THE INVERSE SCATTERING TRANSFORM FOR THE KDV EQUATION WITH STEP-LIKE SINGULAR MIURA INITIAL PROFILES

SERGEI GRUDSKY, CHRISTIAN REMLING, AND ALEXEI RYBKIN

Abstract. We develop the inverse scattering transform for the KdV equation with real singular initial data $q(x)$ of the form $q(x) = r'(x) + r(x)^2$, where $r \in L^2_{loc}$, $r|_{\mathbb{R}_+} = 0$. As a consequence we show that the solution $q(x,t)$ is a meromorphic function with no real poles for any $t > 0$.

Contents

1. Introduction 1
2. The Titchmarsh-Weyl m-function and the reflection coefficient 4
3. Hankel Operators 7
4. The IST Hankel Operator 11
5. Main Results 13
6. Acknowledgement 15
References 15

1. Introduction

This note is motivated by the recent progress in the spectral theory of Schrödinger operators with singular potentials [11] and the long lasting interest in completely integrable systems with low regularity initial data [23].

We are concerned with the Cauchy problem for the Korteweg-de Vries (KdV) equation ($x \in \mathbb{R}, t > 0$)

$$\begin{cases} \partial_t u - 6u\partial_x u + \partial_x^3 u = 0 \\ u(x,0) = q(x) \end{cases} \tag{1.1}$$

with the initial data q satisfying (the terminology will be clarified below)

Hypothesis 1.1. *$q(x)$ is a real-valued $H^{-1}_{\mathrm{loc}}(\mathbb{R})$ distribution subject to*

Date: November, 2014.

1991 *Mathematics Subject Classification.* 34B20, 37K15, 47B35.

Key words and phrases. KdV equation, singular potentials, Titchmarsh-Weyl m-function, Hankel operators, Miura transform.

SG is supported by PROMEP (México) via "Proyecto de Redes", by CONACYT grant 180049, and by Federal program “Scientific and Scientific-Pedagogical Personnel of Innovative Russia for the years 2007-2013” (contract No 14.A18.21.0873) .

CR is supported in part by the NSF grant DMS 1200553 .

AR is supported in part by the NSF grant DMS 1009673.

(1) *(positivity)*
$$\mathbb{L}_q \geq 0 \tag{1.2}$$
(2) *(restricted support)*
$$q|_{\mathbb{R}_+} = 0. \tag{1.3}$$

Here
$$H^{-1}_{\text{loc}}(\mathbb{R}) = \left\{\chi f : \chi \in C_0^\infty(\mathbb{R}), f \in H^{-1}(\mathbb{R})\right\},$$
where $H^s(\mathbb{R})$, $s \in \mathbb{R}$, is the Sobolev space of distributions subject to $(1+|x|)^s \widehat{f}(x) \in L^2(\mathbb{R})$ and $C_0^\infty(\mathbb{R})$ is the space of compactly supported smooth functions.
$$\mathbb{L}_q = -\partial_x^2 + q(x) \tag{1.4}$$
is the Schrödinger operator on $L^2(\mathbb{R})$ associated with the initial profile q in (1.1). As shown in [34] $\mathbb{L}_q$ is well-defined as a selfadjoint operator for large classes of distributional potentials q form $H^{-1}_{\text{loc}}(\mathbb{R})$. Condition (1.2) is understood in the sense that $\langle \mathbb{L}_q \chi, \chi \rangle \geq 0$ if $\chi \in C_0^\infty(\mathbb{R})$. It is one of the main results of [23] that Condition (1.2) holds if and only if
$$q(x) = \partial_x r(x) + r(x)^2 =: B(r) \tag{1.5}$$
with some real $r \in L^2_{\text{loc}}(\mathbb{R})$. The transform (potential) $B(r)$ is referred in [23] to as Miura. To reflect Condition (1.3) we call any q subject to Hypothesis 1.1 a Miura steplike potential.

We emphasize from the beginning that our results can be suitably adjusted to semiboundedness from below in Condition (1.2) and a certain decay assumption at $+\infty$ in Condition (1.3). The numerous complications (some of which are by no means trivial) that arise then are not of principal nature and can all be resolved within our approach. They however seriously aggravate the exposition. We therefore choose here transparency over completeness.

Our main goal is to develop the Inverse Scattering Transform (IST) method for (1.1) under Hypothesis 1.1. We achieve our goal by employing techniques of Hankel operators from [16], [31]-[33]. The version of this approach that we use here makes our considerations particularly transparent. More specifically, with an initial profile subject to Hypothesis 1.1 we associate the Hankel operator (see Definition 3.1) $\mathbb{H}(\varphi_{x,t})$ with the symbol $\varphi_{x,t}(k) = \xi_{x,t}(k) R(k)$ where
$$\xi_{x,t}(k) := \exp\{i(8k^3 t + 2kx)\} \tag{1.6}$$
and $R(k)$ is the reflection coefficient from the right incident (see Definition 2.3). The solution to (1.1) is then given by
$$q(x,t) = -2\partial_x^2 \log\det\left(1 + \mathbb{H}(\varphi_{x,t})\right) \tag{1.7}$$
thus establishing well-posedness of (1.1) in the sense of Definition 5.1. Moreover, $q(x,t)$ is a meromorphic function in x on the entire complex plane for any $t > 0$ with no real poles. We prove (1.7) by first approximating our singular q by C_0^∞-functions for which (1.7) is well-known and then passing to the limit. Justifying the validly of our limiting arguments is the main issue here and it is the techniques of Hankel operators that make it quite effortless.

