrak{b}\}} S_{\mathbf{k+2}}^{[\mathfrak{a};\mathfrak{b};\mathfrak{c};\mathfrak{o}]}(\mathfrak{abc}\mathfrak{M}, \omega)$. $\square$

Finally, when $\mathbf{k} = \mathbf{0}$, we also want to describe the full space of quaternionic modular forms. This is desired, for instance, to construct Eisenstein congruences—see [Mar17], [Mar18].

Proposition 5.6. *We have*

$$E_{\mathbf{0}}(\mathcal{O}, \Omega, \omega) \simeq \bigoplus_{\mu} \mathbb{C} E_{\mathbf{2},\mathfrak{N}_1\mathfrak{N}_2,\mathfrak{M}}(\mu),$$

where μ runs over characters of $F^{\times}\backslash\mathbb{A}_F^{\times}$ such that $\mu^2 = \omega$, $c(\mu)^2|\mathfrak{N}_1\mathfrak{N}_2$, and $\mu_{\mathfrak{p}} \circ N$ agrees with $\Omega_{\mathfrak{p}}$ on $\mathfrak{o}_{E_{\mathfrak{p}}}$ for all $\mathfrak{p}|\mathfrak{D}$, where $E_{\mathfrak{p}}/F_{\mathfrak{p}}$ denotes a quadratic extension such that $\mathcal{O}_{\mathfrak{p}} \simeq \mathcal{O}_{r_{\mathfrak{p}}}(E_{\mathfrak{p}})$.

In the case that $\mathcal{O}$ is of unramified quadratic type and ω is unramified, then μ simply runs over all unramified characters of $F^{\times}\backslash\mathbb{A}_F^{\times}$ such that $\mu^2 = \omega$. If in addition $\omega = 1$ and h_F is odd, then $E_{\mathbf{0}}(\mathcal{O}, \Omega, \omega) \simeq \mathbb{C} E_{\mathbf{2},\mathfrak{N}',\mathfrak{M}}$.

Proof. This is clear from Lemma 3.3, Lemma 4.6 and Proposition 5.2. $\square$

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5.4. **A congruence application.** One application of Corollary 5.5 is that it can be used to refine Eisenstein congruence results from [Mar17]. To us, the main deficiency in the results from [Mar17] is that we could not show we get Eisenstein congruences with newforms when we work with non-maximal orders in the relevant quaternion algebra. At least in some situations, Corollary 5.5 can be used to address this, but we only discuss a very simple case for elliptic modular forms here.

Denote by $E_{2,p}$ the normalized Eisenstein series in $M_2(p)$, and $E_{2,p^2}(z) = E_{2,p}(z) - E_{2,p}(pz)$. The n-th Fourier coefficient of $f \in M_2(N)$ is denoted $a_n(f)$.

Proposition 5.7. *Let* $p \geq 3$. *Then there exists a newform* $f \in S_2^{\mathrm{new}}(p^3)$ *such that* $a_n(f) \equiv a_n(E_{2,p^2}) \bmod p$ *for all* n.

Proof. Since $a_n(E_{2,p^2}) = 0$ when $p|n$, and this is also true for any $a_n(f)$ for a newform $f \in S_2^{\mathrm{new}}(p^3)$, it suffices to prove the above congruence for n prime to p.

In [Mar17, Corollary 2], we proved the existence of an eigenform $f \in S_2(p^3)$ (not necessarily new) satisfying the above congruence for n prime to p. The proof comes via constructing a quaternionic eigenform $\varphi \in S_0(\mathcal{O})$ Hecke congruent mod p to the quaternionic Eisenstein series $\varphi_0 = 1 \in E_0(\mathcal{O})$, where $\mathcal{O}$ is a special order of level p^3 in the quaternion algebra $B/\mathbb{Q}$ with discriminant p, and applying the Jacquet–Langlands correspondence to transfer φ to f.

