Modularity of Hypertetrahedral Representations

Modularité des Représentations Hypertétraèdrales

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Abstract

Let F be a number field, G_F its absolute Galois group, and $\rho : G_F \to \mathrm{GL}_4(\mathbb{C})$ an irreducible continuous Galois representation. Let $\bar{G}$ denote the projective image of ρ in $\mathrm{PGL}_4(\mathbb{C})$. We say that ρ is *hypertetrahedral* if $\bar{G}$ is an extension of A_4 by the Klein group V_4. In this case, we show that ρ is *modular*, i.e., ρ corresponds to an automorphic representation π of $\mathrm{GL}_4(\mathbb{A}_F)$ such that their L-functions are equal. This gives new examples of irreducible 4-dimensional *monomial* representations which are modular, but are not induced from normal extensions and are not essentially self-dual. *To cite this article: K. Martin, C. R. Acad. Sci. Paris, Ser. I 336 (2004).*

Résumé

Soient F un corps de nombres, $G_F = \mathrm{Gal}(\bar{F}/F)$ et $\rho : G_F \to \mathrm{GL}_4(\mathbb{C})$ une représentation irréductible et continue. Soit $\bar{G}$ l'image projective. Nous appellerons une telle représentation *hypertétraèdrale* si $\bar{G}$ est une extension de A_4 par le groupe de Klein V_4. Nous démontrons qu'une représentation hypertétraèdrale est *modulaire*, i.e., il existe une représentation cuspidale π de $\mathrm{GL}_4(\mathbb{A}_F)$ tel que $L(s,\rho) = L(s,\pi)$. Ceci donne de nouveaux exemples de représentations modulaires qui ne sont pas induites par des extensions normales et ne sont pas essentiellement auto-duales. *Pour citer cet article : K. Martin, C. R. Acad. Sci. Paris, Ser. I 336 (2004).*

1. Introduction

Let F be a number field, $G_F = \mathrm{Gal}(\bar{F}/F)$ the absolute Galois group and $\rho : G_F \to \mathrm{GL}_4(\mathbb{C})$ a continuous representation. Let $\bar{\rho} : G_F \to \mathrm{PGL}_4(\mathbb{C})$ denote the composition of ρ with the standard projection from $\mathrm{GL}_4(\mathbb{C})$ to $\mathrm{PGL}_4(\mathbb{C})$ and let $\bar{G}$ be the image of $\bar{\rho}$. We say that ρ is *modular* if there exists an automorphic representation π of $\mathrm{GL}_4(\mathbb{A}_F)$ such that $L(s,\rho) = L(s,\pi)$. We then write $\rho \leftrightarrow \pi$. Thus at all unramified places v of F, $L(s,\rho_v) = L(s,\pi_v)$ and we write $\rho_v \leftrightarrow \pi_v$. Denote the restriction of ρ to a subgroup $\mathrm{Gal}(\bar{F}/E)$ by ρ_E.

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We are interested in the case where $\bar{G}$ is an extension of A_4 by a group of order 4. Let C_n be the cyclic group of order n and V_4 be the Klein 4-group. The extensions of A_4 by C_4 and V_4 can be, for example, easily computed in the computer algebra package GAP. There are 6 possibilities for $\bar{G}$: $C_4 \times A_4$, $V_4 \times A_4$, $\mathrm{SL}_2(\mathbb{F}_3) \times C_2$, $\mathrm{SL}_2(\mathbb{F}_3) \rtimes C_2$, $V_4 \rtimes A_4$ and $V_4 \cdot A_4$, the unique group of order 48 containing both V_4 and A_4 as subgroups which is not a semidirect product of the two. In the first four cases, as will be shown below, ρ is necessarily reducible and therefore modular. If ρ is irreducible (so $\bar{G} = V_4 \rtimes A_4$ or $V_4 \cdot A_4$) then we will say that ρ is *hypertetrahedral*. (Note there exist reducible representations ρ for which $\bar{G} = V_4 \cdot A_4$.)