Let us now put our results in the historic context. The formula (1.7) is a derivation of the classical Dyson (also called Bargman or log-determinant) formula (see, e.g. [10], [29]). For step like (regular) potentials it appeared first in [37] (under assumption that $q(x)$ goes to a constant at $-\infty$) and with no restrictions on $-\infty$

in [16], [31]-[33]. For singular potentials (1.7) is new. In fact, to the best of our knowledge, in the context of singular initial data, the IST is rigorously justified for measure potentials (see e.g. [21]). On the other hand, well-posedness of (1.1) in the Sobolev space $H^s(\mathbb{R})$ with negative index s turned out to be an interesting problem in its own right having drawn enormous attention. The sharpest result says that (1.1) is globally well-posed in $H^{-3/4}(\mathbb{R})$ ([7], [17], [36] and extensive literature therein). Note that the space $H^{-3/4}(\mathbb{R})$ includes such singular functions as $\delta(x)$, $1/x$, etc. However, $s = -3/4$ is the threshold for the harmonic analytical methods commonly used in this circle of issues. On the other hand, if one looks at the KdV as a completely integrable system the Schrödinger operator (1.4) in the Lax pair associated with (1.1) remains well-defined for $s < -3/4$. In fact, the spectral (direct and inverse) theory of Schrödinger (Sturm-Liouville) operators with singular potentials has independently attracted much of interest. The systematic approach to $H^{-1}_{\text{loc}}(\mathbb{R})$ potentials began with the influential paper [34] and has experienced a rapid development culminating in the recent [11], where the completeness of this theory approaches that of the classical Titchmarsh-Weyl theory. We especially mention the recent [19] devoted to the scattering theory for potentials $q \in B(L^2(\mathbb{R}) \cap L^1(\mathbb{R}))$ with the future goal of developing the IST for such initial data[1]. Our methods are completely different.

This suggests that if we use complete integrability of (1.1) the global well-posedness could be pushed across the threshold $s = -3/4$. It is exactly how Kappeler-Topalov [22] were able to extend well-posedness of $H^{-1}(\mathbb{T})$ for periodic q's. One might conjecture that the global well-posedness for (1.1) also holds far beyond $H^{-3/4}(\mathbb{R})$ and could be achieved by a suitable extension of the IST method for $\mathbb{L}_q$ with $q \in H^{-1}(\mathbb{R})$. An important step in this direction was done by Kappeler et al [23] where it was shown that (1.1) is globally well-posed in a certain sense if $q \in B(L^2(\mathbb{R}))$. Of course, $B(L^2(\mathbb{R}))$ doesn't exhaust $H^{-1}(\mathbb{R})$ but, since $H^{-1}(\mathbb{R}) \supset H^{-3/4}(\mathbb{R})$, singularity of such solutions is pushed all the way to[2] $s = -1$.

We note that all functions in $H^{-s}(\mathbb{R})$ exhibit certain decay at $\pm\infty$. On the other hand, there has been a significant interest in non-decaying solutions to (1.1) (other than periodic). The case of the so-called steplike initial profiles (i.e. when $q(x) \to 0$ sufficiently fast as $x \to +\infty$ and $q(x)$ doesn't decay at $-\infty$) is of physical interest and has attracted much attention since the early 70s. We refer to the recent paper [12] for a comprehensive account of the (rigorous) literature on steplike initial profiles with specified behavior at infinity (e.g. q's tending to a constant, periodic function, etc.). In the recent [16], [31]-[33] the case of q's rapidly decaying at $+\infty$ and sufficiently arbitrary at $-\infty$ is studied in great detail. Initial steplike profiles in these papers are at least locally integrable (i.e. regular).

The paper is organized as follows. In Section 2 we discuss the Titchmarsh-Weyl m-function and reflection coefficient in the context of singular potentials. In Section 3 we review Hankel operators and prove some results related to a Hankel operator with a cubic oscillatory symbol. In the last Section 4 we state and prove our main result.

---

[1] To the best of our knowledge this goal has not been realized yet.

[2] As indicated in [34], $\mathbb{L}_q$ with $q \in H^{-s}$ for $s > 1$ is ill-defined.

2. The Titchmarsh-Weyl m-function and the reflection coefficient

It is well-known [34] that any $q \in H^{-1}_{\mathrm{loc}}(\mathbb{R})$ can be represented as $q = \partial_x Q$ with some $Q \in L^2_{\mathrm{loc}}(\mathbb{R})$. We now regularize the Schrödinger differential expression with (formal) potential $q = \partial_x Q$ by introducing the quasi-derivative

$$Dy = \partial_x y - Qy,$$

for (locally) absolutely continuous y.

Following the approach of [34] we then introduce

$$\begin{cases} \mathbb{L}_q y := -\partial_x (Dy) - Q\partial_x y \\ \partial_x Q = q \end{cases} \tag{2.1}$$

the Schrödinger operator with a (singular) potential $q \in H^{-1}_{\mathrm{loc}}(\mathbb{R})$. This may be evaluated on functions y that satisfy $y, Dy \in AC$. Similarly, we regularize the Schrödinger equation by rewriting it as follows:

$$-\partial_x(Dy) - Q\partial_x y = zy. \tag{2.2}$$

As proven in [34], the operator (2.1) is well-defined. Similarly, the classical Titchmarsh-Weyl theory can be developed for $\mathbb{L}_q$ along the usual lines [11]. Essentially, one has to replace regular derivatives ∂_x by the quasi-derivative D where appropriate.

Let us make this more explicit. Rewrite (2.2) as the first order system

$$\partial_x Y = AY, \quad A = \begin{pmatrix} Q & 1 \\ -z - Q^2 & -Q \end{pmatrix}, \quad Y = \begin{pmatrix} y \\ Dy \end{pmatrix}. \tag{2.3}$$

Notice that $\operatorname{tr} A = 0$, so the modified Wronskian $W = y_1 Dy_2 - (Dy_1)y_2$ of two solutions to the same equation is independent of x. In particular, that means that the transfer matrices $T(x,z)$ associated with (2.3) take values in $SL(2,\mathbb{C})$ and if $z \in \mathbb{R}$, then $T(x,z) \in SL(2,\mathbb{R})$. Here we define T as usual as the 2×2 matrix solution of (2.3) with the initial value $T(0,z) = 1$. Moreover, if $z \in \mathbb{C}^+$, then $T(x,z)$ acting as a linear fractional transformation on $w \in \mathbb{C}^+$ is a Herglotz function, that is, the map $w \mapsto Tw$ maps $\mathbb{C}^+$ holomorphically to itself. Recall also that this action is defined as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} w = \frac{aw + b}{cw + d}.$$

To establish this Herglotz property of $w \mapsto T(x,z)w$, use the fact that this property is equivalent to

$$-i(T^* J T - J) \geq 0, \text{ where } J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right);$$

see, for example, [30, Lemma 4.2]. Now this latter property follows from the fact that $-i(JA - (JA)^*) \geq 0$; indeed, a calculation shows that this latter matrix equals $2 \operatorname{Im} z \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

These properties are all we need to have a Titchmarsh-Weyl type theory available. See again [30, Section 4] for more on this abstract interpretation of the theory.