By Corollary 5.5, this means our f as above in fact lies in $S_2(p) \oplus S_2^{\mathrm{new}}(p^3)$. (One can also derive this special case from the results in [Piz80a].) Hence it will suffice to show there is no $f \in S_2(p)$ satisfying this congruence. This follows from Mazur's determination of Eisentein ideals for $S_2(p)$ [Maz77, Proposition 9.7]. $\qquad\square$

We expect that one can use Corollary 5.5 to refine more general Eisenstein congruence results in [Mar17] by analyzing the behavior of ideal classes upon passing to suborders (sidestepping the use of Mazur's result). We hope to address this elsewhere.

6. Theta series

Here we explain how to reinterpret the "classical" Jacquet–Langlands map of the previous section in the more historically classical context of theta series. This will extend the solutions to the basis problem given in [Eic73] and [HPS89b] for $F = \mathbb{Q}$ to both more general quaternion algebras and to totally real F. We keep notation and assumptions as in Section 5.3.

For totally real fields, Eichler studied theta series attached to Brandt matrices (without character) of maximal orders [Eic77], though did not solve the basis problem in this setting. Shimizu [Shi72] effectively gave a representation-theoretic solution to the basis problem in this setting when each $k_v > 2$. Shimizu's solution was in terms of certain adelic theta series, but he did not explicate how to reinterpret these as classical theta series. This issue of realizing Shimizu's theta series as computable, classical theta series was taken up in the thesis [Geb09]. The theta series here, like Eichler's, are given in terms of Brandt matrices, and thus computable (e.g., see [DV13] for how to compute Brandt matrices for Eichler orders), and provide a solution to the basis problem for Hilbert modular forms.

We define the Brandt matrix series to be the matrix of functions of $x \in \mathbb{A}$, $y \in \mathbb{A}^\times$ with $y_\infty \gg 0$ given by

$$(6.1) \quad \Theta \begin{pmatrix} y & x \\ & 1 \end{pmatrix} = A_0 |y|^{\mathbf{k}/2+1} + \sum_{0 \ll \zeta \in F} A_{\zeta y \mathfrak{o}} \cdot N(\zeta y \mathfrak{o})^{-1} (\zeta y_\infty)^{\mathbf{k}/2+1} e_F(\zeta i y_\infty) \chi_F(\zeta x),$$

where the Brandt matrices $A_{\mathfrak{n}}$ are as in Section 4.3 for $\mathfrak{n}$ integral and interpreting $A_{\mathfrak{n}} = 0$ if $\mathfrak{n}$ is not integral. This corresponds to the collection of classical matrix Fourier series given by

$$\Theta^{(m)}(z) = N(t_m) A_0 + N(t_m) \sum_{0 \ll \xi \in t_m} A_{\xi t_m^{-1}} \xi^{\mathbf{k}/2} e_F(\xi z), \quad 1 \le m \le h_F.$$

As in Section 5.1, $t_1, \ldots, t_{h_F}$ are ideal class representatives for F. (To rewrite the constant term, we used that $A_0 = 0$ unless $\mathbf{k} = \mathbf{0}$.)

Consider some entry $\theta(z) = \sum a(\xi) e_F(\xi z)$ in the (i, j)-th $\kappa \times \kappa$ block of $N(t_m)^{-1} \Theta^{(m)}(z)$. Then, for $0 \ll \xi \in t_m$, we can write

$$a(\xi) = \frac{1}{e_j} \sum_\gamma \Omega^{-1}(x_i^{-1} \gamma x_j) r(\gamma) \xi^{\mathbf{k}/2},$$

where γ ranges as in Proposition 4.3 for $\mathfrak{n} = \xi t_m^{-1}$ and r is some matrix coefficient of $\rho_{\mathbf{k}}$. We also have $a(0) = 0$ unless $\mathbf{k} = \mathbf{0}$, $i = j$ and $\Omega = 1$, in which case $a(0) = \frac{1}{e_i}$.

