3. Consequences of Baire's Theorem

In this chapter, we discuss four fundamental functional analytic theorems that are direct descendants of Baire's Theorem (Theorem 1.10). All four results have a somewhat paradoxical character; the assumptions look too weak to give the desired conclusions, but somehow we get these anyway.

Theorem 3.1 (Uniform boundedness principle)**.** *Let X be a Banach space and let Y be a normed space. Assume that $\mathcal{F} \subseteq B(X,Y)$ is a family of bounded linear operators that is bounded* pointwise *in the following sense: For each $x \in X$, there exists $C_x \geq 0$ such that $\|Ax\| \leq C_x$ for all $A \in \mathcal{F}$. Then $\mathcal{F}$ is* uniformly *bounded, that is,* $\sup_{A\in\mathcal{F}} \|A\| < \infty$.

Proof. Let $M_n = \{x \in X : \|Ax\| \leq n \text{ for all } A \in \mathcal{F}\}$. Then M_n is a closed subset X. Indeed, we can write

$$M_n = \bigcap_{A\in\mathcal{F}} \{x \in X : \|Ax\| \leq n\},$$

and these sets are closed because they are the inverse images under A of the closed ball $\overline{B}_n(0)$. Moreover, the assumption that $\mathcal{F}$ is pointwise bounded says that $\bigcup_{n\in\mathbb{N}} M_n = X$. Therefore, by Baire's Theorem, at least one of the M_n's is not nowhere dense. Fix such an n, and let $B_r(x_0)$ be an open ball contained in M_n. In other words, we now know that if $\|y - x_0\| < r$, then $\|Ay\| \leq n$ for all $A \in \mathcal{F}$. In particular, if $x \in X$ is arbitrary with $\|x\| = 1$, then $y = x_0 + (r/2)x$ is such a vector and thus

$$\begin{aligned} \|Ax\| &= \frac{2}{r}\,\|A(y - x_0)\| \leq \frac{2}{r}\,(\|Ay\| + \|Ax_0\|) \\ &\leq \frac{2}{r}\,(n + C_{x_0}) \equiv D. \end{aligned}$$

The constant D is independent of x, so it follows that $\|A\| \leq D$. Since D is also independent of $A \in \mathcal{F}$, this is what we claimed. $\square$

Theorem 3.2 (The open mapping theorem)**.** *Let X, Y be Banach spaces, and assume that $A \in B(X,Y)$ is surjective (that is, $R(A) = Y$). Then A is an open map: if $U \subseteq X$ is open, then $A(U)$ is also open (in Y).*

The condition defining an open map is of course similar to the corresponding property of continuous maps (see Proposition 1.5), but it goes in the other direction. In particular, that means that the inverse of an open map, if it exists, is continuous. Therefore, the open mapping theorem has the following consequence:

Corollary 3.3. *Let X, Y be Banach spaces, and assume that $A \in B(X,Y)$ is bijective. Then $A^{-1} \in B(Y,X)$.*

Exercise 3.1. Prove the following linear algebra fact: The inverse of an invertible linear map is linear.

Proof. By Exercise 3.1, A^{-1} is linear. By the open mapping theorem and the subsequent remarks, A^{-1} is continuous. □

Proof of Theorem 3.2. Let $U \subseteq X$ be an open set, and let $y \in A(U)$, so $y = Ax$ for some $x \in X$ (perhaps there are several such x, but then we just pick one of these). We want to show that there exists an $r > 0$ with $B_r(y) \subseteq A(U)$. Since $y \in A(U)$ was arbitrary, this will prove that $A(U)$ is open.

We know that $B_\epsilon(x) \subseteq U$ for some $\epsilon > 0$, so it actually suffices to discuss the case where $U = B_\epsilon(x)$. In fact, this can be further reduced: it is enough to consider $x, y = 0$, and it then suffices to show that for *some* $R > 0$, the set $A(B_R(0))$ contains a ball $B_r(0)$ for some $r > 0$. Indeed, if this holds, then, using the linearity of A, we will also have

$$A(B_\epsilon(x)) = Ax + \frac{\epsilon}{R} A(B_R(0)) \supseteq Ax + \frac{\epsilon}{R} B_r(0) = B_{\epsilon r/R}(Ax) = B_{\epsilon r/R}(y),$$

and this is exactly what we originally wanted to show.

Since A is surjective, we can write

$$Y = \bigcup_{n \in \mathbb{N}} A(B_n(0)) = \bigcup_{n \in N} \overline{A(B_n(0))}.$$

By Baire's Theorem, one of the closed sets in the second union has to contain an open ball, say $B_r(v) \subseteq \overline{A(B_n(0))}$. In other words, $B_r(0) \subseteq \overline{A(B_n(0))} - v$. Now again $v = Au$ for some $u \in X$, so

$$B_r(0) \subseteq \overline{A(B_n(0))} - Au = \overline{A(B_n(-u))}, \tag{3.1}$$

and if we take $N \geq n + \|u\|$, then $B_N(0) \supseteq B_n(-u)$, so

$$B_r(0) \subseteq \overline{A(B_N(0))}. \tag{3.2}$$

Except for the closure, this is what we wanted to show.

