The spectrum of differential operators of order $2n$ with almost constant coefficients

Horst Behncke[1], Don Hinton[2], and Christian Remling[1]

December 1, 1999

J. Differential Eq. **175** (2001), 130–162

1. Universität Osnabrück, Fachbereich Mathematik/Informatik, 49069 Osnabrück, GERMANY
2. University of Tennessee, Department of Mathematics, Knoxville, TN 37996, USA

E-mail: hinton@math.utk.edu, cremling@mathematik.uni-osnabrueck.de

2000 AMS Subject Classification: 34L05, 34E10

Abstract

We discuss the spectral properties of higher order ordinary differential operators. If the coefficients differ from constants by small perturbations, then the spectral properties are preserved. In this context, "small perturbations" are either short range (i.e., integrable) or long range, but slowly varying. This generalizes classical results on second order operators. Our approach relies on an analysis of the associated differential equations with the help of uniform asymptotic integration techniques.

1 Introduction

In this paper, we will study differential operators of the form

$$(\tau y)(x) = \frac{1}{w(x)} \sum_{k=0}^{n} (-1)^k \frac{d^k}{dx^k} \left(p_k(x) \frac{d^k y}{dx^k} \right). \tag{1}$$

Any self-adjoint expression with sufficiently smooth real valued coefficients can be written in this form (cf. [20, Theorem I.15.2]), so (1) is a natural starting point. The factors $(-1)^k$ ensure that the kth summand is nonnegative (as a quadratic form) if $p_k(x) \geq 0$ pointwise. We are interested in the spectral properties of the self-adjoint operators on the Hilbert space $L_2(0, \infty; w(x)\,dx)$

that are generated by the differential expresssion τ. In particular, we always assume that $w(x) > 0$ almost everywhere in $(0, \infty)$. A second basic assumption is $p_n(x) > 0$ almost everywhere. One then also needs mild regularity conditions on the coefficients, mainly in order to make sure that the intial value problems associated with $\tau y = zy$ have unique solutions. However, we will impose more restrictive assumptions anyway (see Theorem 1.1 below), so these conditions will not be made explicit. Instead, the reader is referred to [33] for the general theory.

If all coefficient functions $w(x), p_i(x)$ are constant, one can of course give a complete analysis of τ. Namely, by taking Fourier transforms, one sees that basically τ is unitarily equivalent to multiplication by a polynomial. As a consequence, the operator always has absolutely continuous spectrum, but there may also be some eigenvalues. Location and multiplicity of the absolutely continuous spectrum can be read off from the polynomial. We will discuss this in more detail below.

Our aim in this paper is to identify classes of perturbations which leave the general picture unchanged. For second order operators ($n = 1$ in (1)), this problem has been studied extensively. In particular, the following well-known result exists, which was first proved by Weidmann in [31]: Consider the Schrödinger operator $(\tau y)(x) = -y''(x) + V(x)y(x)$ on $L_2(0, \infty)$, and suppose that $V = V_1 + V_2$ where $V_1 \in L_1(0, \infty)$ and V_2 is (locally) absolutely continuous, $\lim_{x\to\infty} V_2(x) = 0$, and $V_2' \in L_1(0, \infty)$. Then for all self-adjoint realizations of τ, we have that $\sigma_{ac} = [0, \infty)$ and the spectrum is purely absolutely continuous on $(0, \infty)$. In other words, the part of the operator on $(0, \infty)$ is unitarily equivalent to the corresponding part of the unperturbed operator associated with $\tau_0 y = -y''$. In a sense, this result is almost optimal. For instance, size conditions essentially weaker than $V_1 \in L_1$ are not sufficient to prevent singular spectrum on $(0, \infty)$. Indeed, if $V(x) = O(x^{-1})$ at infinity, positive eigenvalues are possible, as was already recognized in [29]. If V is only of order $V(x) = O(x^{-1/2})$, one can even have *purely* singular spectrum [28]. There are more results; in fact, it seems fair to say that there is now a rather good understanding of Schrödinger operators with conditions only on the size of V; we refer the reader to [7, 9, 24, 27] for recent results and to [26] for an overview.

Continuing the discussion of Weidmann's result, we note that there are two different types of admissible perturbations: Either the perturbation itself is small (in an average sense), or it is slowly varying. It has been known for a long time that such perturbations have controllable effects on the solutions of second order differential equations; this often goes under the name WKB approximation. In fact, these methods can be extended in various directions; see [3, 5, 6, 12, 34] for further information on this topic (with applications to spectral theory).

Compared to this huge set of results, very little is known on analogous problems for higher order operators. We will prove the following generalization of Weidmann's theorem. Actually, our technique can be pushed further to cover larger classes of perturbations. This will be discussed after completing the proof of Theorem 1.1.

Theorem 1.1 *Suppose that every coefficient function from* (1) *is almost constant in the following sense:*

$$\begin{aligned} p_i(x) &= q_i(x) + r_i(x) \qquad (i = 0, 1, \dots, n-1), \\ w(x) &= v(x) + r(x), \\ p_n^{-1}(x) &= q_n^{-1}(x) + r_n(x), \end{aligned}$$

where $r, r_i \in L_1(0,\infty)$ *and the limits* $\lim_{x\to\infty} q_i(x) =: c_i, \lim_{x\to\infty} v(x) = 1$ *exist and* $c_n > 0$*. Moreover,* q_i, v *are locally absolutely continuous and* $q_i', v' \in L_1(0,\infty)$*.*

Then for all self-adjoint realizations of τ *in* $L_2(0,\infty; w(x)\,dx)$*, we have: The singular continuous spectrum is empty, and the absolutely continuous part of the operator is unitarily equivalent to the constant coefficient operator*

$$(\tau_0 y)(x) = \sum_{k=0}^{n} (-1)^k c_k \frac{d^{2k} y}{dx^{2k}}$$

on $L_2(0,\infty; dx)$ *with boundary conditions* $y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0$*. Moreover, the essential spectrum satisfies* $\sigma_{ess} = \sigma_{ac}$*, and the operator is semibounded below.*

Remarks. 1. Basically, this says that the perturbed operator has the same spectral properties as the unperturbed one except that there may be additional point spectrum. As our discussion below will show, this really gives a rather explicit description of the spectral properties. Indeed, location and multiplicity of the spectrum of τ_0 can be read off from the polynomial $\sum c_k \lambda^{2k}$ in a straightforward way – see Sect. 3 for further information on this. Note also that, in contrast to the second order case, multiplicity of the absolutely continuous spectrum is an issue here.

2. The situation where, more generally, $\lim v(x) = c > 0$, can of course be reduced to the case $c = 1$ by a simple normalization.

3. In general, there will also be embedded point spectrum in regions where the multiplicity of the absolutely continuous spectrum is smaller than the maximal possible value n. We do not have very complete results on these embedded eigenvalues, but offer a few remarks in Sect. 9. In particular, we will present an example where these eigenvalues have an accumulation point inside σ_{ac}.

A quick proof of Weidmann's original result (using modern tools) runs as follows: Fix $E > 0$. Then standard asymptotic integration techniques (see, e.g., [12, Chapter 2]) show that the DE $-y'' + Vy = Ey$ has solutions y_+, y_- of the asymptotic form

$$\begin{pmatrix} y_\pm(x) \\ y_\pm'(x) \end{pmatrix} = \begin{pmatrix} 1 \\ \pm i\sqrt{E} \end{pmatrix} \exp\left(\pm i \int_0^x \sqrt{E - V_2(t)}\, dt \right) + o(1) \quad (x \to \infty).$$

Now the desired assertions on the spectral properties follow directly from this and the subordinacy theory [13].

We want to prove Theorem 1.1 using a similar strategy. However, there is no subordinacy theory for higher differential operators, so we must use the information on the solutions of $\tau y = zy$ obtained from asymptotic integration in a different way. Namely, we will approximately compute the Titchmarsh-Weyl M-function of operators on $L_2(a, \infty; w\,dx)$ for large a and then try to deduce the spectral properties of the operators on $L_2(0, \infty; w\,dx)$ from this.

There are several problems which do not occur in the second order case. First of all, the M-function is originally defined only off the spectrum, and the spectral properties depend on the limiting behavior of $M(z)$ as z tends to the spectrum. This means that we must solve the DE $\tau y = zy$ for complex z and then take limits $z \to E \in \mathbb{R}$. Since we need uniform control on the error terms, the usual asymptotic integration theory is insufficient for our purposes. We discussed the extension we need here in [4]. Then, knowledge of the M-function of operators on $L_2(a, \infty; w\,dx)$ does not automatically lead to statements on operators on $L_2(0, \infty; w\,dx)$. To overcome difficulties of this type, we use some results from [25].

The organization of this paper is as follows: In the next section, we compile some facts from the general theory of higher order differential operators. Then, in Sect. 3, we discuss operators with constant coefficients. In Sect. 4–6, we are concerned with the asymptotic integration of the DE $\tau y = zy$. As explained above, special attention has to be paid to the question of obtaining *uniform* estimates on the error terms. To get an overview of the strategy used to prove Theorem 1.1, it is in fact possible to go directly to Theorem 6.1, where we summarize the results of the discussion of Sect. 4–6. We then use Theorem 6.1 in Sect. 7 to conclude the proof of Theorem 1.1. An extension of this result is presented in Sect. 8. Finally, we make some remarks about embedded eigenvalues. We also include the main result of [4] in an Appendix.

Acknowledgments: H.B. and C.R. thank the Deutsche Forschungsgemeinschaft for financial support.

2 Hamiltonian systems

The differential expression τ from (1) gives rise to self-adjoint operators on $L_2(0, \infty; w\,dx)$. This is a classical application of von Neumann's theory of self-adjoint extensions of symmetric operators; a comprehensive discussion can be found in [33]. One first introduces the maximal operator T associated with τ. Loosely speaking, its domain consists of all $y \in L_2(0, \infty; w\,dx)$ for which τy is again in $L_2(0, \infty; w\,dx)$; for these y, one defines $Ty = \tau y$. It turns out that the minimal operator $T_0 = T^*$ is symmetric and has equal deficiency indices, so there are self-adjoint extensions. These self-adjoint restrictions of T can then also be characterized in terms of boundary conditions at $x = 0$ (and possibly also at $x = \infty$).

For our purposes, it will be convenient to write the equation $\tau y = zy$ as a linear Hamiltonian system. In this paper, by a linear Hamiltonian system we mean a differential equation of the form

$$JY'(x) = (zA(x) + B(x))Y(x). \tag{2}$$

Here, $J, A, B \in \mathbb{C}^{2n\times 2n}$, $A(x), B(x)$ are locally integrable and self-adjoint for almost every x, $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and A has block form $A = \left(\begin{smallmatrix} A_1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, with $A_1 \in \mathbb{C}^{r\times r}$ positive definite almost everywhere. The theory of (1) is contained in the general framework of Hamiltonian systems. This follows from a result of Walker [30]; note, however, that Walker uses a slightly different J. His equation can be transformed to the form given above by applying the permutation matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & L \end{smallmatrix}\right)$ to the solution vector Y, where $L \in \mathbb{C}^{n\times n}$ has the matrix elements $L_{ij} = \delta_{j,n+1-i}$. Of course, we can also verify directly that $\tau y = zy$ is equivalent to a system of the form (2) by letting

$$Y_i = y^{(i-1)},\ Y_{n+i} = \sum_{k=i}^{n} (-1)^{k+i} \left(p_k y^{(k)}\right)^{(k-i)} \quad (i = 1, \ldots, n). \tag{3}$$

One then computes that $A_1(x)$ is the 1×1-matrix $w(x)$, and $B = \left(\begin{smallmatrix} -P & K \\ K^* & Q \end{smallmatrix}\right)$, where the non-zero entries of the $n\times n$-matrices P, Q, K are

$$P_{ii} = p_{i-1}, \quad Q_{nn} = p_n^{-1}, \quad K_{i+1,i} = 1.$$

We now recall some facts from the theory of Hamiltonian systems. Some general references for this subject are [2, 10, 15, 16, 17].

Under certain additional assumptions, one can again associate Hilbert space operators with eq. (2); in particular, this can always be done in the case at hand (where (2) comes from a higher order scalar differential equation). The appropriate underlying Hilbert space is the space $L_{2,A}(0,\infty)$ of (equivalence classes of) measurable, $\mathbb{C}^r$-valued functions f satisfying $\int f^* A_1 f < \infty$. Of course, with A_1 as above, this is again the space $L_2(0,\infty; w\,dx)$.