We now define the Titchmarsh-Weyl m function of the problem on $(-\infty, 0)$, with Dirichlet boundary conditions $y(0) = 0$, as follows: For $z \in \mathbb{C}^+$, let $\psi(\cdot, z)$ be the (unique, up to a factor) solution of (2.2) that is square integrable near $-\infty$, and let

$$m(z) = -\frac{D\psi(0,z)}{\psi(0,z)}. \tag{2.4}$$

Then m is a Herglotz function. Moreover, we have continuous dependence on the potential, in the following sense.

Theorem 2.1. *Let $Q_n, Q \in L^2_{\mathrm{loc}}(\mathbb{R})$. Suppose that $\mathbb{L}_q$ is in the limit point case at $-\infty$. If $Q_n \to Q$ in $L^2_{\mathrm{loc}}(\mathbb{R})$, that is, $\|Q_n - Q\|_{L^2(-R,R)} \to 0$ for all $R > 0$, then $m_n \to m$ uniformly on compact subsets of $\mathbb{C}^+$.*

We do not assume limit point case for the operators $\mathbb{L}_{q_n}$ here[3]. If some or all of these are in the limit circle case, then we can make an arbitrary choice of boundary conditions at $-\infty$ in the Theorem 2.1. As the proof below will make clear, what happens far out is in fact quite irrelevant.

Proof. By our limit point assumption, $m(z)$ can be approximated locally uniformly by m functions m_L of problems on $[-L, 0]$, with Dirichlet boundary conditions at $x = -L$ (say), if we send $L \to \infty$ here. This follows from the fact that such m functions lie in the corresponding Weyl disks $D_L(z)$ whose radii go to zero as $L \to \infty$, locally uniformly on $z \in \mathbb{C}^+$. This last statement can be obtained in a general version from a normal families argument (see [30, Theorem 4.4] for such a treatment), or one can use, in more classical style, an explicit formula for the radius of the Weyl disk in terms of entries of the transfer matrix.

Now we have that $m_L(z) = -(Dy_L)(0)/y_L(0)$, where y_L solves (2.2) and $y_L(-L) = 0$, $(Dy_L)(-L) = 1$. We also know that $y_L(0, z)$ is bounded away from zero, uniformly on compact subsets of $\mathbb{C}^+$. It now suffices to show that the values $y_L(0), (Dy_L)(0)$ that are obtained by solving (2.3) across $[-L, 0]$ depend continuously on Q in the sense specified, locally uniformly in z. This is done by a rather routine argument; the key feature that makes things work is the continuous dependence of $A(Q)$ on Q in the L^1 norm on $[-L, 0]$. We include a sketch of the argument for the reader's convenience.

Write (2.3) in integral form:

$$Y_n(x, z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_{-L}^{x} A(t; Q_n(t), z) Y_n(t, z)\, dt \tag{2.5}$$

We want to show that $Y_n(0, z) \to Y(0, z)$, where Y solves the same equation for Q. First of all, by standard ODE theory, the Y_n are uniformly bounded, that is, $|Y_n(x, z)| \le C$ for $n \ge 1$, $-L \le x \le 0$, $z \in K \subset \mathbb{C}$, and here $|\cdot|$ denotes an arbitrary norm on $\mathbb{C}^2$. This implies that for $s < t$[4],

$$|Y_n(s, z) - Y_n(t, z)| \lesssim \int_s^t |A(t; Q_n(t), z)|\, dt \to \int_s^t |A(t; Q(t), z)|\, dt.$$

Here we have used the crucial fact that $A(t; Q_n) \to A(t; Q)$ in $L^1(I)$ for all compact intervals I. The convergence on the right-hand side is uniform in $-L \le s \le t \le 0$ and $z \in K$; moreover $\int_s^t |A(Q)|$ can be made arbitrarily small by taking $|t - s| < \epsilon$. We have verified that $Y_n(\cdot, z)$ is an equicontinuous family. Thus we may pass to a limit in (2.5) along a suitable subsequence. It is easy to verify that the integrals on the right-hand side also approach the expected limit. So, if we write $Z = \lim Y_{n_j}$, then we obtain that

$$Z(x, z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_{-L}^{x} A(t; Q(t), z) Z(t, z)\, dt.$$

[3] In fact, under Hypothesis 1.1 q_n and q are limit point case ([2], [11]).

[4] We write $y \lesssim x$ in place of $y \le Cx$ with some $C > 0$ independent of x.

This identifies $Z = Y$ as the solution of (2.3) for Q; also, since this is the only possible limit, it was not necessary to pass to a subsequence. In particular, we have established that $Y_n(0,z) \to Y(0,z)$, as desired. $\square$

Remark 2.2. *For regular potentials Theorem 2.1 is a folklore. For singular $H^{-1}_{\mathrm{loc}}(\mathbb{R})$ potentials it is new.*

Define now the reflection coefficient R from the right incident of a singular potential $q \in H^{-1}_{\mathrm{loc}}(\mathbb{R})$ such that $q|_{\mathbb{R}_+} = 0$. Note that for such a q, we may alternatively compute m as

$$m(z) = -\lim_{x\to 0+} \frac{\partial_x \psi(x,z)}{\psi(x,z)},$$

and we can similarly replace $D\psi$ with $\partial_x \psi$ in m functions of problems on $(-\infty, x_0)$, with $x_0 > 0$.

Pick a point $x_0 > 0$ and consider a solution to $\mathbb{L}_q y = \lambda^2 y$ which is proportional to the Weyl solution on $(-\infty, x_0)$ and is equal to $e^{-i\lambda x} + re^{i\lambda x}$ on (x_0, ∞). From the continuity of this solution and its derivative at x_0 one has

$$r(\lambda, x_0) = e^{-2i\lambda x_0} \frac{i\lambda - \dfrac{\psi'(x_0,\lambda^2)}{\psi(x_0,\lambda^2)}}{i\lambda + \dfrac{\psi'(x_0,\lambda^2)}{\psi(x_0,\lambda^2)}}.$$

We define the right reflection coefficient by

Definition 2.3 (Reflection coefficient)**.** *We call*

$$R(\lambda) = \lim_{x_0\to 0^+} r(\lambda, x_0) = \frac{i\lambda - m(\lambda^2)}{i\lambda + m(\lambda^2)}. \tag{2.6}$$

the (right) reflection coefficient.