Theorem *Let F be a number field and ρ a hypertetrahedral representation of G_F. Then ρ is* modular*. There are infinitely many such representations with projective image $V_4 \cdot A_4$ which are not essentially self-dual.*

Remarks: (1) A hypertetrahedral representation (irreducible and 4-dimensional) ρ is monomial, so Artin's conjecture is known for ρ. However, ρ is induced from a *non-normal* quartic extension K (i.e., from a degree one character of $\mathrm{Gal}(\bar{F}/K)$) with no intermediate fields, so modularity does not follow from known automorphic induction results.

(2) Recall that ρ is *essentially self-dual* if and only if the image of ρ is contained in $\mathrm{GO}_4(\mathbb{C})$ or $\mathrm{GSp}_4(\mathbb{C})$. The hypertetrahedral representations which are not essentially self-dual give new examples of modular representations. Irreducible solvable representations into $\mathrm{GO}_4(\mathbb{C})$ were shown to be modular in [8]. Also, many cases are known for representations into $\mathrm{GSp}_4(\mathbb{C})$, such as the symmetric cube of a modular 2-dimensional representation ([4]) or when the projective image is an extension of C_2^4 by C_5 ([6]). But very little is known about non-self-dual representations.

Let us elaborate briefly on these remarks. Let $\rho : G_F \to \mathrm{GL}_4(\mathbb{C})$ be a (possibly reducible) representation such that $\bar{G}$ is one of the 6 possible extensions of A_4 by C_4 or V_4. Let L be the fixed field of $\ker(\rho)$, N the fixed field of $\ker(\bar{\rho})$ and $\widetilde{K}/F$ the extension corresponding to the quotient group A_4. Let K be a subextension of $\widetilde{K}/F$ with $\mathrm{Gal}(\widetilde{K}/K) = C_3$. Then K/F is a non-normal quartic extension with Galois closure $\widetilde{K}$. Let E be the subextension of $\widetilde{K}/F$ corresponding to the subgroup V_4. Then E/F is a normal cubic extension. Note that $\mathrm{Gal}(N/E)$ is a 2-group so $\mathrm{Gal}(L/E)$ is the direct product of a 2-group with a cyclic group of odd order. Thus, any irreducible representation of $\mathrm{Gal}(L/E)$ has dimension 2^j for some j.

Consequently, if ρ is reducible, then it is modular. For any 2-dimensional components are modular by [5] and [9]. Also, if ρ has an irreducible 3-dimensional constituent τ, then τ_E is reducible, i.e., τ is induced from the normal cubic extension E, whence modular by [1]. Hence we will assume that ρ is irreducible.

Now we claim that ρ is induced from K, i.e., that ρ_K contains a character. Assume otherwise. Since $\mathrm{Gal}(N/\widetilde{K}) = C_4$ or V_4, any irreducible representation of $\mathrm{Gal}(L/\widetilde{K})$ has dimension 1 or 2. Thus ρ_K cannot be irreducible since the restriction $\rho_{\widetilde{K}}$ to a normal cubic extension is not. So we may assume that ρ_K is a sum of two irreducible 2-dimensionals. Then $\rho_{\widetilde{K}}$ is also sum of two irreducible 2-dimensionals, say $\rho = \sigma \oplus \tau$, and $\mathrm{Gal}(\widetilde{K}/F) = A_4$ acts transitively on on $\{\sigma, \tau\}$. Hence the stabilizer of σ in A_4 is a subgroup of index 2. But A_4 has no subgroups of index 2, a contradiction. This establishes Remark (1).

The Galois group $\mathrm{Gal}(\widetilde{K}/F) = A_4$ acts transitively on the 4 distinct characters occuring in $\rho_{\widetilde{K}}$. This implies that $\mathrm{Gal}(\widetilde{K}/F)$ cannot fix $\mathrm{Gal}(N/\widetilde{K})$ pointwise. However, for each of the four groups $C_4 \times A_4$, $V_4 \times A_4$, $\mathrm{SL}_2(\mathbb{F}_3) \times C_2$ and $\mathrm{SL}_2(\mathbb{F}_3) \rtimes C_2$, any group element fixes pointwise the normal subgroup of order 4. This shows that $\bar{G} = V_4 \rtimes A_4$ or $V_4 \cdot A_4$ (assuming ρ is irreducible).