Note θ can only be nonzero if $N(\mathcal{I}_i \mathcal{I}_j^{-1})$ lies in the same ideal class as t_m, so assume this. The coefficients of $\mathrm{Sym}^{\mathbf{k}}$ are homogenous polynomials of degree $\mathbf{k}$ which are "harmonic with respect to $N_{B/F}$". This means the following. For each infinite place ν_i, let $1, i, j, k$ be the standard basis for $B_{\nu_i} \simeq \mathbb{H}$. Then we can write any element γ of B as $\gamma = x + \epsilon_1 y i + \epsilon_2 z j + \epsilon_1 \epsilon_2 w k$ where $x, y, z, w \in F$ and ϵ_1, ϵ_2 lie in at most quadratic extensions of F. Then the coefficients of $\mathrm{Sym}^{k_i}(\gamma)$ are spherical harmonic polynomials in x, y, z, w of degree k_i (where now harmonic means in the usual sense, i.e., killed by the usual Laplacian). See, e.g., [Eic73, Proposition II.6].

Consequently, $\theta(z)$ is a classical theta series of the type defined by Eichler [Eic77] in the case of maximal orders and trivial character, where he proved that his theta series are classical Hilbert modular forms of suitable level by verifying the appropriate transformation laws. When $F = \mathbb{Q}$, theta series with character were treated in [Eic73] for orders of squarefree level and in [HPS89b] for special orders. However, for a special order of level N, [HPS89b] require some technical conditions to guarantee that their theta series are elliptic modular forms of level N—in general they only show their theta series have level N^2.

It should be possible to extend these approaches for general F to handle special orders and nontrivial character, but we will take a representation-theoretic approach to verifying our theta series are modular forms and have the desired level. However, due to the difference of the definitions of local ramified Hecke operators for $B^\times(\mathbb{A})$ and $\mathrm{GL}_2(\mathbb{A})$ (even at split places), we will only prove this for the "new" cuspidal subspace of theta series, which is sufficient for our solution to the basis problem.

Let $\Theta_{\mathbf{k}+\mathbf{2}}(\mathcal{O}, \Omega, \omega)$ be the subspace of (adelic) theta series generated by the entries of Θ which transform under the center of $\mathrm{GL}_2(\mathbb{A})$ by ω. Algorithmically this subspace can be described as follows. By (4.8), we can simultaneously block diagonalize the Brandt matrices $A_{\mathfrak{n}}$ so that each block is either zero or acts as the restriction of $T_{\mathfrak{n}}$ to

$M_{\mathbf{k}}(\mathcal{O}, \Omega, \omega)$. Call a block of the latter type $A_{\mathfrak{n},\omega}$. This block diagonalizes Θ, giving us block matrices of functions Θ_ω, which can be written as in (6.1) but with $A_{\mathfrak{n},\omega}$'s in place of the $A_{\mathfrak{n}}$'s. Then $\Theta_{\mathbf{k}+\mathbf{2}}(\mathcal{O}, \Omega, \omega)$ is simply the linear span of the entries of Θ_ω.

We can similarly define the new cuspidal subspace as follows. We can further block diagonalize each $A_{\mathfrak{n},\omega}$ (again, simultaneously in $\mathfrak{n}$) into three blocks $A_{\mathfrak{n},\omega}^{\mathrm{new}}$, $A_{\mathfrak{n},\omega}^{\mathrm{old}}$, and $A_{\mathfrak{n},\omega}^{\mathrm{eis}}$ which act by $T_\mathfrak{n}$ on $S_{\mathbf{k}}^{\mathrm{new}}(\mathcal{O}, \Omega, \omega)$, $S_{\mathbf{k}}^{\mathrm{old}}(\mathcal{O}, \Omega, \omega)$ and $E_{\mathbf{k}}(\mathcal{O}, \Omega, \omega)$ respectively. This yields a decomposition of Θ_ω into three blocks $\Theta_\omega^{\mathrm{new}}$, $\Theta_\omega^{\mathrm{old}}$ and $\Theta_\omega^{\mathrm{eis}}$. Define $\Theta_{\mathbf{k}+\mathbf{2}}^{\mathrm{new}}(\mathcal{O}, \Omega, \omega)$ to be the linear span of the entries of $\Theta_\omega^{\mathrm{new}}$.