Theorem 2.1 and Definition 2.3 immediately imply

Theorem 2.4. *Assume that q, q_n are subject to Hypothesis 1.1. Then $R \in H^\infty(\mathbb{C}^+)$,*

$$R(-\overline{\lambda}) = \overline{R(\lambda)}, \quad |R(\lambda)| \le 1 \quad \textit{for } \lambda \in \mathbb{C}^+,$$

and $R_n(\lambda) \to R(\lambda)$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{C}^+$ if $q_n \to q$ in $H^{-1}_{\mathrm{loc}}(\mathbb{R})$.

Note that Hypothesis 1.1 does not rule out the case $|R(\lambda)| = 1$ a. e. on the real line (in the contrast with the short range case when $|R(\lambda)| < 1$ for $\lambda \neq 0$). From the spectral point of view the latter means that the absolutely continuous spectrum of $\mathbb{L}_q$ is supported on $\mathbb{R}_+$ but has uniform multiplicity one (not two as in the short range case). We conclude the section with an explicit example.

Example 2.5. *Let $q(x) = c\delta(x), c > 0$. The Weyl solution corresponding to $-\infty$ can be explicitly computed by ($C \neq 0$)*

$$\psi(x,\lambda^2) = C \begin{cases} e^{-i\lambda x} & , \quad x < 0 \\ \frac{1}{2i\lambda}\left(ce^{i\lambda x} + (2i\lambda - c)e^{-i\lambda x}\right) & , \quad x > 0 \end{cases}$$

and hence by (2.4) *and* (2.6)

$$m(\lambda^2) = i\lambda - c, \quad R(\lambda) = \frac{c}{2i\lambda - c}.$$

3. Hankel Operators

A Hankel operator is an infinitely dimensional analog of a Hankel matrix, a matrix whose (j,k) entry depends only on $j+k$. I.e. a matrix Γ of the form

$$\Gamma=\begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \cdots \\ \gamma_2 & \gamma_3 & \cdots & \\ \gamma_3 & \cdots & & \\ \cdots & & & \gamma_n \end{pmatrix}.$$

Definitions of Hankel operators depend on specific spaces. We consider Hankel operators on the Hardy space $H^2(\mathbb{C}^+)$ (cf. [20], [28]). Here, as usual (but a bit in conflict with our notation of the Sobolev spaces) $H^p(\mathbb{C}^\pm)$ $(0<p\leq\infty)$ denotes the Hardy space of $\mathbb{C}^\pm$. It is well-known (see e.g.[14]) that $L^2(\mathbb{R})=H^2(\mathbb{C}^+)\oplus H^2(\mathbb{C}^-)$, the orthogonal (Riesz) projection $\mathbb{P}_\pm$ onto $H^2(\mathbb{C}^\pm)$ being given by

$$(\mathbb{P}_\pm f)(x)=\pm\frac{1}{2\pi i}\int\frac{f(s)ds}{s-(x\pm i0)}. \tag{3.1}$$

Let $(\mathbb{J}f)(x)=f(-x)$ be the operator of reflection on L^2. It is clearly an isometry and with the obvious property

$$\mathbb{J}\mathbb{P}_\mp=\mathbb{P}_\pm\mathbb{J}. \tag{3.2}$$

Definition 3.1 (Hankel and Toeplitz operators)**.** *Let $\varphi\in L^\infty(\mathbb{R})$. The operators $\mathbb{H}(\varphi)$ and $\mathbb{T}(\varphi)$ defined by*

$$\mathbb{H}(\varphi)f=\mathbb{J}\mathbb{P}_-\varphi f,\ \text{and}\ \mathbb{T}(\varphi)f=\mathbb{P}_+\varphi f,\quad f\in H^2\left(\mathbb{C}^+\right), \tag{3.3}$$

are called respectively the Hankel and Toeplitz operators with the symbol φ.

Due to (3.2), both $\mathbb{H}(\varphi)$ and $\mathbb{T}(\varphi)$ act from $H^2(\mathbb{C}^+)$ to $H^2(\mathbb{C}^+)$. Note that while $\mathbb{H}(\varphi)$ and $\mathbb{T}(\varphi)$ look alike, they are different parts of the multiplication operator

$$\varphi f=\mathbb{J}\mathbb{H}(\varphi)f+\mathbb{T}(\varphi)f,\quad f\in H^2\left(\mathbb{C}^+\right), \tag{3.4}$$

and therefore are quite different. The Toeplitz operator will play only an auxiliary role in our consideration.

As well-know (and also obvious) that $\mathbb{H}(\varphi)$ is selfadjoint if $\mathbb{J}\varphi=\bar{\varphi}$.

Definition 3.1 can be extended to certain unbounded symbols (more exactly, from BMO) which nevertheless produce bounded Hankel operators. In such cases we define $\mathbb{H}(\varphi)$ first on the set

$$\mathfrak{H}_2:=\left\{f\in H^2\left(\mathbb{C}^+\right):f\in C^\infty,\ f(z)=o\left(z^{-2}\right),\ z\to\infty,\ \operatorname{Im}z\geq 0\right\}, \tag{3.5}$$

dense [14] in $H^2(\mathbb{C}^+)$ by

$$\mathbb{H}(\varphi)f=\mathbb{J}\mathbb{P}_-\varphi f,\quad f\in\mathfrak{H}_2, \tag{3.6}$$

and then extend (3.6) to the whole $H^2(\mathbb{C}^+)$ retaining the same notation $\mathbb{H}(\varphi)$ for the extension.

