Relations Between π and e

Khiem Nguyen
Courtney Miller

1. Introduction

Through our research, we were trying to develop a better understanding of the properties of π and e. Delving into the information of these two numbers, we found that there was much misunderstanding between us about transcendental numbers and their properties. This project focuses not distinctly on irrationality, but also it focuses on transcendental number theory because the understanding of transcendental numbers is required in getting a better understanding of π and e.

2. Proof of Irrationality of π

Johann Heinrich Lambert in 1761 was the first to prove the irrationality of π. (Irrationality Proofs). However, we turn to Ivan Niven, who presented a much shorter proof of π in 1947. (Irrationality Proofs) This proof requires a lot of advanced algebra and series knowledge that we did not understand.

A. Niven's polynomials (Irrationality Proofs)

"Polynomials of the form

$$P_n(x) = \frac{x^n(1-x)^n}{n!} = \frac{1}{n!} \sum_{k=n}^{2n} c_k x^k$$

where c_k are integers."

B. The Proof

Theorem: *π^2 is an irrational number.*

Proof: (Irrationality Proofs)

"Suppose $\pi^2 = p/q$, with (p,q) positive integers, consider the function

$$J(x) = q^n(\pi^{2n}P_n(x) - \pi^{2n-2}P_n^{(2)}(x) + \pi^{2n-4}P_n^{(4)}(x) - \ldots (-1)^n P_n^{(2n)}(x)),$$

from the properties of P_n, J(0) and J(1) are integers. But

$$(J'(x)\sin\pi x - J(x)\pi\cos\pi x)' = (J^{(2)}(x)+\pi^2 J(x))\sin\pi x = q^n\pi^{2n+2}P_n(x)\sin\pi x$$
$$= \pi^2 p^n P_n(x)\sin\pi x$$

giving

$$\frac{1}{\pi}[\, J'(x)\sin\pi x - J(x)\pi\cos\pi x]_0^1 = J(0)+J(1) = \pi p^n \int_0^1 P_n(x)\sin\pi x dx$$

hence the last integral is a non zero integer. From the bounds of $P_n(x)$, we have the bounds for the integral

$$0 < \pi p^n \int_0^1 P_n(x)\sin\pi x dx < \frac{\pi p^n}{n!} < 1$$

when n becomes large. This contradicts the fact that the integral was an integer." QED

Corollary: *π is irrational.*

3. Proof of Irrationality of e

Euler in 1744 first established the irrationality of e.

Theorem: *e is irrational.*

Proof (McRae):

"Define e as the sum 1/0! + 1/1! + 1/2! + ...

Suppose that e is equal to some fraction p/q, in lowest terms. Then

$$e = p/q = 1/0! + 1/1! + 1/2! + 1/3! + ... + 1/q! + 1/(q+1)! + ...$$

Multiplying by q!,

$$(q-1)!p = (q!/0!+ q!/1! + ... q!/q!) + 1/(q+1) + 1/[(q+1)(q+2)] + ...$$

and noting that q!/x! is integer as long as x is less than or equal to q, the left side of the following equation is an integer

$$(q-1)!p - (q!/0!+ q!/1! + ... q!/q!) = 1/(q+1) + 1/[(q+1)(q+2)] + ...$$

Therefore, we know that $1/(q+1) + 1/[(q+1)(q+2)] + \ldots$ is some integer, and it's obvious that it is greater than 0. But,

$$1/(q+1) + 1/[(q+1)(q+2)] + \ldots < 1/(q+1) + 1/(q+1)^2 + 1/(q+1)^3 + \ldots = 1/q \text{ £ } 1$$

Therefore, there is an integer between 0 and 1, which is a contradiction." QED

4. Properties of Transcendental Numbers

We found through our research that both e and π are transcendental numbers. Seeing that neither of us knew exactly what transcendental numbers were, we decided it would be important to include this in presentation because of the importance it has with Hilbert's problems.

A. Definitions (Klee)

An algebraic number x is the root of a polynomial equation with integer coefficients:

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0.$$

A number that is not algebraic is called transcendental.

B. Properties

It is also helpful to note that all numbers that can be constructed by ruler and compass are algebraic, and transcendental numbers cannot be constructed in this fashion. From this fact, we can see that the set of transcendental numbers is a subset of the irrational numbers. Because all rational numbers are constructible by ruler and compass, then all rational numbers are algebraic. This implies that numbers that are not constructible cannot be rational. Therefore, transcendental numbers are irrational.