In fact, in this paper we will never use the precise definition of these operators; let us just stress the important point that one recovers precisely the operators associated with (1), so here the theories are equivalent. Now, in the situation of Theorem 1.1, the deficiency indices of the minimal operator associated with (2) (or, equivalently, with (1)) are (n,n) (this will follow from the discussion below), and $x = 0$ is a regular endpoint. Therefore, only a boundary condition at $x = 0$ is needed. In the Hamiltonian system formulation, the admissable boundary conditions are precisely given by

$$(\alpha_1, \alpha_2)Y(0) = 0, \tag{4}$$

where $\alpha_i \in \mathbb{C}^{n\times n}$ satisfy

$$\alpha_1\alpha_1^* + \alpha_2\alpha_2^* = 1, \quad \alpha_1\alpha_2^* - \alpha_2\alpha_1^* = 0.$$

Next, fix a boundary condition $\alpha \equiv (\alpha_1, \alpha_2)$ and $z \in \mathbb{C}$, and define special solutions U_α, V_α of (2) by requiring that

$$(U_\alpha(0,z), V_\alpha(0,z)) = \begin{pmatrix} \alpha_1^* & -\alpha_2^* \\ \alpha_2^* & \alpha_1^* \end{pmatrix}. \tag{5}$$

So $U_\alpha(x,z), V_\alpha(x,z)$ both have n columns and each column solves (2). Moreover, since the matrix on the right hand side of (5) is regular, U, V together span the whole solution space of (2). Note also that V_α satisfies the boundary condition (4).

Now the M-function can be defined for Im $z > 0$ (say) by requiring that $U_\alpha(\cdot, z) + V_\alpha(\cdot, z)M_\alpha(z) \in L_{2,A}(0,\infty)$. (For the definition of M for general deficiency indices, see [15].) M_α is a (matrix valued) Herglotz function, that is, M_α is holomorphic and has positive definite imaginary part. Thus there is a representation of the form

$$M_\alpha(z) = c_1(\alpha) + c_2(\alpha)z + \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) d\rho_\alpha(t),$$

with $c_i^* = c_i, c_2 \geq 0$. The matrix valued measure ρ_α is a spectral measure for the operator with boundary condition α (call this self-adjoint operator H_α). More precisely, H_α is unitarily equivalent to the operator of multiplication by the independent variable in the space $L_2(\mathbb{R}, d\rho_\alpha)$ (for the definition of this space, see, e.g., [1]). The spectral measure ρ_α can be recovered from the boundary behavior of M_α as the weak limit

$$d\rho_\alpha(E) = \frac{1}{\pi} \lim_{\epsilon \to 0+} \text{Im } M_\alpha(E + i\epsilon)\, dE. \tag{6}$$

In other words, (6) holds when integrated against continuous functions of compact support. The pointwise limit $M_\alpha(E) \equiv \lim_{\epsilon\to 0+} M_\alpha(E+i\epsilon)$ also exists for almost every $E \in \mathbb{R}$, and the absolutely continuous part of ρ_α is given by

$$d\rho_\alpha^{(ac)}(E) = \frac{1}{\pi} \text{Im } M_\alpha(E)\, dE. \tag{7}$$

3 Operators with constant coefficients

The differential operators studied in Theorem 1.1 have asymptotically constant coefficients. It is natural to begin the analysis with the unperturbed problem, that is, with operators with constant coefficients. So, consider

$$(\tau_0 y)(x) = \sum_{k=0}^{n} (-1)^k c_k \frac{d^{2k} y}{dx^{2k}}, \tag{8}$$

with $c_k \in \mathbb{R}$, $c_n > 0$. The domain of the corresponding maximal operator T on $L_2(0,\infty)$ is

$$D(T) = \{y \in L_2(0,\infty) : \\ y, \ldots, y^{(2n-1)} \text{ locally absolutely continuous}, y^{(2n)} \in L_2(0,\infty)\}.$$

We first study the self-adjoint operator H_0, whose domain is given by

$$D(H_0) = \{y \in D(T) : y'(0) = y'''(0) = \cdots = y^{(2n-1)}(0) = 0\}.$$

The advantage of this operator lies in the fact that after taking Fourier transforms, the domain is still easy to describe. We could also have chosen the boundary conditions

$$y(0) = y''(0) = \cdots = y^{(2n-2)}(0) = 0$$

instead; in this case, we would have to use the sine transform instead of the cosine transform.

Take Fourier (cosine) transforms

$$y \mapsto (Cy)(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty y(x) \cos \lambda x \, dx.$$

C maps $L_2(0,\infty)$ unitarily onto $L_2(0,\infty; d\lambda)$. We claim that the transformed operator $A = CH_0C^*$ is multiplication by the polynomial

$$Q(i\lambda) \equiv \sum_{k=0}^{n} (-1)^k c_k (i\lambda)^{2k} = \sum_{k=0}^{n} c_k \lambda^{2k};$$

in particular,

$$D(A) = \{f \in L_2(0,\infty) : \lambda^{2n} f(\lambda) \in L_2(0,\infty)\}.$$

This is elementary and can in fact be deduced from the corresponding result for the operator $B = -d^2/dx^2$ on $L_2(0,\infty)$ with Neumann boundary conditions $y'(0) = 0$. Namely, $CBC^* = M_{\lambda^2}$, the operator of multiplication by λ^2 (see [11, p. 1388]), so $Cf(B)C^* = f(CBC^*) = M_{f(\lambda^2)}$. It is a standard fact about self-adjoint operators that if f is a polynomial, the operator $f(B)$ can be defined directly (not using the spectral theorem), and this operator $f(B)$ coincides with the one obtained from the functional calculus (see, e.g., [32]). Hence it is possible to construct $f(B)$ in the following way: Powers of B are (recursively) given by

$$D(B^{r+1}) = \{y \in D(B^r) : B^r y \in D(B)\}, \quad B^{r+1}y = B(B^r y),$$

so $D(f(B)) = D(B^n)$ if the degree of f is n. The action of $f(B)$ on elements from its domain is obvious. Using this, we verify that $f(B) = H_0$ if $f(\lambda) = \sum c_r \lambda^r$; thus $CH_0C^* = M_{f(\lambda^2)} = A$, as claimed.

We now list the spectral properties of A for later reference. Let C be the set of critical values of Q, that is, $C = \{Q(z) : z \in \mathbb{C}, Q'(z) = 0\}$. (This is consistent with the notation that will be used in Lemma 3.3 below.)

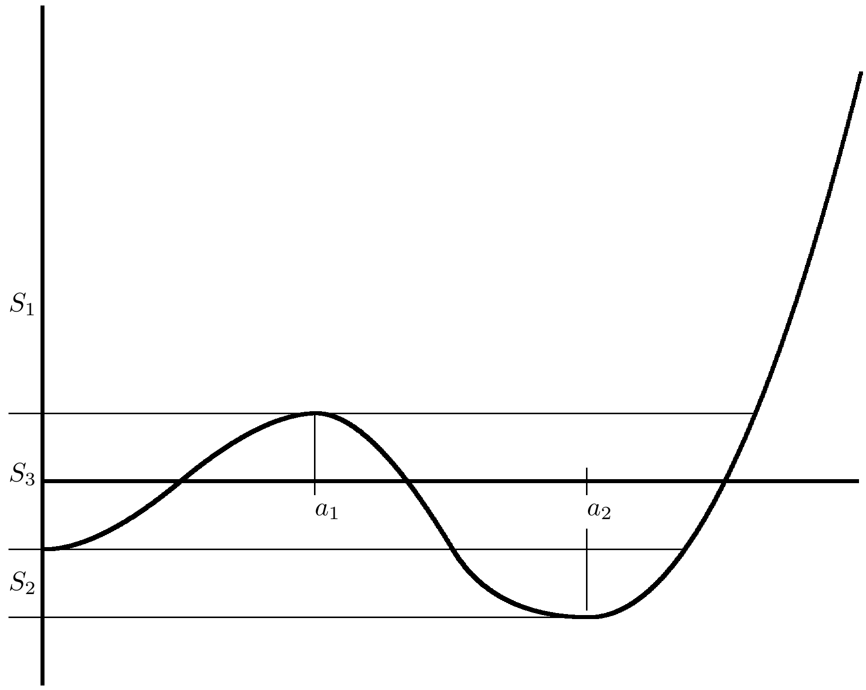
Proposition 3.1 *A has purely absolutely continuous spectrum and $\sigma(A) = \sigma_{ac}(A) = \{Q(i\lambda) : \lambda \geq 0\}$. Let*

$$S_m = \{E \in \sigma(A) \setminus C : \#\{\lambda \geq 0 : Q(i\lambda) = E\} = m\}.$$

Then $A\chi_{S_m}(A)$ is unitarily equivalent to the orthogonal sum of m copies of the operator of multiplication by λ in the space $L_2(S_m; d\lambda)$.

Remark. Roughly speaking, the last part says that S_m is the part of the spectrum on which A has exact multiplicity m. It is not really necessary to exclude the critical values of Q, but this makes things somewhat cleaner because then the sets S_m are finite unions of open intervals. The following proof together with the figure below should clarify things further.

Proof. Pick $0 = a_0 < a_1 < \cdots < a_{N-1} < a_N = \infty$ so that $Q(i\lambda)$ is strictly monotone on each interval $a_{i-1} < \lambda < a_i$. The subspaces $L_2(a_{i-1}, a_i)$ reduce A. By monotonicity, we can use a transformation of the independent variable to see that $A \restriction L_2(a_{i-1}, a_i)$ is unitarily equivalent to M_λ in $L_2(I_i; d\lambda)$, where $I_i = \{Q(i\lambda) : a_{i-1} < \lambda < a_i\}$. $\square$



We can now analyze self-adjoint realizations of τ_0 from (8) with arbitrary boundary conditions. Denote these operators by H_α, where the index α refers to the boundary condition at $x = 0$.

Proposition 3.2 *a) For every α, the absolutely continuous part of H_α is unitarily equivalent to the operator A from Proposition 3.1.*
b) $\sigma_{sc}(H_\alpha) = \emptyset$ for every α.
c) For every α, the point spectrum $\sigma_p(H_\alpha)$ is finite.

d) The operator with boundary conditions

$$y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0$$

has empty point spectrum.

Remark. In the framework of Hamiltonian systems, the boundary conditions of d) correspond to taking $\alpha_1 = 1, \alpha_2 = 0$. These boundary conditions will be particularly convenient in the discussion of the perturbed problem. Note also that part d) holds for the operator H_0 as well; indeed, this is part of what Proposition 3.1 states. Part b) will only be used for the boundary condition of part d), and part c) will not be used at all. They are just stated for completeness.

Proof. a) A change of boundary conditions is a finite rank perturbation of the resolvent, so the claim follows at once from general results of scattering theory (see, e.g., [22, Theorem XI.9]).

For the remaining parts, we work with the solutions of the DE $\tau_0 y = zy$. If the roots $\lambda_i = \lambda_i(z)$ of the characteristic polynomial $Q(\lambda) - z$ are distinct, then the functions $y_i(x, z) = e^{\lambda_i x}$ span the space of solutions (see [8] or any other ODE text). In the general case, denote the multiplicity of λ_i (as a zero of $Q(\lambda) - z$) by ν_i; then a basis consisting of solutions of the form $x^{m_i} e^{\lambda_i x}$ can be found, where m_i takes the values $0, 1, \ldots, \nu_i - 1$ (see again [8]).

To exploit these formulae, we also need some information about the zeros of polynomials, that is, about algebraic functions. A careful discussion of this subject is given in [18]. We extract the facts we need here and state them as

Lemma 3.3 *Let $\lambda_1, \ldots, \lambda_m$ be the zeros of a polynomial*

$$p(\lambda, z) = \lambda^m + a_{m-1}(z)\lambda^{m-1} + \cdots + a_0(z)$$

with coefficients which are themselves polynomial functions of a parameter z.

a) Every $\lambda_i(z)$ is a holomorphic function in any simply connected region in which there are no multiple roots.

b) The critical set

$$C = \{z \in \mathbb{C} : p(\cdot, z) \text{ has multiple roots}\}$$

is either finite or else $C = \mathbb{C}$. In the first case, if $z_0 \in C$, then in a neighborhood of z_0, the λ_i can be represented as Puiseux series $\sum_{n=0}^{\infty} c_n (z - z_0)^{n/p}$, with $1 \leq p \leq m$.

c) A similar statement holds at $z_0 = \infty$: There is $R > 0$, so that for $|z| > R$, the $\lambda_i(z)$ admit representations of the form $\sum_{n=-N}^{\infty} c_n z^{-n/p}$, with $N \in \mathbb{N}_0$ and $1 \leq p \leq m$.