Now we want to know when ρ will be not essentially self-dual. If ρ is induced from a normal extension, then it is modular by [1]. So we will assume it is not. Then we claim that ρ cannot be of symplectic type. Observe dimensionality requires that if $\Lambda^2(\rho)$ contains a character, it contains two (counting multiplicity), which implies that ρ is induced from a 2-dimensional representation, whence the claim.

The case where $\bar{G} = V_4 \rtimes A_4$ yields examples of irreducible monomial 4-dimensional representations of orthogonal type, which are modular by [8]. However in the case where $\bar{G} = V_4 \cdot A_4$, we obtain below irreducible monomial representations ρ which are not of orthogonal type, whence not essentially self-dual. Then ρ is not a tensor product of two 2-dimensionals since its image does not lie in $\mathrm{GO}_4(\mathbb{C})$. Nor is ρ a symmetric cube lift of a 2-dimensional representation because $\bar{G}$ is not a subgroup of $\mathrm{PGL}_2(\mathbb{C})$.

Example. Take the group G_{192} of order 192 generated by

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

As this is solvable, it occurs as a Galois group over $\mathbb{Q}$ by a theorem of Shafarevich ([7]) and has a hyper-tetrahedral representation ρ which is not essentially self-dual and not induced from a normal extension. Such examples exist of orders $192 \cdot k$, $k = 1, 2, 3, ...$ This can easily be seen by taking central products of G_{192} with cyclic groups.

2. Proof of Theorem

The proof of modularity is similar to Langlands' original tetrahedral argument ([5]), which relied upon normal cubic base change for GL_2 ([5]), the symmetric square lift of Gelbart and Jacquet from GL_2 to GL_3 ([2]), and the structure of A_4. We use normal cubic base change for GL_4, the exterior square of Kim from GL_4 to GL_6 ([3]), and the structure of $\bar{G}$, in a manner similar to the argument in [6].

As observed in the remarks following the theorem, we may assume that ρ is irreducible and $\bar{G} = V_4 \rtimes A_4$ or $V_4 \cdot A_4$. Let the extensions $L \supseteq N \supseteq \widetilde{K} \supseteq K \supseteq F$ and $\widetilde{K} \supseteq E \supseteq F$ be as in the previous section.

Lemma 2.1 *The representations ρ_E and $\Lambda^2(\rho)$ are modular.*

Proof. As remarked in the previous section, $\mathrm{Gal}(L/E)$ is a direct product of a 2-group P_2 with a cyclic group C of odd order. Therefore $\mathrm{Gal}(L/E)$ is nilpotent. By a theorem of Arthur and Clozel ([1]), all representations of nilpotent groups are modular. In particular ρ_E is modular.

We now show $\Lambda^2(\rho)$ is modular. First note $\Lambda^2(\rho)$ does not contain any characters because ρ cannot be symplectic, as mentioned above. Thus $\Lambda^2(\rho)$ cannot contain an irreducible 5-dimensional representation either. Any 2-dimensional representation inside $\Lambda^2(\rho)$ is modular by Langlands and Tunnell ([5], [9]).

Now suppose $\Lambda^2(\rho)$ contains an irreducible τ of dimension 3 or 6. We know that all irreducible representations of $\mathrm{Gal}(L/E)$ have dimension a power of two because $\mathrm{Gal}(L/E) = P_2 \times C$. Thus τ_E must be reducible, whence τ is induced from the normal extension E and therefore modular.

Finally, consider the case where $\Lambda^2(\rho)$ contains an irreducible 4-dimensional representation σ. Since there is a natural symmetric pairing $\Lambda^2(\rho) \times \Lambda^2(\rho) \to \Lambda^4(\rho)$, σ maps into $\mathrm{GO}_6(\mathbb{C})$. The dimension of σ implies that its image lies in $\mathrm{GO}_4(\mathbb{C})$. Hence σ is modular by [8].

Thus all irreducible components of $\Lambda^2(\rho)$ must be modular, so $\Lambda^2(\rho)$ is also. QED.