Introduce the regularized Riesz projection

$$(\widetilde{\mathbb{P}}_\pm f)(x)=\pm\frac{1}{2\pi i}\int_{\mathbb{R}}\left(\frac{1}{s-(x\pm i0)}-\frac{1}{s+i}\right)f(s)\ ds,\quad f\in L^\infty(\mathbb{R}). \tag{3.7}$$

As well-known

$$\widetilde{\mathbb{P}}_\pm f\in\operatorname{BMOA}(\mathbb{C}^\pm)\ \text{if}\ \ f\in L^\infty(\mathbb{R}), \tag{3.8}$$

where $\mathrm{BMOA}(\mathbb{C}^{\pm})$ is the class of analytic in $\mathbb{C}^{\pm}$ functions having bounded mean oscillation:

$$\sup_{I\in\mathbb{R}} \frac{1}{|I|}\int_I |f(x)-f_I|\ dx<\infty,\quad f_I:=\frac{1}{|I|}\int_I f(x)\ dx,$$

for any bounded interval I. One has

$$\widetilde{\mathbb{P}}_+f+\widetilde{\mathbb{P}}_-f=f,\ f\in L^\infty(\mathbb{R}). \tag{3.9}$$

The next elementary statement will play a important role in our consideration.

Theorem 3.2. *If $\varphi\in L^\infty(\mathbb{R})$ then*

$$\mathbb{H}(\varphi)=\mathbb{H}(\widetilde{\mathbb{P}}_-\varphi). \tag{3.10}$$

Proof. As well-known [14] every BMOA function h is subject to $h(x)/\left(1+x^2\right)\in L^1(\mathbb{R})$ and one can easily see that $hf\in H^2(\mathbb{C}^+)$ if $f\in\mathfrak{H}_2$. Hence $\mathbb{P}_-hf=0$ and

$$\mathbb{H}(\varphi+h)f=\mathbb{H}(\varphi)f\quad\text{for any}\quad f\in\mathfrak{H}_2. \tag{3.11}$$

Therefore (3.11) can be closed to the whole $H^2(\mathbb{C}^+)$ and $\mathbb{H}(\varphi+h)$ is well-defined in the sense discussed above, bounded and

$$\mathbb{H}(\varphi+h)=\mathbb{H}(\varphi)$$

holds. By (3.9) $\varphi=\widetilde{\mathbb{P}}_-\varphi+\widetilde{\mathbb{P}}_+\varphi$ and (3.10) follows from (3.11) with $h=-\widetilde{\mathbb{P}}_+\varphi$ which, by (3.8), is in $\mathrm{BMOA}(\mathbb{C}^+)$. □

Directly from Definition 3.1, $\|\mathbb{H}(\varphi)\|\le\|\varphi\|_\infty$ but we will need much stronger statements.

Let us now introduce the Sarason algebra

$$H^\infty\left(\mathbb{C}^+\right)+C(\mathbb{R}):=\{f:f=h+g,\ h\in H^\infty\left(\mathbb{C}^+\right),\ g\in C(\mathbb{R})\},$$

where

$$C(\mathbb{R})=\left\{f:f\text{ is continuous on }\mathbb{R},\ \lim_{x\to\infty}f(x)=\lim_{x\to-\infty}f(x)\ne\pm\infty\right\}.$$

Theorem 3.3 (Sarason, 1967)**.** *$H^\infty(\mathbb{C}^+)+C(\mathbb{R})$ is a closed sub-algebra of $L^\infty(\mathbb{R})$.*

The set $H^\infty+C$ is one of the most common function classes in the theory of Hankel (and Toeplitz) operators due to the following fundamental theorem.

Theorem 3.4 (Hartman, 1958)**.** *Let $\varphi\in L^\infty(\mathbb{R})$. Then $\mathbb{H}(\varphi)$ is compact if and only if $\varphi\in H^\infty(\mathbb{C}^+)+C(\mathbb{R})$. I.e. $\mathbb{H}(\varphi)$ is compact if and only if $\mathbb{H}(\varphi)=\mathbb{H}(g)$ with some $g\in C(\mathbb{R})$.*

For Hankel operators appearing in completely integrable systems the membership of the symbol in $H^\infty(\mathbb{C}^+)+C(\mathbb{R})$ is far from being obvious. The following statement will be crucial to our approach.

Theorem 3.5 (Grudsky, 2001)**.** *Let $p(x)$ be a real polynomial with a positive leading coefficient such that*

$$p(-x)=-p(x). \tag{3.12}$$

Then

$$e^{ip}\in H^\infty\left(\mathbb{C}^+\right)+C(\mathbb{R}). \tag{3.13}$$

Moreover, there exist an infinite Blaschke product B and a unimodular function $u\in C(\mathbb{R})$ such that

$$e^{ip}=Bu. \tag{3.14}$$

In our case $p(\lambda) = t\lambda^3 + x\lambda$ with real x (spatial variable) and positive t (time). Note that for polynomials p of even order, Theorem 3.5 fails.

Definition 3.6. *A function $f \in H^\infty(\mathbb{C}^+) + C(\mathbb{R})$ is said invertible in $H^\infty(\mathbb{C}^+) + C(\mathbb{R})$ if $1/f \in H^\infty(\mathbb{C}^+) + C(\mathbb{R})$. Similarly, f is not invertible in $H^\infty(\mathbb{C}^+) + C(\mathbb{R})$ if $1/f \notin H^\infty(\mathbb{C}^+) + C(\mathbb{R})$.*

This concept is very important in the connection with invertibility of Toeplitz operators, as the following theorem suggests (see, e.g. [5], [9]).

Theorem 3.7. *Let $\varphi \in H^\infty(\mathbb{C}^+) + C(\mathbb{R})$ and $1/\varphi \in L^\infty(\mathbb{R})$. Then*

$$1/\varphi \notin H^\infty\left(\mathbb{C}^+\right) + C(\mathbb{R}) \Longrightarrow \mathbb{T}(\varphi) \text{ is left-invertible,} \tag{3.15}$$

$$1/\varphi \in H^\infty\left(\mathbb{C}^+\right) + C(\mathbb{R}) \Longrightarrow \mathbb{T}(\varphi) \text{ is Fredholm.} \tag{3.16}$$

Lemma 3.8. *Let B be an infinite Blaschke product, $u \in H^\infty(\mathbb{C}^+) + C(\mathbb{R})$ and unimodular. Then $\varphi = Bu$ is not invertible in $H^\infty(\mathbb{C}^+) + C(\mathbb{R})$.*