Remarks. 1. In part a) (and also in Lemma 5.1 below) it is of course tacitly assumed that the λ_i are numbered appropriately. In fact, it is clear (either from part b) or by considering a simple example like $p(\lambda, z) = \lambda^2 - z$) that the labeling of the zeros must in general depend on the region. More precisely, if $D, \widetilde{D} \subset \mathbb{C}$ are as in part a), then it may be impossible to define the corresponding holomorphic functions $\lambda_i(z), \widetilde{\lambda}_i(z)$ so that $\lambda_i(z) = \widetilde{\lambda}_i(z)$ for every i and all $z \in D \cap \widetilde{D}$.

2. In part b), each Puiseux series gives precisely p different roots $\lambda_i(z)$ for $z \neq z_0$, but close to z_0. Note that at $z_0 = \infty$ (= part c)), the statement is weaker because the λ_i may have poles (as functions of $z^{-1/p}$).

3. Part a) will also play an important role in the discussion of Sect. 5.

We now resume the proof of Proposition 3.2. Lemma 3.3 applies to $p(\lambda, z) = (-1)^n c_n^{-1}(Q(\lambda) - z)$. So off the critical set, which can also be described here as $C = \{Q(\lambda) : Q'(\lambda) = 0\}$, the $\lambda_i(z)$ are distinct and holomorphic. It can be worked out now (using the above representation of the solutions of $\tau_0 y = zy$) that the M-function M_0 of the operator with the boundary conditions of part d) of the lemma can be holomorphically continued across any interval $(a, b) \subset \mathbb{R}$ that does not meet C. This fact and related problems will be discussed again in Sect. 7, to which we refer for more details. For arbitrary α, there is the transformation formula

$$M_\alpha(z) = (-\alpha_2 + \alpha_1 M_0(z))(\alpha_1 + \alpha_2 M_0(z))^{-1}.$$

Hence M_α can be meromorphically continued through (a, b) if $(a, b) \cap C = \emptyset$. The singular part of the spectral measure ρ_α is supported by

$$\{E \in \mathbb{R} : \limsup_{\epsilon \to 0+} \|M_\alpha(E + i\epsilon)\| = \infty\},$$

so $\sigma_{sc}(H_\alpha) = \emptyset$, as claimed.

c) We first prove that no $E \in \mathbb{R}$ can be an accumulation point of the set of eigenvalues. The easiest case is $E \in \mathbb{R} \setminus C$. Then the space of solutions of $\tau_0 y = Ey$ is spanned by the $y_i = e^{\lambda_i x}$ ($i = 1, \ldots, 2n$); by Lemma 3.3a), we can number so that the $\lambda_i(z)$ are holomorphic close to E. The subspace of square integrable solutions is spanned by the y_i with Re $\lambda_i < 0$. For non-real z, there are precisely n linearly independent square integrable solutions to $\tau_0 y = zy$. Moreover, if $\lambda_i(z)$ is continuous in some region contained in the upper (or lower) half-plane, then Re $\lambda_i(z)$ does not change sign there. Indeed, if such a sign change occured, then, by the mean value theorem, Re $\lambda_i(z_0) = 0$ for some non-real z_0. This is a contradiction, since Q evaluated at a purely imaginary number is real and thus cannot be equal to z_0.

So, by relabeling if necessary, we may assume that $y_1(\cdot, z), \ldots, y_n(\cdot, z) \in L_2$ for $z \in \mathbb{C}^+$ and close to E. It is then also true that for real z from a neighborhood of E, every L_2 solution of $\tau_0 y = zy$ belongs to $L(y_1(\cdot, z), \ldots, y_n(\cdot, z))$. Indeed, for $i = 1, \ldots, n$ and such z, the function $y_i(\cdot, z)$ may or may not be square integrable, but if $i \geq n + 1$, then Re $\lambda_i(z) \geq 0$ and hence no linear combination containing one of these $y_i(\cdot, z)$'s is in L_2.

So for z (close to E) to be an eigenvalue, it is necessary that the $n \times n$-matrix $(\alpha_1, \alpha_2)(Y_1(0, z), \ldots, Y_n(0, z))$ be singular. Now, the $2n$-vectors Y_i are obtained from the y_i by the transformation given in (3), with p_k replaced by c_k. However, since non-real z's cannot be eigenvalues, this matrix is certainly regular if $z \notin \mathbb{R}$. It is also holomorphic, so E cannot be an accumulation point of eigenvalues.

The case $E \in C$ is not much different: Here, Lemma 3.3b) shows that the λ_i depend holomorphically on $\zeta \equiv (z-E)^{1/N}$ for suitable $N \in \mathbb{N}$ (N is the least common multiple of the p's from the Puiseux series), and with some determination of the root. Now the reasoning of the preceding paragraph is still valid, with ζ taking the role of z.

Finally, a similar modification shows that eigenvalues do not accumulate at ∞, either. Namely, we use $\zeta \equiv z^{-1/N}$ with appropriate N as the variable this time and invoke Lemma 3.3c). In this way, we see that the crucial quantity, namely the determinant of $(\alpha_1, \alpha_2)(Y_1(0,z), \dots, Y_n(0,z))$, viewed as a function of ζ, has either a pole or a removable singularity at $\zeta = 0$. In either case, it is impossible that zeros accumulate at $\zeta = 0$. This concludes the proof of c).

d) Let y be solution of $\tau_0 y = Ey$; so y is of the form

$$y(x) = \sum_{r=1}^{m} e^{\lambda_r x} \sum_{s=0}^{\nu_r - 1} a_{rs} x^s, \tag{9}$$

where $\lambda_1, \dots, \lambda_m$ are distinct solutions of $Q(\lambda) = E$, and $\nu_1, \dots, \nu_m$ are the corresponding multiplicities. We can of course assume that for every $r \in \{1, \dots, m\}$, there is at least one $s \in \{0, \dots, \nu_r - 1\}$ with $a_{rs} \neq 0$. Now, if y is also square integrable, then we must have $\operatorname{Re} \lambda_r < 0$ for $r = 1, \dots, m$. Since Q is an even polynomial of order $2n$, it follows that $\sum_{r=1}^{m} \nu_r \leq n$. We see directly from (9) that y also solves

$$\prod_{r=1}^{m} \left(\frac{d}{dx} - \lambda_r \right)^{\nu_r} y = 0.$$

By the above remarks, this is an nth order equation, so if in addition y satisfies the boundary condition $y(0) = \cdots = y^{(n-1)}(0) = 0$, then $y \equiv 0$. $\square$

4 Transformation of the DE

In this section, we write the eigenvalue equation $\tau y = zy$ as a first order system and transform this system to be able to apply asymptotic integration techniques. Since this material is rather standard and may be found, for instance, in [12, Sect. 3.1], we will only sketch the main steps.

The solution vector Y was introduced in (3). Recall that Y solves (2), with A, B as in Sect. 2. We multiply from the left by $J^{-1} = -J$ to obtain the equation $Y' = CY$, where

$$\begin{aligned} C_{i,i+1} &= -C_{n+i+1,n+i} = 1 && (i = 1, \dots, n-1), \\ C_{n+i,i} &= p_{i-1} && (i = 2, \dots, n), \\ C_{n,2n} &= p_n^{-1},\ C_{n+1,1} = p_0 - zw, && \end{aligned}$$

and $C_{ij} = 0$ otherwise. To solve the equation $Y' = CY$ asymptotically, we first split off the L_1 terms. That is, we write $C = D + R$, where the non-zero entries of R are

$$R_{n+i,i} = r_{i-1} \qquad\qquad (i = 2, \dots, n),$$
$$R_{n,2n} = r_n, \quad R_{n+1,1} = r_0 - zr.$$

It is clear that the hypotheses of Theorem 1.1 ensure that $\|R(x,z)\| \le \rho(x) \in L_1(0,\infty)$, locally uniformly in $z \in \mathbb{C}$. Next, we want to diagonalize $D \equiv C - R$. The characteristic polynomial of this matrix is

$$(-1)^n q_n^{-1}(x) \left(\sum_{k=0}^{n} (-1)^k q_k(x) \lambda^{2k} - zv(x) \right).$$

So, letting

$$P(\lambda; x) \equiv \sum_{k=0}^{n} (-1)^k q_k(x) \lambda^{2k}, \tag{10}$$

we get the eigenvalues of $D = D(x,z)$ as the solutions of $P(\lambda; x) = zv(x)$. Now, let us assume that these roots $\lambda_1(x,z), \dots, \lambda_{2n}(x,z)$ of $P - zv$ are distinct for some fixed z and for all large enough x. (It is a simple consequence of Lemma 3.3 that this can fail only for finitely many z's – we will see this in the next section.) We can then diagonalize $D(x,z)$; a diagonalizing transformation T is given by $T = ((K^{-1/2}f)(\lambda_1), \dots, (K^{-1/2}f)(\lambda_{2n}))$, where the column vectors f have the following components:

$$f_i(\lambda) = \lambda^{i-1}, \; f_{n+i}(\lambda) = \sum_{k=i}^{n} (-1)^{k+i} q_k \lambda^{2k-i} \quad (i = 1, \dots, n).$$

The numbers K are defined by

$$K(\lambda) = \frac{\partial P(\lambda; x)}{\partial \lambda}. \tag{11}$$

In particular, $K(\lambda_i) \ne 0$ since the λ_i are simple roots of $P - zv$ by our assumption above. There is also a similar formula for the inverse T^{-1}. Namely, the kth row of T^{-1} is the vector $K^{-1/2} g(\lambda_k)$, where the entries of the row vector g are as follows:

$$g_i(\lambda) = \sum_{k=i}^{n} (-1)^k q_k \lambda^{2k-i}, \; g_{n+i}(\lambda) = (-1)^i \lambda^{i-1}, \quad (i = 1, \dots, n).$$

The reader should keep in mind that both the λ_k's and the various quantities introduced above depend on x and z; this was largely suppressed in the notation.

Introduce $U = T^{-1}Y$; then U solves

$$U' = (\Lambda + S + T^{-1}RT)U. \tag{12}$$

Here $\Lambda_{ij} = (T^{-1}CT)_{ij} = \delta_{ij}\lambda_i$ and $S = -T^{-1}\partial T/\partial x$. In fact, the matrix elements of S can be calculated explicitly (cf. [12, Lemma 3.1.1]): We have $S_{ii} = 0$, and if $i \neq j$, then

$$S_{ij} = (\lambda_j - \lambda_i)^{-1} \left(K(\lambda_i)K(\lambda_j)\right)^{-1/2} \left[\sum_{k=0}^{n} (-1)^k q_k'(\lambda_i\lambda_j)^k - zv' \right]. \tag{13}$$

We now want to solve (12) with the help of Theorem A.1. Control of the perturbations $S, T^{-1}RT$ will be easy; the main issue is the verification of what we called the weak uniform dichotomy condition (see the Appendix). That is, we need to study the differences Re $(\lambda_i - \lambda_j)$, and we need detailed information on the z dependence of these quantitites. Therefore, we now turn to investigating the λ_i.