We define an $\mathcal{H}^S$-action on $\Theta_{\mathbf{k}+\mathbf{2}}(\mathcal{O}, \Omega, \omega)$ as follows. Consider the decomposition (4.2). We take a basis Φ of $\mathbb{C}^{h\kappa}$ consisting of $\dim \pi_{\mathbf{k}}^\Omega$ forms for each π in this decomposition and vectors spanning $\ker \Xi$, with Ξ as in Section 4.3. Block diagonalizing with respect to Φ gives a decomposition $\Theta_{\mathbf{k}+\mathbf{2}}(\mathcal{O}, \Omega, \omega) = \bigoplus \Theta_\pi$ where Θ_π is generated by at most $\dim \pi_{\mathbf{k}}^\Omega$ theta series whose normalized Fourier coefficients are the Hecke eigenvalues of any nonzero $\varphi \in \pi$ away from $\mathfrak{N}$. For $\mathfrak{n}$ coprime to $\mathfrak{N}$, we let $T_\mathfrak{n}$ act on Θ_π by the corresponding unramified Hecke eigenvalue for π. Extend this action linearly to $\Theta_{\mathbf{k}+\mathbf{2}}(\mathcal{O}, \Omega, \omega)$.

Proposition 6.1. *There are $\mathcal{H}^S$-module epimorphisms $M_{\mathbf{k}}(\mathcal{O}, \Omega, \omega) \to \Theta_{\mathbf{k}+\mathbf{2}}(\mathcal{O}, \Omega, \omega)$ and $S_{\mathbf{k}}^{\mathrm{new}}(\mathcal{O}, \Omega, \omega) \to \Theta_{\mathbf{k}+\mathbf{2}}^{\mathrm{new}}(\mathcal{O}, \Omega, \omega)$. The latter map is an isomorphism if $\mathcal{O}$ is of unramified quadratic type.*

Proof. The first statement is clear from the above decomposition $\Theta_{\mathbf{k}+\mathbf{2}}(\mathcal{O}, \Omega, \omega) = \bigoplus \Theta_\pi$. The second statement follows as $\dim \Theta_\pi = \dim \pi_{\mathbf{k}}^\Omega = 1$ for each such π when $\mathcal{O}$ is of unramified quadratic type. $\qquad\square$

Theorem 6.2. *The full theta space $\Theta_{\mathbf{k}+\mathbf{2}}(\mathcal{O}, \Omega, \omega)$ embeds as an $\mathcal{H}^S$-submodule of $M_{\mathbf{k}+\mathbf{2}}(\mathfrak{N}, \omega)$. In particular, for every $\theta \in \Theta_{\mathbf{k}+\mathbf{2}}(\mathcal{O}, \Omega, \omega)$, there exists $f \in M_{\mathbf{k}+\mathbf{2}}(\mathfrak{N}, \omega)$ such that their (nonzero) Fourier coefficients agree away from $\mathfrak{N}$.*

Proof. Consider the decomposition $\Theta_{\mathbf{k}+\mathbf{2}}(\mathcal{O}, \Omega, \omega) = \bigoplus \Theta_\pi$ as above. For π appearing in this decomposition, there exists an eigenform $f_\pi \in M_{\mathbf{k}+\mathbf{2}}(\mathfrak{N}, \omega)$ whose Hecke eigenvalues agree with those of any nonzero $\varphi \in \pi$ outside of $\mathfrak{N}$. This follows from Theorem 5.4(i) when $\dim \pi > 1$ and Proposition 5.6 when $\dim \pi = 1$. Let $\mathfrak{N}'$ be the exact level of f_π, i.e., the conductor of the associated automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. The number of linearly independent f_π with this property is $d_\pi := \prod(v_\mathfrak{p}(\mathfrak{N}) - v_\mathfrak{p}(\mathfrak{N}') + 1)$.