Proof. (By contradiction). Since $B \in H^\infty(\mathbb{C}^+)$, due to the algebraic property (Theorem 3.3) of $H^\infty(\mathbb{C}^+) + C(\mathbb{R})$, one has $\varphi \in H^\infty(\mathbb{C}^+) + C(\mathbb{R})$. Assume that φ is invertible in $H^\infty(\mathbb{C}^+) + C(\mathbb{R})$, i.e. $1/\varphi \in H^\infty(\mathbb{C}^+) + C(\mathbb{R})$. Then by (3.16) $\mathbb{T}(\varphi)$ is Fredholm that forces $\mathbb{T}(B)$ to be Fredholm too. Indeed, $B \in H^\infty(\mathbb{C}^+)$ and, since $\varphi = Bu$,

$$1/B = u \cdot 1/\varphi \in H^\infty\left(\mathbb{C}^+\right) + C(\mathbb{R}).$$

Thus B is invertible in $H^\infty(\mathbb{C}^+) + C(\mathbb{R})$ and (3.16) holds. Hence $\mathbb{T}(\overline{B}) = \mathbb{T}(1/B)$ is also Fredholm and therefore by definition

$$\dim\ker\mathbb{T}(\overline{B}) < \infty. \tag{3.17}$$

We now show that (3.17) may not hold for B with infinitely many zeros $\{z_k\}$, which creates a desired contradiction. To this end consider the Blaschke product

$$B(x) = \prod b_n(x),\ b_n = c_n\left(\frac{x - z_n}{x - \overline{z_n}}\right)$$

and set

$$f_n(x) = c_n(x - \overline{z_n})^{-1}.$$

Clearly $f_n \in H^2(\mathbb{C}^+)$ and

$$\mathbb{T}(\overline{B})f_n = \mathbb{P}_+\overline{B}f_n = \mathbb{P}_+\overline{\overline{c_n}(\cdot - z_n)^{-1}B} = \mathbb{P}_+(\cdot - z_n)^{-1}\overline{B_n},$$

where $B_n = B/b_n$. But $\overline{B_n} \in H^\infty(\mathbb{C}^-)$ and $(x - z_n)^{-1} \in H^2(\mathbb{C}^-)$. Hence

$$(x - z_n)^{-1}\overline{B_n(x)} \in H^2\left(\mathbb{C}^-\right)$$

and

$$\mathbb{T}(\overline{B})f_n = 0.$$

Therefore $f_n \in \ker\mathbb{T}(\overline{B})$ and the lemma is proven as $\{f_n\}$ are linearly independent. □

Note that Lemma 3.8 is entirely about $H^\infty(\mathbb{C}^+) + C(\mathbb{R})$ but its proof, as often happens in this circle of issues, relies on operator theoretical arguments.

The next important claim directly follows from Theorem 3.5 and Lemma 3.8.

Theorem 3.9. *Let u be a unimodular function from $H^\infty(\mathbb{C}^+) + C(\mathbb{R})$ and e^{ip} as in Theorem 3.5. Then $e^{ip}u$ is not invertible in $H^\infty(\mathbb{C}^+) + C(\mathbb{R})$.*

Combining Theorems 3.10 and

Theorem 3.10 (Widom, 1960). *Let φ be unimodular. Then $\|\mathbb{H}(\varphi)\| < 1$ if and only if $\mathbb{T}(\varphi)$ is left invertible.*

yields

Theorem 3.11. *If $\varphi \in H^\infty(\mathbb{C}^+) + C(\mathbb{R})$ and unimodular but not invertible then*

$$\|\mathbb{H}(\varphi)\| < 1. \tag{3.18}$$

Proof. By Theorem 3.7, $\mathbb{T}(\varphi)$ is left-invertible. By Theorem 3.10 we have (3.18). □

While an immediate consequence of Theorems 3.11 and 3.5, the following theorem is vital to our approach.

Theorem 3.12. *If $u \in H^\infty(\mathbb{C}^+) + C(\mathbb{R})$, $|u| = 1$, and p is as in Theorem 3.5, then*

$$\|\mathbb{H}(e^{ip}u)\| < 1.$$

Theorem 3.13. *If $\varphi \in H^\infty(\mathbb{C}^+) + C(\mathbb{R})$ is not unimodular but $\|\varphi\|_\infty \le 1$ and $\mathbb{J}\varphi = \bar{\varphi}$ then (3.18) holds.*

Proof. (By contradiction) Assume that $\|\mathbb{H}(\varphi)\| = 1$. Since $\mathbb{H}(\varphi)$ is selfadjoint and compact (by Theorems 3.5 and 3.4), $\mathbb{H}(\varphi)$ has a unimodular eigenvalue λ ($\lambda = \pm 1$). For the associated normalized eigenfunction $f \in H^2(\mathbb{C}^+)$ we have by (3.2)

$$\langle \mathbb{H}(\varphi)f, f\rangle = \langle \varphi f, \mathbb{P}_-\mathbb{J}f\rangle = \langle \varphi f, \mathbb{J}f\rangle$$

and hence by the Cauchy inequality

$$\begin{aligned} |\langle \mathbb{H}(\varphi)f, f\rangle|^2 &\le \left(\int_{\mathbb{R}} |\varphi(x) f(x) f(-x)| \; dx\right)^2 \\ &\le \int_{\mathbb{R}} |\varphi(x)| |f(x)|^2 dx \; \int_{\mathbb{R}} |f(-x)|^2 \; dx \\ &= \int_{\mathbb{R}} |\varphi(x)| |f(x)|^2 \; dx \\ &= \int_S |\varphi(x)| |f(x)|^2 \; dx + \int_{\mathbb{R}\setminus S} |\varphi(x)| |f(x)|^2 \; dx \qquad (3.19) \\ &< \int_{\mathbb{R}} |\varphi(x)| |f(x)|^2 \; dx = 1. \end{aligned}$$

Here S is a set of positive Lebesgue measure where $|\varphi(x)| < 1$ a.e. Here we have used the fact that $f \in H^2(\mathbb{C}^+)$ and hence cannot vanish on S. The inequality (3.19) implies that $|\lambda| < 1$ which is a contradiction. □

Finally, we note that the Hankel operator can also be defined as an integral operator on $L^2(\mathbb{R}_+)$ whose kernel depends on the sum of the arguments

$$(\mathbb{H}f)(x) = \int_{\mathbb{R}_+} h(x+y)f(y)dy, \; f \in L^2(\mathbb{R}_+), \; x \ge 0 \tag{3.20}$$

and it is this form that Hankel operators typically appear in the inverse scattering formalism. One can show that the Hankel operator $\mathbb{H}$ defined by (3.20) is unitary equivalent to $\mathbb{H}(\varphi)$ with the symbol φ equal to the Fourier transform of h. We emphasize though that the form (3.20) does not prove to be convenient for our purposes and also h is in general not a function but a distribution.