5 The dichotomy condition

The key technical result that makes asymptotic integration techniques applicable is the following statement on the solutions $\lambda_i(x, z)$ of $P(\lambda; x) = zv(x)$. It will be convenient to introduce, for easier reference, the (rectangular) sets

$$S_\delta(E) = \{z \in \mathbb{C} : |\text{Re } z - E| \leq \delta, 0 \leq \text{Im } z \leq \delta\}. \tag{14}$$

Lemma 5.1 *There is an exceptional set $\mathcal{E} \subset \mathbb{R}$ which has only finitely many accumulation points such that the following holds. If $E_0 \in \mathbb{R} \setminus \mathcal{E}$, there are $\delta, x_0 > 0$, so that:*

1. *For fixed $x \geq x_0$, $\lambda_i(x, z)$ is a holomorphic function of z in $|z - E_0| < \delta$. The limits $\lim_{x\to\infty} \lambda_i(x, z)$ exist, uniformly in $|z - E_0| < \delta$. Moreover, $\lambda_i(x, z) \neq \lambda_j(x, z)$ if $i \neq j$, and in fact*

   $$\inf\{|\lambda_i(x, z) - \lambda_j(x, z)| : i \neq j, x \geq x_0, |z - E_0| < \delta\} > 0.$$

2. Re $\lambda_i(x, z)$ *does not change sign if $x \geq x_0$ and $z \in S_\delta(E_0)$. Moreover, either*
   (i) Re $\lambda_i(x, E) = 0$ *for all $x \geq x_0$, $E \in S_\delta(E_0) \cap \mathbb{R}$, and* $|\text{Re } \lambda_i(x, z)| \geq c \,\text{Im } z$ *($c > 0$) for $x \geq x_0, z \in S_\delta(E_0)$, or*
   (ii) $|\text{Re } \lambda_i(x, z)| \geq c > 0$ *for all $x \geq x_0, z \in S_\delta(E_0)$.*

3. Re $(\lambda_i(x, z) - \lambda_j(x, z))$ *also has constant sign in $x \geq x_0, z \in S_\delta(E_0)$. Moreover, either*
   (i) Re $(\lambda_i(x, E) - \lambda_j(x, E)) = 0$ *for all $x \geq x_0$, $E \in \mathbb{R} \cap S_\delta(E_0)$, and* $|\text{Re } (\lambda_i(x, z) - \lambda_j(x, z))| \geq c \,\text{Im } z$ *($c > 0$) for $x \geq x_0, z \in S_\delta(E_0)$, or*
   (ii) $|\text{Re } (\lambda_i(x, z) - \lambda_j(x, z)| \geq c > 0$ *for all $x \geq x_0, z \in S_\delta(E_0)$.*

Remarks. 1. Since these statements admittedly look somewhat technical, we would like to comment on their significance. As we will see later, the asymptotic behavior of the solutions of $\tau y = zy$ is governed by the exponential factors

$\exp\left(\int_{x_0}^{x} \lambda_i(t,z)\,dt\right)$. Now part 1. of the Lemma establishes regularity properties of the λ_i, and 2. tells us that the exponential factors are either not very big or not very small throughout. Finally, the crucial last part helps to control the ratio of two such factors. Basically, it says that the worst that can happen (namely, case (i)) is that the ratio is ≈ 1 for real z but tends to 0 (or infinity, respectively) off the real axis. But this change of behavior takes place in a very controlled way, and this is exactly the type of situation we can deal with using Theorem A.1.

2. Unfortunately, there will be three different (though closely related) polynomials associated with (1), each playing a slightly different role. The reader should keep in mind the following: $P(\lambda; x)$, defined in (10), determines the eigenvalues of the matrix D in $Y' = (D+R)Y$ and thus the asymptotic behavior of the solutions to this equation. Letting $x \to \infty$, we obtain the limiting polynomial $Q(\lambda) = \sum (-1)^k c_k \lambda^{2k}$; in particular, the solutions of $Q(\lambda) = z$ are the limits of the eigenvalues $\lambda_i(x,z)$ from above. Finally, $Q(i\lambda)$, which is the polynomial introduced in Sect. 3, describes the spectral properties. We will often use the fact that all three polynomials are even and have real coefficients. So, if λ solves an equation of the form $Q(\lambda) = z$ (say), then so does $-\lambda$. Moreover, if $z \in \mathbb{R}$, $\overline{\lambda}$ and $-\overline{\lambda}$ are solutions, too.

3. We can give a more explicit description of the accumulation points of $\mathcal{E}$. The proof below will show that the only possible accumulation points are the real values of the limiting polynomial Q at its critical points.

Proof of Lemma 5.1. Basically, it will be sufficient to study the solutions $\lambda_i(z)$ of the limiting equation $Q(\lambda) = z$. Here, as explained in Remark 2,

$$Q(\lambda) = \sum_{k=0}^{n} (-1)^k c_k \lambda^{2k} \tag{15}$$

and $Q(\lambda) = \lim_{x\to\infty} P(\lambda; x)$. To be able to apply Lemma 3.3 to the polynomials we are concerned with here (namely, $Q(\lambda) - z$ and $P(\lambda; x) - zv(x)$), we have to normalize to make sure that the leading coefficient is equal to one. However, to keep the notation transparent, we will not carry this out explicitly in the sequel.

Let $w_1 < w_2 < \cdots < w_m$ be the elements of $C \cap \mathbb{R}$, where C is the critical set of $Q(\lambda) - z$, as introduced in Lemma 3.3b). So, more explicitly, the w_k are the real numbers among the values $Q(\mu_i)$, where the μ_i are the zeros of Q'. Also, put $w_0 = -\infty, w_{m+1} = \infty$. Then, if $z \notin \{w_k\}$ is sufficiently close to the real axis, the $\lambda_i(z)$ are distinct. Moreover, these functions are holomorphic there by Lemma 3.3a).

We now let, for $i \neq j \in \{1, 2, \dots, 2n\}$,

$$\begin{aligned}
S_i &= \left\{ E \in \bigcup (w_k, w_{k+1}) : \text{Im } \lambda_i(E) \neq 0, \text{Im } \lambda_i'(E) = 0 \right\}, \\
T_{ij} &= \left\{ E \in \bigcup (w_k, w_{k+1}) : \text{Re } \lambda_i(E) = \text{Re } \lambda_j(E), \lambda_i(E) \neq \overline{\lambda_j(E)} \right\}, \\
U_{ij} &= \left\{ E \in \bigcup (w_k, w_{k+1}) : \lambda_i'(E) = \lambda_j'(E) \right\}
\end{aligned}$$

and define

$$\mathcal{E} = \{w_1, \dots, w_m\} \cup \bigcup S_i \cup \bigcup T_{ij} \cup \bigcup U_{ij}.$$

We first show that the S_i, T_{ij}, U_{ij} cannot have accumulation points different from the w_k's. In other words, a "generic" choice of E removes unnecessary degeneracies: There is no reason for $\lambda_i'(E)$ to be real unless $\lambda_i(E)$ itself is real, and similarly for the other sets.

The discussion is easier for S_i, so we start with these sets. Note that if $E_0 \in S_i$, then by the first condition, there is an index $j \neq i$ so that $\lambda_j(E_0) = \overline{\lambda_i(E_0)}$. The λ's are distinct at E_0, so by continuity, if $E \in \mathbb{R}$ is sufficiently close to E_0, we still have $\lambda_j(E) = \overline{\lambda_i(E)}$. Now assume, to obtain a contradiction, that points of S_i accumulate at some point inside some (w_k, w_{k+1}). Then, with j as above, we have that $\lambda_j'(E) = \overline{\lambda_i'(E)} = \lambda_i'(E)$ on a set with an accumulation point in (w_k, w_{k+1}), and hence by analyticity for all $E \in (w_k, w_{k+1})$. Differentiating the equation $Q(\lambda_i(E)) = E$, we get $Q'(\lambda_i(E))\lambda_i'(E) = 1$, thus $Q'(\lambda_i(E)) = Q'(\lambda_j(E))$ for all $E \in (w_k, w_{k+1})$. We can differentiate again to deduce that also $Q''(\lambda_i(E)) = Q''(\lambda_j(E))$. Continuing in this way, we finally arrive at $Q^{(2n-1)}(\lambda_i(E)) = Q^{(2n-1)}(\lambda_j(E))$, which is the desired contradiction since $Q^{(2n-1)}(\lambda) = (2n)!\,(-1)^n c_n \lambda$ and $\lambda_j = \overline{\lambda_i} \neq \lambda_i$.

Consider now T_{ij}, and assume again that this set has an accumulation point in (w_k, w_{k+1}). Then $\text{Re } \lambda_i(E) = \text{Re } \lambda_j(E)$ for all $E \in (w_k, w_{k+1})$ by analyticity. Write, for $m = i, j$, $\lambda_m(E) = \mu(E) + i\nu_m(E)$, with $\mu(E), \nu_m(E) \in \mathbb{R}$ for $E \in (w_k, w_{k+1})$. Then $\mu(E) - i\nu_m(E)$ are also solutions of $Q(\lambda) = E$. The functions μ, ν_m can be holomorphically continued along any curve that avoids the (finite) critical set C from Lemma 3.3b). To see this, simply observe that for $z \in (w_k, w_{k+1})$, we have

$$\mu(z) = \frac{1}{2}\left(\lambda_m(z) + \overline{\lambda_m(\overline{z})}\right), \quad \nu_m(z) = \frac{1}{2i}\left(\lambda_m(z) - \overline{\lambda_m(\overline{z})}\right),$$

and apply Lemma 3.3a). Furthermore, these continuations $\mu(z) \pm i\nu_m(z)$ still solve $Q(\mu(z) \pm i\nu_m(z)) - z = 0$ because the left-hand side of this equation is a holomorphic continuation of the zero function and thus indeed equal to zero for all z for which it has been defined. (However, $\mu(z), \nu_m(z)$ will not, in general, be real, even if $z \in \mathbb{R} \setminus (w_k, w_{k+1})$. This reflects the fact that the continuation of the real part (say) of the holomorphic function λ_m is not necessarily equal to the real part of the continuation.)

We can now use this continuation procedure to define the μ, ν_m for big positive values of z. For such z, write $z = k^{2n}$ with $k > 0$ (and large). Then an elementary discussion (see also [18]) shows that the solutions of $Q(\lambda) = k^{2n}$ are of the asymptotic form

$$\lambda_r(k^{2n}) = ake^{i\pi r/n} + O(1) \quad (r = 1, 2, \dots, 2n) \tag{16}$$

as $k \to \infty$, with $a = ic_n^{-1/2n}$ (with some fixed choice of the root). In particular, the solutions constructed above by holomorphic continuation must be of the form (16); that is, there must be indices $r_1, \dots, r_4 \in \{1, 2, \dots, 2n\}$ so that

$$\mu(k^{2n}) + i\nu_i(k^{2n}) = ake^{i\pi r_1/n} + O(1) \quad (k \to \infty),$$

and r_2, r_3, r_4 correspond in the same way to $\mu - i\nu_i$, $\mu + i\nu_j$, and $\mu - i\nu_j$, respectively. By adding the asymptotic representations of $\mu \pm i\nu_m$ for $m = i, j$ and equating the results, we obtain

$$ak\left(e^{i\pi r_1/n} + e^{i\pi r_2/n}\right) = ak\left(e^{i\pi r_3/n} + e^{i\pi r_4/n}\right) + O(1).$$

Letting $k \to \infty$, we deduce that

$$e^{i\pi r_1/n} + e^{i\pi r_2/n} = e^{i\pi r_3/n} + e^{i\pi r_4/n}.$$

There are three possibilities to satisfy this equation, namely $r_1 = r_3, r_2 = r_4$ or $r_1 = r_4, r_2 = r_3$ or both sides equal zero. In the first case, it follows that $\mu + i\nu_i$ and $\mu + i\nu_j$ differ only by a term of order $O(1)$. But (16) clearly implies that if $r \neq s$, then $|\lambda_r - \lambda_s| \geq ck$ as $k \to \infty$, with $c > 0$, so in fact $\mu + i\nu_i = \mu + i\nu_j$ for large k. Now these two functions were holomorphic continuations of λ_i and λ_j, respectively, thus also $\lambda_i(E) = \lambda_j(E)$ for all $E \in (w_k, w_{k+1})$. This is a contradiction since $i \neq j$.

If $r_1 = r_4$ (and $r_2 = r_3$, but this will not be used), then similar reasoning shows that $\lambda_i(E) = \overline{\lambda_j(E)}$ on (w_k, w_{k+1}), which is also impossible in view of the definition of T_{ij}. Finally, in the third case it follows that $\mu \equiv 0$, in contradiction to Re $\lambda_i \neq 0$.

The proof that the U_{ij} have no accumulation points outside $\{w_k\}$ is analogous, but simpler.