To get our embedding, it suffices to show that $\dim \Theta_\pi \leq d_\pi$. By Lemma 3.2 and (4.10), $\dim \Theta_\pi \leq \dim \pi_{\mathbf{k}}^\Omega \leq 2^j \prod_{\mathfrak{p}|\mathfrak{M}}(v_\mathfrak{p}(\mathfrak{N}) - v_\mathfrak{p}(\mathfrak{N}') + 1)$, where j is the number of primes $\mathfrak{p}|\mathfrak{N}_2$ such that $\dim \pi_\mathfrak{p} > 1$. Hence it suffices to remove a factor of 2 in this latter bound for each $\mathfrak{p}|\mathfrak{N}_2$ such that $\dim \pi_\mathfrak{p} > 1$ and $v_\mathfrak{p}(\mathfrak{N}) = v_\mathfrak{p}(\mathfrak{N}')$. Consider such a $\mathfrak{p}$, and suppose $\varphi, \varphi' \in \pi_{\mathbf{k}}^\Omega$ are linearly independent factorizable functions such that $\varphi_v = \varphi'_v$ for $v \neq \mathfrak{p}$. By assuming φ, φ' lie in our basis Φ, we see they will only contribute at most a 1-dimensional space to Θ_π if all Hecke operators $T_\mathfrak{n}$ act by scalar matrices on $\langle \varphi, \varphi' \rangle$. This is obvious if $\mathfrak{p} \nmid \mathfrak{n}$, so it remains to show this for each $T_{\mathfrak{p}^m}$. But $T_{\mathfrak{p}^m}$ kills both φ and φ' by Proposition 4.1(iii). $\qquad\square$

Remark 6.1. Note the last paragraph of the proof describes the kernel of the (non-canonical) map $S_{\mathbf{k}}^{\mathrm{new}}(\mathcal{O}, \Omega, \omega) \to \Theta_{\mathbf{k}+\mathbf{2}}^{\mathrm{new}}(\mathcal{O}, \Omega, \omega)$ from Proposition 6.1. (The kernel is canonical up to isomorphism as an $\mathcal{H}^S$-module.) Namely, its dimension is $\sum(2^{s(\pi)} - 1)$

where π runs over representations occurring in $S_{\mathbf{k}}^{\mathrm{new}}(\mathcal{O}, \Omega, \omega)$ and $s(\pi)$ is the number of primes $\mathfrak{p} | \mathfrak{N}_2$ for which $\dim \pi_{\mathfrak{p}} > 1$.

We have not shown that the Fourier coefficients of our theta series for $\mathfrak{n}$ not prime to $\mathfrak{N}$ are the Fourier coefficients of a corresponding form in $M_{\mathbf{k+2}}(\mathfrak{N}, \omega)$, however we can conclude this for the new cuspidal theta series:

Theorem 6.3. *We have $\Theta_{\mathbf{k+2}}^{\mathrm{new}}(\mathcal{O}, \Omega, \omega) \subset S_{\mathbf{k+2}}^{\mathrm{new}}(\mathfrak{N}, \omega)$.*

Proof. Consider π in the decomposition of $S_{\mathbf{k}}^{\mathrm{new}}(\mathcal{O}, \Omega, \omega)$ from Proposition 4.4(i). For each π, we can associate an $f_\pi \in S_{\mathbf{k+2}}^{\mathrm{new}}(\mathfrak{N}, \omega)$ as in the previous proof, which now is unique up to scalars.