4. The IST Hankel Operator

For a reason which will become clear in the next section we introduce

Definition 4.1 (IST Hankel operator)**.** *Assume that initial data q is subject to Hypothesis 1.1. Let R be as in Definition 2.3. We call the Hankel operator*

$$\mathbb{H}(x,t) := \mathbb{H}(\varphi_{x,t}),$$

with the symbol

$$\varphi_{x,t}(k) = \xi_{x,t}(k)R(k), \tag{4.1}$$

the IST Hankel operator associated with q.

We need some general statements on singular numbers of Hankel operators. We recall that the n-th singular value $s_n(\mathbb{A})$ of a compact Hilbert space operator $\mathbb{A}$ is defined as the n-th eigenvalue of the operator $(\mathbb{A}^*\mathbb{A})^{1/2}$.

The following theorems are fundamental in the study of singular numbers of Hankel operators.

Theorem 4.2 (Adamyan-Arov-Krein, 1971)**.** *Let $\varphi \in L^\infty(\mathbb{R})$. Then*

$$s_n(\mathbb{H}(\varphi)) = \operatorname{dist}_{L^\infty}\left(\varphi, \mathcal{R}_n + H^\infty\left(\mathbb{C}^+\right)\right),$$

where $\mathcal{R}_n$ is the set of rational functions bounded at infinity with all poles in $\mathbb{C}^+$ of total multiplicity $\leq n$.

Theorem 4.3 (Jackson, 1910)**.** *Let $\varphi \in C^m(\mathbb{R})$. Then*

$$\operatorname{dist}_{L^\infty}\left(\varphi, \mathcal{R}_n + H^\infty\left(\mathbb{C}^+\right)\right) \lesssim \left\|\varphi^{(m)}\right\|_\infty / n^m.$$

Theorems 4.2 and 4.3 immediately yield the following observation.

Lemma 4.4. *Let $f \in L^1(\mathbb{R})$, $h > 0$ and*

$$\varphi(k) = \int_{\mathbb{R}} \frac{f(s)}{s-k+ih} ds. \tag{4.2}$$

Then

$$s_n(\mathbb{H}(\varphi)) \lesssim (2/h)\|f\|_1 \exp\{-(h/2)\,n\}.$$

Proof. Differentiating (4.2) one has

$$\left\|\varphi^{(m)}\right\|_\infty \leq \|f\|_1 \frac{m!}{h^{m+1}}$$

and hence by Theorems 4.2 and 4.3 for any $m = 0, 1, 2, \ldots$

$$s_n(\mathbb{H}(\varphi)) \lesssim \frac{\|f\|_1}{h} \frac{m!}{(hn)^m}.$$

Rewriting the last estimate as $s_n(\mathbb{H}(\varphi)) \dfrac{m!}{(2hn)^m} \lesssim \dfrac{\|f\|_1}{h} 2^{-m}$ and summing up on m implies the desired result. □

Here is the main result of this section

Theorem 4.5 (Properties of the IST Hankel operator)**.** *Under Hypothesis 1.1 the IST Hankel operator $\mathbb{H}(x,t)$ is well-defined and has the properties: for any $x \in \mathbb{R}$, $t > 0$*

(1) *$\mathbb{H}(x,t)$ is selfadjoint,*

(2) $\mathbb{H}(x,t)$ *is compact and its singular numbers* $s_n(\mathbb{H}(x,t))$ *satisfy*

$$s_n(\mathbb{H}(x,t)) \lesssim \frac{2}{h}\left\{\int_{\mathbb{R}} |\xi_{x,t}(\lambda+ih)R(\lambda+ih)|\,d\lambda\right\}\ \exp\left\{-\frac{2n}{h}\right\} \tag{4.3}$$

for any $h>0$.

(3) $\|\mathbb{H}(x,t)\|<1$.

(4) $\partial_t^m \mathbb{H}(x,t)$, $m=0,1$, *is an entire in* x *operator-valued function.*

Proof. Statement (1) is obvious as $\mathbb{J}\varphi_{x,t}=\overline{\varphi_{x,t}}$. We now prove statement (2). By Theorem 3.2 ($\varphi_{x,t}=\xi_{x,t}R$)

$$\mathbb{H}(x,t)=\mathbb{H}(\varphi_{x,t})=\mathbb{H}(\widetilde{\mathbb{P}}_{-}\varphi_{x,t})$$

where

$$\left(\widetilde{\mathbb{P}}_{-}\varphi_{x,t}\right)(k)=-\frac{1}{2\pi i}\int_{\mathbb{R}}\left(\frac{1}{\lambda-(k-i0)}-\frac{1}{\lambda+i}\right)\varphi_{x,t}(\lambda)\ d\lambda. \tag{4.4}$$

The function $\varphi_{x,t}$ is clearly analytic in $\mathbb{C}^+$. Since $R\in H^\infty(\mathbb{C}^+)$ with $\|R\|_\infty\le 1$ and $\xi_{x,t}(\lambda+ih)$ rapidly decays as $\lambda\to\pm\infty$ for any $t>0$ and arbitrary $h>0$, we can deform the contour of integration in (4.4) to $R+ih,\ h>0$. Thus

$$\left(\widetilde{\mathbb{P}}_{-}\varphi_{x,t}\right)(k)=-\frac{1}{2\pi i}\int_{\mathbb{R}+ih}\frac{\varphi_{x,t}(\lambda)}{\lambda-k}\ d\lambda-\int_{\mathbb{R}+ih}\frac{\varphi_{x,t}(\lambda)}{\lambda+i}\ d\lambda.$$

Since the last term is a constant, we conclude that

$$\mathbb{H}(x,t)=\mathbb{H}(\Phi_{x,t}) \tag{4.5}$$

with an entire function

$$\Phi_{x,t}(k):=-\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{\xi_{x,t}(\lambda+ih)R(\lambda+ih)}{\lambda-k+ih}\ d\lambda. \tag{4.6}$$

By Theorem 3.4 operator $\mathbb{H}(x,t)$ is compact. By Lemma 4.4 yields (4.3).