Let us say $\rho_E \leftrightarrow \Pi$. We claim that ρ_E is irreducible. Indeed, the irreducibility of ρ implies that $\mathrm{Gal}(E/F) = C_3$ acts transitively on the irreducible components of ρ_E. This action has order dividing 3. Thus if there is more than one irreducible component of ρ_E, there must be three or a multiple thereof. However $\dim \rho_E = 4$, so that is impossible. Therefore ρ_E is irreducible, whence Π is cuspidal.

Let $\delta = \delta_{E/F}$ be a non-trivial idele class character of $F^* \mathfrak{N}_{E/F}(\mathbb{A}_E^*) \backslash \mathbb{A}_F^* = \mathrm{Gal}(E/F) = C_3$. Base change results ([1]) tell us that there are precisely three cuspidal representations, $\pi_0, \pi_1 = \pi_0 \otimes \delta$ and $\pi_2 = \pi_0 \otimes \delta^2$, of $\mathrm{GL}_4(\mathbb{A}_F)$ whose base change to E is Π.

Lemma 2.2 *There is a unique π_i such that $\Lambda^2(\pi_i) \leftrightarrow \Lambda^2(\rho)$.*

Proof. All the representations $\Lambda^2(\pi_i)$ base change to $\Lambda^2(\pi_0 \otimes \delta^i)_E = \Lambda^2(\pi_0)_E$. They are all distinct because they have distinct central characters $\omega_{\Lambda^2(\pi_i)} = \omega_{\Lambda^2(\pi_0)}\delta^{2i}$. Therefore these are the only representations of W_F which base change to $\Lambda^2(\pi_0)_E$. We also know that $\Lambda^2(\rho)$ corresponds to some automorphic representation β on $\mathrm{GL}_6(\mathbb{A}_F)$. But then $\beta_E = \Lambda^2(\pi_0)_E$ implies that β must equal some $\Lambda^2(\pi_i)$. QED.

Denote the π_i of the lemma by π. We claim now that in fact $\rho \leftrightarrow \pi$. It will suffice to show for all unramified places that $\rho_v \leftrightarrow \pi_v$. Say ρ_v has Frobenius eigenvalues $\{a, b, c, d\}$ and π_v has Satake parameters $\{e, f, g, h\}$. We want to show $\{a, b, c, d\} = \{e, f, g, h\}$. For a diagonal element D of GL_4, we have $\Lambda^2(D) = 1$ if and only if $D = \pm I$. Hence $\Lambda^2(\rho_v) \leftrightarrow \Lambda^2(\pi_v)$ implies $\{a, b, c, d\} = \pm\{e, f, g, h\}$. If they are equal, we are done. Assume therefore

$$\{a, b, c, d\} = -\{e, f, g, h\}. \tag{1}$$

Now we can use base change to E. In our projective image $\bar{G}$, any element cubed lies inside the normal subgroup of index 3, $\mathrm{Gal}(N/E)$. Thus any element of $G(L/F)$ cubed lies inside $\mathrm{Gal}(L/E)$. In particular $Fr_v^3 \in \mathcal{O}_{E_w}$, where w is a prime of E above v and Fr_v is the Frobenius. Then $\rho_{v,E} \leftrightarrow \pi_{v,E}$ implies $\{a^3, b^3, c^3, d^3\} = \{e^3, f^3, g^3, h^3\}$. Combining this with (1) yields,

$$\{a^3, b^3, c^3, d^3\} = \{-a^3, -b^3, -c^3, -d^3\}. \tag{2}$$

Without loss of generality, assume $a^3 = -b^3$ and $c^3 = -d^3$. Then either $b = -\zeta_3 a$ or $d = -\zeta_3 c$, for otherwise $a = -b$, $c = -d$ which would imply $\{a, b, c, d\} = \{e, f, g, h\}$. Let us say $b = -\zeta_3 a$. Then $\rho(Fr_v) \sim \mathrm{diag}(a, -\zeta_3 a, c, d)$ so $\bar{\rho}(\mathrm{Fr}_v) \sim \mathrm{diag}(1, -\zeta_3, c/a, d/a)$ is an element of order divisible by 6 in $\bar{G} = Im(\bar{\rho}) \subseteq \mathrm{PGL}_4(\mathbb{C})$. But $\bar{G}$ has no elements of order 6, a contradiction! Therefore ρ is modular.

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