Using a basis of $S_{\mathbf{k}}^{\mathrm{new}}(\mathcal{O}, \Omega, \omega)$ composed of bases for each π as above, we can block diagonalize $\Theta_\omega^{\mathrm{new}}$ with one block for Θ_π for each π. Then it suffices to show each $T_{\mathfrak{p}^m}$ acts on $\pi_{\mathbf{k}}^{\Omega}$ by $C(\mathfrak{p}^m, f_\pi)$ for each π, $\mathfrak{p}$ and m. This is clear for $\mathfrak{p} \nmid \mathfrak{N}$. For $\mathfrak{p} | \mathfrak{D}$, this follows from Proposition 4.1 and the standard computations of Hecke operators on local newforms of $\mathrm{GL}_2(F_{\mathfrak{p}})$. For $\mathfrak{p} | \mathfrak{M}$, this follows from Proposition 4.2. $\square$

This proves that the map $S_{\mathbf{k}}^{\mathrm{new}}(\mathcal{O}, \Omega, \omega) \to \Theta_{\mathbf{k+2}}^{\mathrm{new}}(\mathcal{O}, \Omega, \omega)$ from Proposition 6.1 is a realization of the map JL in terms of theta series.

Specializing to the original case of trivial nebentypus, we get the following solution to the basis problem. Note that for any level $\mathfrak{N}$, we may choose our quaternion algebra B and special order $\mathcal{O}$ such that there is at most one prime dividing $\mathfrak{N}_2$.

For a space S of modular forms, S^ψ denotes the space of twists by ψ.

Corollary 6.4. *If $\mathcal{O}$ is unramified quadratic type, i.e., $\mathfrak{N}_2 = \mathfrak{o}$ then*

$$S_{\mathbf{k+2}}^{\mathrm{new}}(\mathfrak{N}, 1) = \Theta_{\mathbf{k+2}}^{\mathrm{new}}(\mathcal{O}, 1, 1).$$

If $\mathfrak{N}_2 = \mathfrak{p}^{2e}$, then

$$S_{\mathbf{k+2}}^{\mathrm{new}}(\mathfrak{N}, 1) \subset \Theta_{\mathbf{k+2}}^{\mathrm{new}}(\mathcal{O}, 1, 1) + \sum_\psi \sum_{j=1}^{e} \Theta_{\mathbf{k+2}}^{\mathrm{new}}(\mathcal{O}(\mathfrak{N}_1, \mathfrak{p}^j, \mathfrak{M}), \Omega_\psi, \psi^{-2})^\psi + \sum_\psi S_{\mathbf{k+2}}^{\mathrm{new}}(\mathfrak{N}_1 \mathfrak{M}, \psi^{-2})^\psi,$$

where ψ runs over primitive characters of conductor $\mathfrak{p}^e$, $\mathcal{O}(\mathfrak{N}_1, \mathfrak{p}^j, \mathfrak{M})$ denotes a special order of level $\mathfrak{N}_1 \mathfrak{p}^j \mathfrak{M}$ in B locally isomorphic to $\mathcal{O}$ at all places away from $\mathfrak{p}$, which is of unramified quadratic type at $\mathfrak{p}$ (and thus globally) if j is odd, and Ω_ψ is a character for this order extending ψ^{-2} as in Section 4.1.

Proof. Since we have trivial central character, any cuspidal π of conductor $\mathfrak{N}$ must be discrete series at $\mathfrak{p}$ when $v_{\mathfrak{p}}(\mathfrak{N})$ is odd. Thus, by Theorem 5.4, $\Theta_{\mathbf{k+2}}^{\mathrm{new}}(\mathcal{O}, 1, 1)$ contains all newforms $f \in S_{\mathbf{k+2}}^{\mathrm{new}}(\mathfrak{N}, 1)$ except possibly those corresponding to automorphic representations π which are not minimal at $\mathfrak{p} | \mathfrak{N}_2$. In particular, the first statement follows.