Although people believed that transcendental numbers existed, it was not until 1844 that Louiville proved their existence. Following was Hermite's proof of the transcendence of e in 1873 and then Lindemann's proof of the transcendence of π in 1882.

But more interestingly, Cantor in 1874 proved that almost all numbers are transcendental (Classifications).

In order to do this, Cantor showed that the set of algebraic numbers is denumerable and the set of transcendental numbers is uncountable. A set S is said to be denumerable (or countable infinite) if there exists a bijection of the natural numbers onto S. A set S is said to be uncountable if it is neither finite nor denumerable. (Bartle)

Theorem: The set of algebraic numbers is denumerable.

Proof (Hardy):

"We call a number algebraic if it satisfies the equation:

$$a_0x^n + a_1x^{n-1} + \ldots + a_n = 0.$$

We define the rank of the equation as $N = n + |a_0| + |a_1| + \ldots + |a_n|$.

The minimum value of N is 2. It is plain that there are only a finite number of equations

$$E_{N,1}, E_{N,,2}, \ldots, E_{N,,k\_N}$$

of rank N. We can arrange the equations in the sequence

$$E_{2,1}, E_{2,2}, \ldots, E_{2,,k\_2}, E_{3,1}, E_{3,2}, \ldots, E_{3,k\_3}, E_{4,1}, \ldots$$

And so correlate them with the numbers 1, 2, 3, ... Hence the aggregate of equations is countable. But every algebraic number corresponds to at least one of these equations, and the number of algebraic numbers corresponding to any equation is finite." QED

Now, to prove that the set of transcendental numbers is uncountable, whose elements are not algebraic, we need only show that the set of real numbers between 0 and 1 is uncountable. This was done in Discrete Math. So because the algebraic set is a subset of the real numbers that is uncountable, but the algebraic set is countable, and numbers can only be one or the other, it follows that the transcendental set is uncountable. This implies that there are far many more transcendental numbers than algebraic numbers, although both are infinite. (Hardy)

In 1900, David Hilbert presented 1900 in Paris 20 problems that he wanted solved in the upcoming century. Problem 7 dealt with the transcendence and irrationalities of certain numbers. Specifically, he presented the problem:

"*The expression α^β, for an algebraic base α and an irrational algebraic exponent β, e.g., the number $2^{\sqrt{2}}$ or $e^\pi = i^{-2i}$, always represents a transcendental or at least an irrational number.*" (Hilbert)

He also stated that though this problem has such simplicity, he did not expect it to be solved within his lifetime. 34 years later Gel-fond and Schneider independently solved this problem. (Tijdeman)

Theorem: If a, b are real numbers algebraic over the rational numbers, if a $\neq$ 0 or 1, and b is irrational, then a^b is transcendental.

Corollary: *$2^{\sqrt{2}}$ is transcendental.*

5. Relations of π and e

A. π + e and π - e

Today, it cannot be proven that $\pi + e$ or $\pi - e$ is independently irrational. By intuition, we may believe that they both are irrational, but today, we can only prove as far as saying that one or the other *must* be irrational.

Proof (Klee and Wagon):
Assume that both $\pi + e$ and $\pi - e$ are rational numbers. Because the sum of two rational numbers is a rational number, it follows that

$(\pi + e) + (\pi - e) = q$ where q is an element of the rational numbers.

So

$2\pi = q$

which cannot happen because 2π is irrational.

Therefore we have a contradiction and conclude that $\pi + e$ or $\pi - e$ must be irrational. QED

It is possible that both are, but we cannot establish this fact. A similar argument can be used to show that $e + \pi$ or $e - \pi$ must be irrational.

B. πe and π/e

It is also not proven that πe and π/e are individually irrational. However, like the above, we can say that πe or π/e must be irrational.

Proof (Klee and Wagon):
So we assume that πe and π/e are both rational numbers. Because the product of two rational numbers must be rational, it follows that

$(\pi e)(\pi/e) = q$ where q is a rational number.

So

$\pi^2 = q$

which contradicts the irrationality of π^2. QED

The same can be stated about the relationship between πe and e/π.

Work Cited

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