Exercise 3.2. In (3.1), we used the following fact: If $M \subseteq X$ and $x \in X$, then $\overline{M} + x = \overline{M + x}$. Prove this and also the analogous property that $\overline{cM} = c\overline{M}$ ($c \in \mathbb{C}$).

We will now finish the proof by showing that $\overline{A(B_N(0))} \subseteq A(B_{2N}(0))$. So let $y \in \overline{A(B_N)}$ (since all balls will be centered at 0, we will use this simplified notation).

We can find an $x_1 \in B_N$ with $\|y - Ax_1\| < r/2$. Since, by (3.2) and Exercise 3.2,

$$B_{r/2} = \frac{1}{2}B_r \subseteq \frac{1}{2}\overline{A(B_N)} = \overline{A(B_{N/2})},$$

we then also see that $y - Ax_1 \in \overline{A(B_{N/2})}$. Thus there exists an $x_2 \in B_{N/2}$ with $\|y - Ax_1 - Ax_2\| < 2^{-2}r$. We continue in this way and obtain a sequence x_n with the following properties:

$$x_n \in B_{2^{-n+1}N}, \qquad \left\| y - \sum_{j=1}^{n} Ax_j \right\| < 2^{-n}r \tag{3.3}$$

This shows, first of all, that the series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent. Indeed, $\sum_{n=1}^{\infty} \|x_n\| < 2N \sum_{n=1}^{\infty} 2^{-n} = 2N < \infty$. By Exercise 2.22, $x := \sum_{n=1}^{\infty} x_n$ exists. Moreover, by the calculation just carried out, $\|x\| \le \sum \|x_n\| < 2N$, so $x \in B_{2N}$. Since A is continuous, we have $Ax = \lim_{n\to\infty} \sum_{j=1}^{n} Ax_j$, and the second property from (3.3) now shows that $Ax = y$. In other words, $y \in A(B_{2N})$, as desired. □

The *graph* of an operator $A : X \to Y$ is defined as the set $\mathcal{G}(A) = \{(x, Ax) : x \in X\}$. We can think of $\mathcal{G}(A)$ as a subset of the Banach space $X \oplus Y$ that was introduced in Chapter 2; see especially Theorem 2.17.

Exercise 3.3. Show that $\mathcal{G}(A)$ is a (linear) subspace of $X \oplus Y$ if A is a linear operator.

Definition 3.4. Let X, Y be Banach spaces. A linear operator $A : X \to Y$ is called *closed* if $\mathcal{G}(A)$ is closed in $X \oplus Y$.

If we recall how the norm on $X \oplus Y$ was defined, we see that $(x_n, y_n) \to (x, y)$ in $X \oplus Y$ precisely if $x_n \to x$ and $y_n \to y$. Therefore, using sequences, we can characterize closed operators as follows: $A : X \to Y$ is closed precisely if the following holds: If $x_n \to x$ and $Ax_n \to y$, then $y = Ax$.

On the other hand, A is continuous precisely if $x_n \to x$ implies that $Ax_n \to y$ and $y = Ax$ (formulated in a slightly roundabout way here to facilitate the comparison). This looks clearly stronger than the condition from above: what was part of the hypothesis has become part of the conclusion. In particular, continuous operators are always closed. When viewed against this background, the following result is quite stunning.

Theorem 3.5 (The closed graph theorem)**.** *Let X, Y be Banach spaces and assume that $A : X \to Y$ is linear and closed. Then $A \in B(X, Y)$.*

Proof. We introduce the projections $P_1 : X \oplus Y \to X$, $P_2 : X \oplus Y \to Y$, $P_1(x, y) = x$, $P_2(x, y) = y$. It is clear that P_1, P_2 are linear and continuous. By hypothesis and Exercise 3.3, $\mathcal{G}(A)$ is a closed linear subspace of $X \oplus Y$. By Proposition 2.7, it is therefore a Banach space itself (with the same norm as $X \oplus Y$). Now P_1, restricted to $\mathcal{G}(A)$ is a bijection onto X. Corollary 3.3 shows that the inverse $P_1^{-1} : X \to \mathcal{G}(A)$, $P_1^{-1}x = (x, Ax)$ is continuous. It follows that $A = P_2P_1^{-1}$ is a composition of continuous maps and thus continuous itself. □

Exercise 3.4. Let X, Y be Banach spaces and $A_n, A \in B(X, Y)$. We say that A_n converges (to A) *strongly* if $A_n x \to Ax$ for all $x \in X$. In this case, we write $A_n \xrightarrow{s} A$. Prove that strong convergence has the following properties:
(a) $\|A_n - A\| \to 0 \Longrightarrow A_n \xrightarrow{s} A$;
(b) The converse of part (a) does not hold;
(c) If $A_n \xrightarrow{s} A$, then $\sup_n \|A_n\| < \infty$ (*Hint:* use the uniform boundedness principle).

Exercise 3.5. Suppose that for some measure space (X, μ) and exponents p, q, we have $L^p(X, \mu) \subseteq L^q(X, \mu)$. Show that then there exists a constant $C > 0$ such that $\|f\|_q \le C\|f\|_p$ for all $f \in L^p(X, \mu)$.

Suggested strategy: If $L^p \subseteq L^q$, we can define the inclusion map $I : L^p \to L^q$, $If = f$. Use Corollary 2.9 to show that this map is closed, and then apply the closed graph theorem.