Now assume $\mathfrak{N}_2 = \mathfrak{p}^{2e}$ and $\pi_{\mathfrak{p}}$ is not minimal. Then there exists a local character $\psi_{\mathfrak{p}}$ such that $c(\psi_{\mathfrak{p}}) = e$ and $\pi_{\mathfrak{p}} \otimes \psi_{\mathfrak{p}}^{-1}$ minimal. Let ψ^{-1} be a finite-order globalization of $\psi_{\mathfrak{p}}^{-1}$ unramified at all other finite places. If $\pi_{\mathfrak{p}}$ is a (ramified) principal series, then $c(\psi_{\mathfrak{p}}) = \frac{v_{\mathfrak{p}}(\mathfrak{N}_2)}{2}$ and $\pi_{\mathfrak{p}} \otimes \psi_{\mathfrak{p}}^{-1}$ is unramified. Otherwise, $j = c(\pi_{\mathfrak{p}} \otimes \psi_{\mathfrak{p}}^{-1}) \geq 1$, and we can replace $\mathcal{O}_{\mathfrak{p}}$ by a special order $\mathcal{O}_{\mathfrak{p}}'$ of level $\mathfrak{p}^j$ (of unramified quadratic type if j is odd) to pick up the twist $\pi \otimes \psi^{-1}$ in a suitable $S_{\mathbf{k}}^{\mathrm{new}}(\mathcal{O}', \Omega', \psi^{-2})$. $\square$

The proof of this, without assuming anything on $\mathfrak{N}_2$, tells us the effectively weaker result that $S_{\mathbf{k+2}}^{\text{new}}(\mathfrak{N}, 1)$ is generated by $\Theta_{\mathbf{k+2}}^{\text{new}}(\mathcal{O}, 1, 1)$ and twists of Hilbert newforms of smaller levels at some primes dividing $\mathfrak{N}_2$. This is a more precise version of Theorem 1.2, which we think of as a "weak solution" to the basis problem.

Moreover, by varying the quaternion algebra B, we see that we will get every newform in $S_{\mathbf{k+2}}^{\text{new}}(\mathfrak{N}, 1)$ via some quaternionic theta series unless $[F : \mathbb{Q}]$ is odd (so we cannot take B to be unramified at all finite places) and the associated representation π is a principal series at all ramified places, i.e., π is a twist of an unramified representation. This proves Corollary 1.3, our solution to the basis problem for trivial nebentypus.

The same argument can be applied to modular forms with character, however now even a weak solution to the basis problem is not as clean in general: the space $S_{\mathbf{k+2}}^{\text{new}}(\mathfrak{N}, \psi)$ is generated by $\Theta_{\mathbf{k+2}}^{\text{new}}(\mathcal{O}, \Omega, \psi)$ together with Hilbert newforms of level $\mathfrak{N}$ which are minimal ramified principal series at some primes dividing $\mathfrak{N}_1 \mathfrak{N}_2$ and twists of Hilbert newforms of smaller levels at some primes dividing $\mathfrak{N}_2$. Namely, if $\mathfrak{p}|\mathfrak{N}_1\mathfrak{N}_2$ such that $c_{\mathfrak{p}}(\psi) = v_{\mathfrak{p}}(\mathfrak{N})$, then there is a minimal ramified principal series, $\pi(1, \psi_{\mathfrak{p}})$ in standard notation, with local conductor $v_{\mathfrak{p}}(\mathfrak{N})$. Hence one will not get all newforms in general spaces $S_{\mathbf{k+2}}^{\text{new}}(\mathfrak{N}, \psi)$ by theta series attached to $\mathcal{O}$ and twists of newforms of smaller level, rather only when $c_{\mathfrak{p}}(\psi) < v_{\mathfrak{p}}(\mathfrak{N})$ for all $\mathfrak{p}|\mathfrak{N}_1\mathfrak{N}_2$.

However, by choosing B to be ramified at at most one finite prime, and taking appropriate $\mathcal{O}$ and Ω, we see that we can generate $S_{\mathbf{k+2}}^{\text{new}}(\mathfrak{N}, \psi)$ by quaternionic theta series and twist of newforms of smaller level if $[F : \mathbb{Q}]$ is even (where no twists are needed) or if $c(\psi) \neq \mathfrak{N}$. Note the exception $c(\psi) \neq \mathfrak{N}$ is the analogue of the classical situation with $F = \mathbb{Q}$ and $\psi = 1$ where $S_k(1)$ is not generated by theta series.

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