Summing up, we have shown that the accumulation points of $\mathcal{E}$ are contained in $\{w_1, \dots, w_m\}$. It remains to check assertions 1.-3. of the Lemma for fixed $E_0 \notin \mathcal{E}$ and δ, x_0 sufficiently small and big, respectively. It may be necessary to change the values of δ, x_0 several times in the following arguments, but we will not mention this explicitly.

With this understanding, statement 1. is a consequence of Lemma 3.3a). In particular, we use the fact that the λ_i depend continuously on the coefficients $q_k(x), v(x)$ of the polynomial $P - zv$ which tend to limits as $x \to \infty$. For later use, note that we indeed have much more regularity: The λ_i are (multiple) power series in the q_k, v and z.

Moving on to 2., we observe that if Re $\lambda_i(E_0) \neq 0$, then, again by continuity, (ii) holds. On the other hand, we will prove below that if Re $\lambda_i(E_0) = 0$, then Re $\lambda_i(x, E) = 0$ for all $x \geq x_0, E \in \mathbb{R} \cap S_\delta(E_0)$. Accepting this for the moment, we then see that Re $(d\lambda_i/dE)(x, E) = 0$. But $(d\lambda_i/dE)(x, E) \neq 0$, so the Taylor expansion

$$\lambda_i(x, E + i\epsilon) = \lambda_i(x, E) + i\epsilon\frac{d\lambda_i}{dE}(x, E) + O(\epsilon^2)$$

shows that (i) holds. ($d\lambda_i/dE$ is bounded away from zero and the control on the error term $O(\epsilon^2)$ is uniform in x, E.)

We now prove the claim made above. So suppose that Re $\lambda_j(E_0) = 0$ for some j and some $E_0 \in \mathbb{R} \setminus \mathcal{E}$. Think of $Q = Q(\lambda; c_0, \ldots, c_n)$ as a function of λ and of the coefficients c_i. In this notation, $P(\lambda; x) = Q(\lambda; q_0(x), \ldots, q_n(x))$. Recall that the polynomial $Q(i\mu)$ has real coefficients. Finally, $(dQ/d\mu)(i\mu) \neq 0$ if μ solves $Q(i\mu) = E_0$. Indeed, the values of E_0 where this fails are precisely the w_k's. So the implicit function theorem (for real valued functions!) applies: In a neighborhood of $(E_0, c_0, \ldots, c_n, 1)$, the equation $Q(i\mu; d_0, \ldots, d_n) = Ew$ has a real valued, continuous solution $\mu = \mu(E; d_0, \ldots, d_n, w)$ with $\mu(E_0; c_0, \ldots, c_n, 1) = -i\lambda_j(E_0)$. In particular, if x is sufficiently large, we can take $d_i = q_i(x), w = v(x)$, and we obtain a real solution μ of $P(i\mu; x) = Ev(x)$. Since the λ_i are distinct, this solution μ is of course nothing but $-i\lambda_j(x, E)$ itself. So $-i\lambda_j(x, E) \in \mathbb{R}$ for E sufficiently close to E_0 and $x \geq x_0$.

Finally, we come to assertion 3. If i, j are such that Re $\lambda_i(E_0) \neq$ Re $\lambda_j(E_0)$, then, again, a straightforward continuity argument shows that 3.(ii) holds. So assume that Re $\lambda_i(E_0) =$ Re $\lambda_j(E_0)$. The point E_0 was not in $\mathcal{E}$, so by the definition of this set, we must have either $\lambda_j(E_0) = \overline{\lambda_i(E_0)}$ and Im $\lambda_i'(E_0) \neq 0$ or Re $\lambda_i(E_0) =$ Re $\lambda_j(E_0) = 0$ and $\lambda_i'(E_0) \neq \lambda_j'(E_0)$. In the first case, by continuity, these two conditions will also hold for $\lambda_i(x, E)$ and $\lambda_j(x, E)$, provided $E \in \mathbb{R}$ is close to E_0 and x is sufficiently large. Hence a Taylor expansion gives

$$\text{Re } \left(\lambda_i(x, E + i\epsilon) - \lambda_j(x, E + i\epsilon)\right) = -2\epsilon \, \text{Im} \, \frac{d\lambda_i}{dE}(x, E) + O(\epsilon^2). \tag{17}$$

The constant implicit in $O(\epsilon^2)$ can be estimated uniformly with respect to x, E. Moreover, Im $(d\lambda_i/dE)(x, E)$ is bounded away from zero. We thus see from (17) that condition 3.(i) of the Lemma holds. The proof in the second case is similar. □

Lemma 5.1 gives us enough information to verify the weak uniform dichotomy condition of Theorem A.1 on the sets $S_\delta(E_0)$ from the Lemma (assuming, of course, $E_0 \notin \mathcal{E}$). We leave the details of this verification to the reader and just give the result: If, for some pair $i \neq j$, statement 3.(i) of Lemma 5.1 holds, then, referring to the list of conditions given in the Appendix, 4. or 5. holds, depending on the sign of Re $(\lambda_i - \lambda_j)$ in 3.(i). In the case of 3.(ii), we have 1. or 2., again depending on the sign of Re $(\lambda_i - \lambda_j)$. The last possibility allowed by Theorem A.1, namely condition 3., does not occur here.

6 Asymptotic integration

Having discussed the dichotomy condition, we now turn to the required uniform estimates on the remainders $S, T^{-1}RT$ from (12). As usual, z runs over some $S_\delta(E_0)$, and, according to Theorem A.1, our goal is to establish estimates of the form $\|S(x, z)\|, \|T^{-1}(x, z)R(x, z)T(x, z)\| \leq \rho(x)$ for $x \geq x_0$ with $\rho \in L_1(x_0, \infty)$. It follows from Lemma 5.1, part 1., and the formulae of Sect. 3 that $\|T\|$ and $\|T^{-1}\|$ are uniformly bounded. We have already observed that R itself satisfies an estimate of the above form, thus also $\|T^{-1}RT\| \leq \|T^{-1}\| \, \|R\| \, \|T\| \leq \rho$, as desired. To bound S, we use similar arguments together with the hypotheses of Theorem 1.1 and (13). Finally, it is also easy to see that Λ, S, T, R are continuous functions of z for fixed $x \geq x_0$ and that $\|\Lambda(x,z)\| \leq f(x) \in L_{1,loc}$. This concludes the verification of the hypotheses of Theorem A.1.

Applying this result to (12), we get solutions of the form

$$U_k(x,z) = (e_k + r_k(x,z)) \exp\left(\int_{x_0}^{x} \lambda_k(t,z)\, dt \right). \tag{18}$$

Here, k runs over $\{1, 2, \ldots, 2n\}$, e_k denotes the kth unit vector, and r_k is jointly continuous in x, z and tends to zero as $x \to \infty$, uniformly in z. Of course, we were originally interested in solutions of $\tau y = zy$, so we transform back to $Y = TU$. For later reference, we formulate the results of our discussion as a theorem:

Theorem 6.1 *Assume the hypotheses of Theorem 1.1, and let $\mathcal{E}$ be the set from Lemma 5.1. Then, for every $E \in \mathbb{R} \setminus \mathcal{E}$, there is a $\delta > 0$, so that for all $z \in S_\delta(E)$ (this set was defined in* (14)*), the following (vector) functions Y form a basis of the space of solutions of* (2)*:*

$$Y_k(x,z) = (\widetilde{f}(\lambda_k(z)) + r_k(x,z)) \exp\left(\int_{x_0}^{x} \lambda_k(t,z)\, dt \right).$$

Here, the $\lambda_k(z)$ are the zeros of the limiting polynomial $Q(\cdot) - z$ (Q was defined in (15)*), while $\lambda_k(x,z)$ denotes the corresponding zero of $P(\cdot; x) - zv(x)$, with P defined in* (10)*. Furthermore,*

$$\widetilde{f}_i(\lambda) = \lambda^{i-1},\ \widetilde{f}_{n+i}(\lambda) = \sum_{k=i}^{n} (-1)^{k+i} c_k \lambda^{2k-i} \quad (i = 1, \ldots, n).$$

The remainders r_k are (jointly) continuous and tend to zero as $x \to \infty$, uniformly in $z \in S_\delta(E)$.

Remark. By (3), the first components of the Y_k, for $k = 1, \ldots, 2n$, give a basis of the solution space of the original equation $\tau y = zy$. Moreover, (3) obviously also yields asymptotic formulae for the (quasi-)derivatives of these solutions y.

Sketch of proof. We just have to carry out the transformation $Y = TU$. Of course, we then get certain new combinations of the original remainders r_k from (18), but these new combinations still have the same properties. This follows from the fact that all matrix elements of $T(x,z)$ tend to limits as $x \to \infty$, uniformly in z. By the same token, we can replace the vectors $f(\lambda_k(x,z))$ by their limits $\widetilde{f}(\lambda_k(z))$. $\square$

7 Spectral analysis

In this section, we want to use Theorem 6.1 to prove Theorem 1.1. We need two criteria from [25] (namely, Theorems 5.1 and 6.3 of that paper); we consider general differential operators of the form (1) which are regular at $x = 0$ and have deficiency indices (n, n) (minimal possible).

Theorem 7.1 ([25]) *Suppose that the equation $\tau y = Ey$ has r linearly independent solutions $y \in L_2(0, \infty; w\,dx)$ for all E in some Borel set $S \subset \mathbb{R}$. Then for all self-adjoint realizations of τ, the multiplicity of the continuous spectrum of the part of the operator in S is $\leq n - r$.*

This is a consequence of

Lemma 7.2 ([25]) *Suppose there are r linearly independent solutions $y_1, \ldots, y_r \in L_2(0, \infty; w\,dx)$ of $\tau y = Ey$ and $D(H_\alpha) \cap L(y_1, \ldots, y_r) = \{0\}$ (in other words, no nontrivial linear combination of the y_i's satisfies the boundary condition). Then there are r linearly independent vectors $v_1, \ldots, v_r \in \mathbb{C}^n$ such that $v_i^* \mathrm{Im}\ M_\alpha(E + i\epsilon) v_i = O(\epsilon)$ as $\epsilon \to 0+$.*

We remark parenthetically that it seems to be an interesting open problem to determine to what extent a converse to Lemma 7.2 holds. If $\lim_{\epsilon \to 0+} M_\alpha(E+i\epsilon)$ exists, then every v_i as above yields (in the Hamiltonian system formulation) an $L_{2,A}$ solution f_i by letting

$$f_i(x, E) \equiv \left(U_\alpha(x, E) + V_\alpha(x, E) \lim_{\epsilon \to 0+} M_\alpha(E + i\epsilon) \right) v_i,$$

but it is not clear what happens in the general case.

The second criterion alluded to above is

Theorem 7.3 ([25]) *Fix a boundary condition α and $a \geq 0$, and let $M(z)$ be the M-function of the operator on $L_2(a, \infty; w\,dx)$ with boundary condition α at $x = a$. Let $S \subset \mathbb{R}$ be a Borel set and $r \in \{0, 1, \ldots, n\}$, such that the following holds for every $E \in S$:*

1. $\limsup_{\epsilon \to 0+} \|M(E + i\epsilon)\| < \infty$.
2. *There are r linearly independent solutions in $L_2(a, \infty; w\,dx)$ to $\tau y = Ey$, but no L_2 solution satisfies the boundary condition at $x = a$.*
3. $\liminf_{\epsilon \to 0+} w^* \mathrm{Im}\ M(E + i\epsilon) w > 0$ *for all $w \in \mathbb{C}^n \setminus L(v_1(E), \ldots, v_r(E))$, where $v_1(E), \ldots, v_r(E) \in \mathbb{C}^n$ are linearly independent vectors with $v_i^* \mathrm{Im}\ M(E + i\epsilon) v_i = O(\epsilon)$. (The existence of such v_i's follows from assumption 2. together with Lemma 7.2.)*

Then for all boundary conditions β, the singular continuous part of the spectral measure ρ_β of the operator on $L_2(0, \infty; w\,dx)$ with boundary condition β at $x = 0$ gives zero weight to S, i.e. $\rho_\beta^{(sc)}(S) = 0$.

Actually, we could also allow r to depend on E, but this is not really a more general result because we can always decompose S according to the value of r. Roughly speaking, Theorem 7.3 says that if the limiting behavior of a particular M corresponds to either L_2 solutions ($\mathrm{Im}\ M(E + i\epsilon) \sim \epsilon$) or absolutely continuous spectrum ($\mathrm{Im}\ M(E) > 0$), then there can never be singular continuous spectrum, no matter what boundary condition and left endpoint are chosen.

The additional hypothesis 1. is essential: in general, absence of singular continuous spectrum is not a property that is stable under a change of boundary conditions and/or left endpoint.

We now turn to proving Theorem 1.1. Large parts of the argument will depend on the simple but important observation that $Y_k(\cdot, z) \in L_{2,A}$ precisely if Re $\lambda_k(z) < 0$ (where the Y_k are the solutions from Theorem 6.1). Indeed, if Re $\lambda_k(z) = 0$, then z must be real because Im $Q(\lambda) = 0$ for purely imaginary λ. But then Re $\lambda_k(x, z) = 0$ for all sufficiently large x by Lemma 5.1, part 2., and hence $Y_k \notin L_{2,A}$. This establishes our claim since the assertion is obvious in all other cases. Moreover, it is also easy to see that a linear combination of the Y_k's is square integrable if and only if Re $\lambda_k < 0$ for every k occuring in this linear combination. Also, we showed already (see the proof of Proposition 3.2) that Re $\lambda_k(z)$ is non-zero and does not change sign for z from the upper half-plane.

We can thus label the λ's in such a way that Re $\lambda_k(z) < 0$ for $k = 1, \ldots, n$ if Im $z > 0$, and if $z \in \mathbb{R}$, we have Re $\lambda_k(z) = 0$ for $k = 1, \ldots, r$ and Re $\lambda_k(z) < 0$ for $k = r+1, \ldots, n$. Here, $r \in \{0, \ldots, n\}$ depends on z but is locally constant as long as the exceptional set $\mathcal{E}$ from Lemma 5.1 is avoided. Finally, the remaining n roots are $\lambda_{n+i} = -\lambda_i$ (since Q is even, zero can never be a simple root of $Q - z = 0$).

We now also see that there are precisely n linearly independent solutions $Y(\cdot, z) \in L_{2,A}$ if Im $z > 0$; the corresponding space is spanned by $Y_1, \ldots, Y_n$. In particular, the deficiency indices are (n, n), and the theory of Sect. 2 applies.

Having made these preliminary remarks, let us now first discuss semiboundedness and the location of σ_{ess}. (Regarding σ_{ess}, the reader should recall that by the results of Sect. 3, the assertion of Theorem 1.1 is that σ_{ess} is equal to the range of the polynomial $Q(i\lambda)$. Also, since the leading coefficient of $Q(i\lambda)$ is positive, $\{Q(i\lambda) : \lambda \in \mathbb{R}\} = [\min Q(i\lambda), \infty)$.)

We will need the following result from what is usually called "oscillation theory". We consider differential operators generated by expressions of the type (1) on the interval $x \in [0, \infty)$:

Theorem 7.4 *Suppose that the range of the spectral projection on $(-\infty, \lambda_0)$ is of infinite dimension. Fix $E \geq \lambda_0$. Then, for arbitrarily large a and L, there exists $b \geq a + L$ so that the DE $\tau y = Ey$ has a non-trivial solution y with $y^{(k)}(a) = y^{(k)}(b) = 0$ for $k = 0, 1, \ldots, n-1$.*

Although this is only a slight variation of statements presented in [14, Sect. 2.12], we give the full proof below. It is based on

Lemma 7.5 *Let $E_0(b) \leq E_1(b) \leq \cdots$ be the eigenvalues of the self-adjoint realization of τ on $[a, b]$ with boundary conditions $y^{(k)}(a) = y^{(k)}(b) = 0$ ($k = 0, 1 \ldots, n-1$). Then $E_n(b)$ is a continuous, decreasing function of $b \in (a, \infty)$.*

The dependence of the eigenvalues on the parameters of the problem has been studied in much greater generality by Kwong, Wu, and Zettl in [19]. It is not completely clear, however, if this lemma can be derived from their results. In the special case we are interested in here, the following routine argument based on the min-max principle seems more appropriate anyway.

Proof. The operators H_b on finite intervals $[a,b]$ are semibounded below (see [21]). So, the min-max principle for quadratic forms [23, Theorem XIII.2] applies, and we have

$$E_n(b) = \sup_{g_1,\dots,g_n\in L_2(a,b)} \inf \{\langle f, H_b f\rangle : f \in Q(H_b), \|f\| = 1, f \perp g_1,\dots,g_n\}.$$

Note that by $L_2(a,b)$, we really mean the weighted space $L_2(a,b;w(x)\,dx)$, but w is fixed, so this has not been made explicit in the notation. The form domain $Q(H_b)$ is given by

$$Q(H_b) = \{f \in L_2(a,b) : f, f', \dots, f^{(n-1)} \text{ absolutely continuous,} \\ \int_a^b p_n|f^{(n)}|^2 < \infty, f^{(k)}(a) = f^{(k)}(b) = 0 \text{ for } k = 0,1,\dots,n-1\}.$$

For $b' > b$, we can also interpret $H_b = H_b \oplus 0$ as an operator in the space $L_2(a,b') = L_2(a,b) \oplus L_2(b,b')$, and correspondingly for the quadratic forms. With this understanding $Q(H_b) \oplus \{0\} \subset Q(H_{b'})$ and $\langle f, H_{b'}f\rangle = \langle f, H_b f\rangle$ for all $f \in Q(H_b)$. It now follows that

$$\begin{aligned} E_n(b') &= \sup_{g_i\in L_2(a,b')} \inf \{\langle f, H_{b'} f\rangle : f \in Q(H_{b'}), \|f\| = 1, f \perp g_1,\dots,g_n\} \\ &\le \sup_{g_i\in L_2(a,b')} \inf \{\langle f, H_{b'} f\rangle : f \in Q(H_b)\oplus\{0\}, \|f\| = 1, f \perp g_1,\dots,g_n\} \\ &= \sup_{g_i\in L_2(a,b)} \inf \{\langle f, H_b f\rangle : f \in Q(H_b), \|f\| = 1, f \perp g_1,\dots,g_n\} = E_n(b). \end{aligned}$$

The continuity of E_n is a consequence of the norm convergence of the resolvents as $b' \to b$; for example, one can use a min-max principle for the resolvents (see also [19]). □

Proof of Theorem 7.4. Consider τ on the interval $[a,b']$ with boundary conditions $y^{(k)}(a) = y^{(k)}(b') = 0$ $(k = 0,\dots,n-1)$. Put $b' = a + L$ and pick n (large enough) so that $E_n(b') \ge E$. As $b' \to \infty$, the operators on $[a,b']$ converge to the operator on $[a,\infty)$ in the sense of strong resolvent convergence. So, by assumption and Lemma 7.5, $E_n(b') < \lambda_0$ for sufficiently large b'. By continuity, there exists $b \ge a + L$ so that $E_n(b) = E$, and the sought y can be taken as an eigenfunction associated with this eigenvalue. □

We will now prove that the assertion of Theorem 7.4 cannot hold if $E \notin \{Q(ix) : x \in \mathbb{R}\} \cup \mathcal{E}$. Since such an E can still be chosen arbitrarily close to $\min Q(ix)$, this will show at one stroke that the operators are semibounded below and that $\sigma_{ess} \subset \{Q(ix) : x \in \mathbb{R}\}$. So fix E as above. Then $Q(\lambda) = E$ has no purely imaginary solution, so Re $\lambda_k(E) \ne 0$ for all k. More precisely, by the way the λ's were labeled, Re $\lambda_k(E) < 0$ if $k \le n$ and Re $\lambda_k(E) > 0$ if $k > n$. We now use the asymptotic formulae from Theorem 6.1. (What we need in this part of the proof is actually only a very special case of Theorem 6.1: We only need to solve the DE $\tau y = Ey$ for *fixed* E.)

Transforming back from the vectors Y_k of Theorem 6.1 to the original functions (compare (3)), we get solutions $y_1, \dots, y_{2n}$ of the form

$$y_i^{(k)}(x,E) = (\lambda_i^k(E) + r_{k+1,i}(x)) \exp\left(\int_a^x \lambda_i(t,E)\, dt \right) \quad (k = 0, 1, \dots, n-1),$$

with $\lim_{x\to\infty} r_{k+1,i}(x) = 0$. (As already observed in the remark following Theorem 6.1, there are also asymptotic formulae for the higher quasi-derivatives, but we do not need them here.) The argument E will from now on be dropped in this part of the proof. We want to show that there are $a, L > 0$ so that if $y(x) = \sum c_i y_i(x)$ and $y^{(k)}(a) = y^{(k)}(b) = 0$ for all $k = 0, 1, \dots, n-1$ and for some $b \geq a + L$, then $c_1 = \cdots = c_{2n} = 0$. Introduce the matrices $A(x), B(x) \in \mathbb{C}^{n\times n}$ by letting

$$\begin{aligned} A_{ij}(x) &= \lambda_j^{i-1} + r_{ij}(x), \\ B_{ij}(x) &= \lambda_{n+j}^{i-1} + r_{i,n+j}(x). \end{aligned}$$

We also associate with $c \in \mathbb{C}^{2n}$ the vector $d \in \mathbb{C}^{2n}$ with the components $d_i = c_i \exp\left(\int_a^b \lambda_i(t)\, dt \right)$. Then, in view of the above formulae, we must show that there are a, L so that if $b \geq a + L$, no non-zero vector $c \in \mathbb{C}^{2n}$ satisfies simultaneously $(A(a), B(a))c = 0$ and $(A(b), B(b))d = 0$. The matrices $A(x), B(x)$ are, for large x, small perturbations of Vandermonde matrices. Thus, by taking a sufficiently large, we can ensure that

$$\begin{aligned} m &\equiv \inf\{\|A(x)v\|, \|B(x)v\| : x \geq a, v \in \mathbb{C}^n, \|v\| = 1\} > 0, \\ M &\equiv \sup\{\|A(x)v\|, \|B(x)v\| : x \geq a, v \in \mathbb{C}^n, \|v\| = 1\} < \infty. \end{aligned}$$

By the remarks on the $\lambda_i(t)$ made at the beginning of the argument, we can further achieve that for some $\delta > 0$,

$$\left| \exp\left(\int_a^b \lambda_i(t)\, dt \right) \right| \begin{cases} \leq e^{-\delta L} & \text{if } i \leq n, \\ \geq e^{\delta L} & \text{if } i > n. \end{cases}$$

Writing $c = (c_1, c_2)^t$ with $c_i \in \mathbb{C}^n$, and similarly for d, we now see that $\|A(b)d_1\| \leq M e^{-\delta L}\|c_1\|$ and $\|B(b)d_2\| \geq m e^{\delta L}\|c_2\|$, so it follows from $(A(b), B(b))d = 0$ that

$$\|c_2\| \leq \frac{M}{m} e^{-2\delta L} \|c_1\|.$$

Similarly, the condition at a implies $\|c_1\| \leq (M/m)\|c_2\|$. Now the parameter L is still at our disposal, so these two inequalities together yield the desired conclusion $c_1 = c_2 = 0$, provided we take L large enough.

We have now established semiboundedness and we have shown that $\sigma_{ess} \subset \{Q(ix)\}$. The converse inclusion will follow from the results on the absolutely continuous spectrum (to be proved below), which include $\sigma_{ac} \supset \{Q(ix)\}$.

Next, we study the M-function of the perturbed problem on $x \in [a, \infty)$ with boundary conditions

$$y(a) = y'(a) = \cdots = y^{(n-1)}(a) = 0.$$

Call this M-function M_a; also, denote the M-function of the unperturbed operator (8) with the same boundary conditions by M_0 (of course, M_0 is independent of the left endpoint a). We will show that M_a is close to M_0 in the following sense:

Lemma 7.6 *Fix $E_0 \in \mathbb{R} \backslash \mathcal{E}$. Then there are $\delta, a_0 > 0$ and a continuous function $\eta(a, z)$, defined for $a \geq a_0, z \in S_\delta(E_0)$ and satisfying*

$$\lim_{a\to\infty} \max_{z\in S_\delta(E_0)} \|\eta(a, z)\| = 0,$$

so that $M_a(z) = M_0(z) + \eta(a, z)$ if $\text{Im } z > 0$.

Moreover, the limit $M_a(E) \equiv \lim_{\epsilon\to 0+} M_a(E + i\epsilon)$ exists for every $E \in (E_0 - \delta, E_0 + \delta)$, and $M_a(E) = M_0(E) + \eta(a, E)$.

Proof. For $\text{Im } z > 0$, let $F(x, z)$ be a basis of the space of $L_{2,A}$ solutions of (2). By this we mean that F has $2n$ rows and n columns, and the columns span the space of $L_{2,A}$ solutions of (2). Then, by definition of M, there is $C(z) \in \mathbb{C}^{n\times n}$ so that $F(x, z)C(z) = U_a(x, z) + V_a(x, z)M_a(z)$. Write $F = \binom{F_1}{F_2}$ with $F_i \in \mathbb{C}^{n\times n}$. By eliminating C and taking into account the initial values of U, V (compare (5)!), we get

$$M_a(z) = F_2(a, z) \left(F_1(a, z)\right)^{-1}.$$

Of course, there is an analogous formula for M_0. In this case, we can plug in the explicit form of the L_2 solutions to obtain the representation $M_0 = AB^{-1}$, where

$$A_{ij}(z) = \sum_{k=i}^{n} (-1)^{k+i} c_k \lambda_j^{2k-i}(z), \quad B_{ij}(z) = \lambda_j^{i-1}(z).$$

Note that B is again a Vandermonde matrix. Thus, it is invertible precisely if the λ_j are distinct; as a consequence, the above formulae may only be used in this case.

Fix $E_0 \in \mathbb{R} \setminus \mathcal{E}$, and pick δ small enough, so that the assertions of Theorem 6.1 hold for $z \in S_\delta(E_0)$. By the above remarks on M_0, we may also require that $M_0(E) \equiv \lim M_0(E + i\epsilon)$ exists for all $|E - E_0| < \delta$. Now, to compute $M_a(z)$ for $z \in S_\delta(E_0) \setminus \mathbb{R}$, we can simply take $F = (Y_1, \ldots, Y_n)$. It follows from Theorem 6.1 that $F_2(a, z) = A(z) + R_1(a, z)$ and $F_1(a, z) = B(z) + R_2(a, z)$. Here, the remainders R_i are defined for all $z \in S_\delta(E_0)$ (so z may be real), they are continuous functions of (a, z), and $\lim_{a\to\infty} \max_{z\in S_\delta(E_0)} \|R_i(a, z)\| = 0$.

By Lemma 5.1, part 1., the $\lambda_j(z)$ are separated from one another by a positive distance as z runs over $S_\delta(E_0)$. So, $B(z)$ is invertible for all $z \in S_\delta(E_0)$, and in fact $\sup_{z\in S_\delta(E_0)}\|B^{-1}(z)\| < \infty$. Hence we can find an a_0 so that $B(z)+R_2(a,z)$ is also invertible if $a \ge a_0$. (Note that for complex z, the matrix $F_1(a,z)$ must be invertible for general reasons; the point is that here this is still true on the real line.) Obviously, we also have that $\sup_{z\in S_\delta(E_0)}\|A(z)\| < \infty$, so we can indeed write $M_a = (A+R_1)(B+R_2)^{-1}$ in the form $M_a = AB^{-1}+\eta$, with a remainder η that is easily checked to have the stated properties. The last assertion is now also obvious because both $M_0(E+i\epsilon)$ and $\eta(a,E+i\epsilon)$ tend to limits as $\epsilon \to 0+$. □

With this Lemma, we can now complete the proof of Theorem 1.1. Let S_m be the sets defined in Proposition 3.1. Lemma 7.6 shows that for every $E_0 \in S_m\setminus\mathcal{E}$, the limit $M_a(E) = \lim M_a(E+i\epsilon)$ exists for all E from a neighborhood of E_0 and big enough a. We pick $\delta > 0$ so that $(E_0-2\delta, E_0+2\delta)$ is contained in this neighborhood and also in S_m. Moreover, we see from the discussion above that we can also achieve that $M_0(z)$ has a holomorphic continuation across $(E_0-2\delta, E_0+2\delta)$. This together with Propositions 3.1, 3.2a) and (7) imply that for all but finitely many $E\in[E_0-\delta,E_0+\delta]$, the rank of $\text{Im}\, M_0(E)$ is m. Let $E_1,\dots,E_s$ be these exceptional points. Since the rank of a matrix can only increase under small perturbations, Lemma 7.6 shows that we can find, for any $\epsilon > 0$, an $a_0 > 0$ so that rank $\text{Im}\, M_a(E) \ge m$ if $a \ge a_0$ and $E \in [E_0-\delta, E_0+\delta]$, $|E-E_i| \ge \epsilon$ $(i=1,\dots,s)$. Lemma 7.6 also shows that $\text{Im}\, M_a(E)$ is continuous on $E\in[E_0-\delta,E_0+\delta]$.

The absolutely continuous part of the operator is independent of a. More precisely (using self-explanatory notation), $H_{(0,\infty)}$ differs from $H_{(0,a)}\oplus H_{(a,\infty)}$ by a finite rank perturbation of the resolvent, so the absolutely continuous parts are unitarily equivalent. Furthermore, $\sigma_{ac}(H_{(0,a)}) = \emptyset$. We can therefore conclude, using the results of the discussion above and also (7), that the absolutely continuous part of an arbitrary self-adjoint realization of (1) contains some part which is unitarily equivalent to the orthogonal sum of m copies of the operator of multiplication by the variable in the space $L_2(E_0-\delta,E_0+\delta)$. Finally, $E_0\in S_m$ was almost arbitrary: only the countable set $\mathcal{E}$ had to be excluded. So we have actually shown that the absolutely continuous part contains some part which is unitarily equivalent to the operator A from Proposition 3.1 and thus also to the realization of τ_0 with boundary conditions $y(0)=\cdots=y^{(n-1)}(0)=0$.

On the other hand, we will see in a moment that the multiplicity of the absolutely continuous spectrum on $(E_0-\delta,E_0+\delta)$ is also $\le m$ if this interval is contained in S_m. Combining this with the result from the preceding paragraph, we obtain the assertion of Theorem 1.1 on the absolutely continuous part of (1).

To prove the claim made above, we will use Theorem 7.1. By definition of S_m, there are precisely $2m$ purely imaginary solutions to $Q(\lambda)=E$ for every $E\in(E_0-\delta,E_0+\delta)\setminus\mathcal{E}$ if, as we assumed, $(E_0-\delta,E_0+\delta)\subset S_m$. Hence, since $\lambda=0$ is not a solution, the remaining $2n-2m$ solutions have non-zero real parts, and thus there are $n-m$ solutions $\lambda_i(E)$ with negative real parts. So there are $n-m$ $L_{2,A}$ solutions of $\tau y = Ey$, as desired.

Having discussed the absolutely continuous part, it only remains to show that $\sigma_{sc}=\emptyset$. To this end, we check the hypotheses of Theorem 7.3. We again fix $E_0 \in S_m \setminus \mathcal{E}$, and take $S = (E_0 - \delta, E_0 + \delta)$ with small $\delta > 0$. Actually, by throwing away an at most countable set of E_0's, we can further require that rank Im $M_0 = m$ in S. Of course, as α in Theorem 7.3 we take again the standard boundary condition $\alpha_1 = 1, \alpha_2 = 0$. Then hypotheses 1.–3. in fact immediately follow from the discussion above, with $r = n - m$. So an application of Theorem 7.3 concludes the proof of Theorem 1.1. □

It is perhaps interesting to pause for a moment and take a bird's eye view on the proof: We started out by analyzing the spectral representation of τ_0. This knowledge was later used to deduce properties of M_0. Since we had detailed information about the solutions of $\tau y = zy$, we could conclude that the M-function of the full problem must be a small perturbation of M_0. Finally, we could then go back again to the spectral properties, but this time those of the perturbed operator τ.

8 Extensions

The asymptotic integration techniques leading to Theorem 6.1 can be carried further. This leads to extensions of Theorem 1.1. Since only the technical details of the argument change, we will be extremely sketchy in this section.

The idea is to diagonalize the coefficient matrix of the DE a second time. More precisely, introduce $V = T_2^{-1} U$ (using the notation from Sect. 4), where T_2 is picked so that $T_2^{-1}(\Lambda + S)T_2$ is diagonal modulo integrable terms. Since Λ is already diagonal and $S(x, z) \to 0$ as $x \to \infty$, one can take $T_2 = 1 + Q$ with $Q \to 0$ also.

This procedure is carried out in [12, Sect. 1.6], of course for fixed z. However, the same proof gives an analogous result on *uniform* asymptotic integration if the application of Levinson's Theorem is replaced by an application of Theorem A.1. This also goes for the simplified version of [12, Theorem 1.6.1] that is presented in [12] following the proof of the original theorem. The net result is that the condition $\|S(x, z)\| \le \rho(x) \in L_1$ is now replaced by the set of conditions

$$\begin{gathered}
\lim_{x \to \infty} \sup_z \left(\lambda_i(x, z) - \lambda_j(x, z)\right)^{-1} \|S(x, z)\| = 0, \\
\left\| \frac{\partial}{\partial x} \left(\left(\lambda_i(x, z) - \lambda_j(x, z)\right)^{-1} S(x, z) \right) \right\| \le \rho_1(x), \\
|\lambda_i(x, z) - \lambda_j(x, z)|^{-1} \|S(x, z)\|^2 \le \rho_2(x),
\end{gathered} \tag{19}$$

with $\rho_i \in L_1$.

This allows us to treat, more generally, coefficients of the following form:

$$\begin{aligned} p_k(x) &= c_k + \sum_{i=1}^{3} p_{k,i}(x) \qquad\qquad (k = 0, \ldots, n-1), \\ p_n^{-1}(x) &= (c_n + p_{n,2}(x) + p_{n,3}(x))^{-1} + p_{n,1}(x), \\ w(x) &= 1 + \sum_{i=1}^{3} v_i(x), \end{aligned}$$

with $p_{k,1}, p'_{k,2}, p''_{k,3}, p'^2_{k,3} \in L_1$ and $p_{k,2}, p_{k,3} \to 0$, and similarly for the v_i.

We proceed as outlined above. We first split off the contributions coming from the L_1 terms (that is, coming from the $p_{k,1}$'s), then diagonalize as in Sect. 4. The matrix S thus obtained contains again integrable terms (associated with the $p_{k,2}$'s) which can be absorbed by a remainder. Finally, we apply the transformation discussed above; so we need to check conditions (19), where S is still given by (13), but with q_k, v replaced by $p_{k,3}, v_3$ (the other terms have already been separated!). To this end, we have to recall that the $\lambda_i(x, z)$ tend to distinct limits as $x \to \infty$, uniformly in z. Also, since λ solves $P(\lambda; x) = z(w(x) - v_1(x))$, we get from (11) that

$$\frac{\partial \lambda}{\partial x} = \frac{z(v'_2 + v'_3) - \sum_{k=0}^{n} (-1)^k (p'_{k,2} + p'_{k,3}) \lambda^{2k}}{K(\lambda)}.$$

The details are left to the reader.

In conclusion, we see that the statement of Theorem 6.1 still holds. Then, by the discussion of Sect. 7, we also still have the assertions of Theorem 1.1.

9 Embedded eigenvalues

It is clear that if $E \in S_m \setminus \mathcal{E}$ with $1 \le m \le n-1$ (using the notation from Proposition 3.1), then the equation $\tau y = Ey$ has L_2 solutions, so any such E is an eigenvalue for suitable boundary conditions. Since also $S_m \subset \sigma_{ac}$, these eigenvalues are embedded in the absolutely continuous spectrum. So far, the situation is as in the unperturbed case. However, it turns out that the perturbations can even produce non-discrete embedded point spectrum:

Example. Let

$$S = -\frac{d^2}{dx^2} - \frac{1}{x}$$

be the Schrödinger operator of the hydrogen atom. To avoid a singular left endpoint, we consider S on $L_2(1, \infty)$. Then, for every boundary condition at $x = 1$, the corresponding operator has infinitely many eigenvalues $E_1 < E_2 < \cdots < 0$, and $E_n \to 0$. An elementary calculation shows that the differential expression τ defined as $\tau = (S-1)^2$ has the form

$$(\tau y)(x) = \frac{d^4 y}{dx^4} + 2\frac{d}{dx}\left((x^{-1} + 1)\frac{dy}{dx}\right) + \left(2x^{-3} + x^{-2} + 2x^{-1} + 1\right) y.$$

Clearly, this falls under the scope of Theorem 1.1. The unperturbed problem is $\tau_0 y = y^{(4)} + 2y'' + y$, and thus the polynomial describing the spectral properties of τ is given by $Q(i\lambda) = \lambda^4 - 2\lambda^2 + 1 = (\lambda^2 - 1)^2$. So in particular, $\sigma_{ess} = \sigma_{ac} = [0, \infty)$. However, for suitable self-adjoint realizations of τ, eigenvalues accumulate at $1 \in \sigma_{ac}$.

This can be seen as follows: Fix a self-adjoint boundary condition for the second order differential expression S (one can take $y(1) = 0$, say). Denote the domain of the corresponding operator by D_0. Then the operator H which acts as $Hy = \tau y$ on the domain

$$D(H) = \{y \in D_0 : Sy \in D_0\}$$

is a self-adjoint realization of τ in $L_2(1, \infty)$ (see [32] and also the discussion of Sect. 3). But clearly, the eigenfunctions of S that are in D_0 also lie in $D(H)$, so H has eigenvalues $(E_n - 1)^2$. $\square$

This counterexample, however, is of a rather special type. Namely, the accumulation point 1 is a critical value of the polynomial Q. Indeed, one can prove that these critical values are the only possible accumulation points for operators of order four! We give a rough outline of this argument: If $E \in S_2$, there are no L_2 solutions (this follows from Theorem 6.1 if $E \notin \mathcal{E}$, and even if $E \in \mathcal{E}$, one can still verify the hypotheses of the original Levinson theorem to obtain the claim in this case as well). So suppose $E \in S_1$. Recall that the $\lambda_i(E)$ are the zeros of $Q(\lambda) - E$, which is an even polynomial with real coefficients. The λ corresponding to the L_2 solution has negative real part, so the four λ's must be of the form $\pm\mu(E), \pm i\nu(E)$, with $\mu, \nu > 0$. Not surprisingly, the analysis of Lemma 5.1 can be improved in this very special situation. It is not hard to show that the eigenvalue associated with the L_2 solution y (namely, $-\mu$) satisfies a much stronger form of the dichotomy condition. This, in turn, implies that the corresponding solution vector $Y(0, z)$ is an analytic function of z in a neighborhood of E. (See also [4] for a more comprehensive discussion of these issues.) Since the eigenvalues are the solutions of an equation of the form $(\alpha_1, \alpha_2)Y(0, z) = 0$, we see that E cannot be an accumulation point of eigenvalues.

For operators of order six or higher, this argument breaks down. More specifically, it is possible that $\text{Re } \lambda_i = \text{Re } \lambda_j < 0$, so one does not obtain analytic solutions from the asymptotic integration machine. We always get continuous dependence on z outside $\mathcal{E}$, but this only implies that an accumulation of eigenvalues must itself be an eigenvalue. (As a consequence, any fixed $E \notin \mathcal{E}$ is not an accumulation point of the eigenvalues of H_α for lots of boundary conditions α.) But of course this remark does not clarify the situation completely, so we conclude with the following

Open Question. In the situation of Theorem 1.1 with $n \geq 3$, what are the possible accumulation points of the point spectrum?

If something positive can be said, then the two obvious candidates are the sets $\mathcal{E}$ and $\{Q(i\lambda) : \lambda \in \mathbb{R}, Q'(i\lambda) = 0\}$ (which is a subset of $\mathcal{E}$).

A Uniform asymptotic integration

In this appendix, we state the result from [4] which is used in this paper. So consider a linear differential system of the form

$$Y'(x,z) = (\Lambda(x,z) + R(x,z))Y(x,z) \quad (x \in [c,\infty)), \tag{20}$$

where Λ, R are $m \times m$ matrices (so $m = 2n$ in the applications in this paper) and Λ is assumed to be diagonal: $\Lambda_{ij}(x,z) = \delta_{ij}\lambda_i(x,z)$. The parameter $z = E + i\epsilon$ runs through a set M of the form $M = \{z : a \le E \le b, 0 \le \epsilon \le \delta\}$. The asymptotic integration result below requires a peculiar form of the dichotomy condition; one might call this a weak uniform dichotomy condition (although this term is not especially fortunate since there are five alternatives and not two). Put $\nu_{ij}(x,z) = \int_c^x \mathrm{Re}\,(\lambda_i - \lambda_j)(s,z)\,ds$; then we say that Λ satisfies the dichotomy condition if for any two indices $i, j \in \{1, \ldots, m\}$, $i \ne j$, one of the following (mutually exclusive) conditions holds:

1. There exists a constant K so that $\nu_{ij}(x,z) - \nu_{ij}(t,z) \le K$ for all $z \in M$ and $x \ge t \ge c$. Moreover, $\lim_{x\to\infty} \sup_{z\in M} \nu_{ij}(x,z) = -\infty$.
2. Condition 1. holds with ν_{ij} replaced by ν_{ji}.
3. There exists a constant K so that $-K \le \nu_{ij}(x,z) - \nu_{ij}(t,z) \le K$ for all $z \in M$ and $x \ge t \ge c$.
4. There exists a constant K so that $\nu_{ij}(x,z) - \nu_{ij}(t,z) \le K$ for all $z \in M$ and $x \ge t \ge c$. For all $\epsilon_0 \in (0, \delta]$, we have
$$\lim_{x\to\infty} \sup_{\{z\in M: \mathrm{Im}\, z \ge \epsilon_0\}} \nu_{ij}(x,z) = -\infty.$$
Moreover, if $z \in M \cap \mathbb{R} = [a,b]$, then also $-K \le \nu_{ij}(x,z) - \nu_{ij}(t,z) \le K$ for all $x \ge t \ge c$.
5. Condition 4. holds with ν_{ij} replaced by ν_{ji}.

Roughly speaking, this gives uniform control off the real line, while allowing "discontinuities" as z approaches $[a,b]$ (if conditions 4. or 5. hold). In a sense 4./5. "interpolate" between 1./2. and 3. This dichotomy condition just abstract the situation encountered in the analysis of higher order differential equations, as discussed in the body of this paper.

The result from [4] is:

Theorem A.1 ([4]) *Suppose that $\Lambda(x,z), R(x,z)$ are continuous functions of z for (almost every) fixed x and $\|\Lambda(x,z)\| \le a(x)$ and $\|R(x,z)\| \le \rho(x)$ with $a \in L_{1,loc}([c,\infty))$ and $\rho \in L_1(c,\infty)$. Suppose further that Λ satisfies the dichotomy condition, in the sense discussed above. Then* (20) *has solutions of the asymptotic form*

$$Y_k(x,z) = (e_k + r_k(x,z)) \exp\left(\int_c^x \lambda_i(t,z)\,dt\right) \quad (k = 1, \ldots, m).$$

Here, e_k is the kth unit vector, and the error terms r_k are (jointly) continuous in (x, z) and tend to zero as $x \to \infty$, uniformly in $z \in M$.

References

[1] N.I. Akhiezer and I.M. Glazman, Theory of Linear Operators in Hilbert Space I, II, Pitman, London, 1981.

[2] F.V. Atkinson, Discrete and Continuous Boundary Problems, Academic Press, New York, 1964.

[3] H. Behncke, Absolute continuity of Hamiltonians with von Neumann-Wigner potentials II, Manuscr. Math. **71** (1991), 163–181.

[4] H. Behncke and C. Remling, Uniform asymptotic integration of a family of linear differential systems, Math. Nachr. **225** (2001), 5–17.

[5] M. Ben-Artzi, On the absolute continuity of Schrödinger operators with spherically symmetric, long-range potentials I, J. Diff. Eq. **38** (1980), 41–50.

[6] V.S. Buslaev and V.B. Matveev, Wave operators for Schrödinger equation with a slowly decreasing potential, Theor. Math. Phys. **2** (1970), 266–274.

[7] M. Christ and A. Kiselev, Absolutely continuous spectrum for one-dimensional Schrödinger operators with slowly decaying potentials: some optimal results, J. Amer. Math. Soc. **11** (1998), 771–797.

[8] E.A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.

[9] P. Deift and R. Killip, On the absolutely continuous spectrum of one-dimensional Schrödinger operators with square summable potentials, Commun. Math. Phys. **203** (1999), 341–347.

[10] A. Dijksma, H.S.V. de Snoo, and A.A. El Sabbagh, Selfadjoint extensions of regular canonical systems with Stieltjes boundary conditions, J. Math. Anal. Appl. **152** (1990), 546–583.

[11] N. Dunford and J.T. Schwartz, Linear Operators II, Interscience, New York, 1963.

[12] M.S.P. Eastham, The Asymptotic Solution of Linear Differential Systems, London Math. Soc. Monographs New Series, Vol. 4, Oxford University Press, Oxford, 1989.

[13] D.J. Gilbert and D.B. Pearson, On subordinacy and analysis of the spectrum of one-dimensional Schrödinger operators, J. Math. Anal. Appl. **128** (1987), 30–56.

[14] I.M. Glazman, Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators, Jerusalem 1965.

[15] D.B. Hinton and A. Schneider, On the Titchmarsh-Weyl coefficients for singular S-Hermitian systems I, Math. Nachr. **163** (1993), 323–342.

[16] D.B. Hinton and J.K. Shaw, On the Titchmarsh-Weyl $M(\lambda)$-functions for linear Hamiltonian systems, J. Diff. Eq. **40** (1981), 316–342.

[17] D.B. Hinton and J.K. Shaw, On the spectrum of a singular Hamiltonian system, Quaest. Math. **5** (1982), 29–81.

[18] K. Knopp, Theory of Functions II, Dover, New York, 1947.

[19] Q. Kwong, H. Wu, and A. Zettl, Dependence of eigenvalues on the problem, Math. Nachr. **188** (1997), 173–201.

[20] B.M. Levitan and I.S. Sargsjan, Introduction to Spectral Theory, Transl. of Math. Monographs Vol. 39, Amer. Math. Soc. Providence, 1975.

[21] M.A. Naimark, Linear Differential Operators, Ungar, New York 1967.

[22] M. Reed and B. Simon, Methods of Modern Mathematical Physics, III. Scattering Theory, Academic Press, San Diego, 1979.

[23] M. Reed and B. Simon, Methods of Modern Mathematical Physics, IV. Analysis of Operators, Academic Press, San Diego, 1978.

[24] C. Remling, The absolutely continuous spectrum of one-dimensional Schrödinger operators with decaying potentials, Commun. Math. Phys. **193** (1998), 151–170.

[25] C. Remling, Spectral analysis of higher order differential operators, I. General properties of the M-function, J. London Math. Soc. **58** (1998), 367–380.

[26] C. Remling, One-dimensional Schrödinger operators with decaying potentials, in Operator Theory: Advances and Applications, Vol. 108: Mathematical Results in Quantum Mechanics (Birkhäuser, 1999), pp. 343–349.

[27] C. Remling, Bounds on embedded singular spectrum for one-dimensional Schrödinger operators, Proc. Amer. Math. Soc. **128** (2000), 161–171.

[28] B. Simon, Some Jacobi matrices with decaying potentials and dense point spectrum, Commun. Math. Phys. **87** (1982), 253–258.

[29] J. von Neumann and E. Wigner, Über merkwürdige diskrete Eigenwerte, Z. Phys. **30** (1929), 465–467.

[30] P.W. Walker, A vector-matrix formulation for formally symmetric ordinary differential equations with applications to solutions of integrable square, J. London Math. Soc. **9** (1974), 151–159.

[31] J. Weidmann, Zur Spektraltheorie von Sturm-Liouville-Operatoren, Math. Z. **119** (1967), 268–302.

[32] J. Weidmann, Linear Operators in Hilbert Spaces, Graduate Texts in Mathematics, Vol. 68, Springer-Verlag, Berlin, 1980.

[33] J. Weidmann, Spectral Theory of Ordinary Differential Operators, Springer Lect. Notes **1258**, Springer-Verlag, Berlin, 1987.

[34] D.A.W. White, Schrödinger operators with rapidly oscillating central potentials, Trans. Amer. Math. Soc. **275** (1983), 641–677.