Statement (3) follows from Theorem 3.13 if $|R(\lambda)|<1$ on a set of positive Lebesgue measure, or Theorem 3.12 if $|R(\lambda)|=1$ a.e.

It remains to prove statement (4). One can easily see from the straightforward formula ($t>0$)

$$|\xi_{z,t}(\lambda+ih)|=\exp\left\{8h^3t-2h\operatorname{Re}z+\frac{\operatorname{Im}^2 z}{24ht}-\left(\sqrt{24ht}\lambda+\frac{\operatorname{Im}z}{\sqrt{24ht}}\right)^2\right\} \tag{4.7}$$

and (4.6) that $\partial_t^m\Phi_{z,t}(k)$ is well-defined for any complex z. I.e. it is also entire in z for any $t>0$. Therefore the operator-valued function $\partial_t^m\mathbb{H}(x,t)$ defined by

$$\partial_t^m\mathbb{H}(x,t)=\mathbb{H}(\partial_t^m\Phi_{x,t})$$

is also entire. □

We conclude this section with a few remarks.

Remark 4.6. *Statement (3) of Theorem 4.5 says that* $(\mathbb{I}+\mathbb{H}(x,t))^{-1}$ *is a bounded operator on* $H^2(\mathbb{C}^+)$ *for any* $x\in\mathbb{R}$ *and* $t>0$*, which is of course of a particular importance for validation of the IST. A weaker versions of this theorem (stated in different terms) was proven in* [16] *(which in turn improved* [33]*).*

5. Main Results

In this section we finally state and prove our main results. With all the preparations done in the previous sections, the actual proof will be quite short.

Note that while the interest to well-posedness of integrable systems has been generated by the progress in soliton theory, well-posedness issues are typically approached by means of PDEs techniques [36] (norm estimates, etc.) and the IST is not usually employed. In soliton theory, in turn, well-posedness is commonly assumed (frequently even by default) and one applies the IST method to study the unique solution to (1.1) or any other integrable system. The paper [22] represents a rather rare example where the complete integrability of (1.1) with periodic initial data was used in a crucial way to prove some subtle well-posedness results for irregular q which are not accessible by harmonic analysis means. In our case neither a priori well-posedness nor IST are readily available and we have to deal with both at the same time.

Solutions of the KdV can be understood in a number of different ways [36] (classical, strong, weak, etc.) resulting in a variety of different well-posedness results.

Definition 5.1 (Natural solution)**.** *We call $q\,(x,t)$ a global natural solution to (1.1) if for any sequence of $C_0^\infty\,(\mathbb{R})$ potentials $\{q_n\,(x)\}$ converging to $q\,(x)$ in $H^{-1}_{\mathrm{loc}}\,(\mathbb{R})$, the corresponding sequence of (classical) solutions $\{q_n\,(x,t)\}$ to (1.1) with initial data $q_n\,(x)$ converges to $q\,(x,t)$ for any $t>0$ uniformly in x on compacts of $\mathbb{R}$.*

Our definition is a stronger version of that in [22]. It also looks quite natural from the computational and physical point of view. Another feature of Definition 5.1 is that existence implies uniqueness[5]

Theorem 5.2 (Main Theorem)**.** *Assume that the initial data q in (1.1) is subject to Hypothesis 1.1. Then the Cauchy problem (1.1) has a global natural solution $q(x,t)$ (Definition 5.1) given by*

$$q(x,t) = -2\partial_x^2 \log\det\left(1+\mathbb{H}(x,t)\right), \tag{5.1}$$

where $\mathbb{H}(x,t)$ is the IST Hankel operator associated with q (Definition 4.1). The solution $q(x,t)$ has no singularities and admits a meromorphic continuation $q(z,t)$ to the whole $\mathbb{C}$ with no poles in parabolic domains

$$D\,(\delta,t) := \left\{ z : \frac{\operatorname{Im}^2 z}{12} < \delta \operatorname{Re} z - \delta^2 + \frac{\sqrt{\delta t}}{4}\log\frac{t}{\delta^3} \right\} \tag{5.2}$$

for any $t,\delta>0$.

Proof. Let $\{q_n\,(x)\}$ be any real $C_0^\infty\,(\mathbb{R})$ sequence converging to $q\,(x)$ in $H^{-1}_{\mathrm{loc}}\,(\mathbb{R})$. Without loss of generality we may assume that $q_n\,(x)$ is of form (1.5). The problem (1.1) with initial data $q_n\,(x)$ is classical and its (unique) classical solution $q_n\,(x,t)$ and can be computed by the Dyson formula

$$q_n\,(x,t) = -2\partial_x^2 \log\det\left(\mathbb{I}+\mathbb{H}_n(x,t)\right),$$

where $\mathbb{H}_n(x,t)$ is the IST Hankel operator corresponding to q_n. By Theorem 4.5, $q_n\,(x,t)$ is a meromorphic function in x on the entire complex plane. Consider the function $q\,(x,t)$ given by (5.1). By Theorem 4.5, it is well defined and entire in x

[5] and certain continuous dependence on the initial data which we don't discuss here.

for any $t>0$. It remains to prove that $q(x,t)=\lim q_n(x,t)$, $n\to\infty$, solves (1.1). Inserting $q=q_n+\Delta q_n$ into (1.1) one gets

$$\begin{aligned}&\partial_t q-6q\partial_x q+\partial_x^3 q\\&=\partial_t\Delta q_n+3\partial_x\left[\left(\Delta q_n-2q\right)\Delta q_n\right]+\partial_x^3\Delta q_n.\end{aligned}\tag{5.3}$$

For Δq_n we have (dropping subscript x,t)

$$\Delta q_n=-2\partial_x^2\log\det\left(\mathbb{I}-\left(\mathbb{I}+\mathbb{H}\right)^{-1}\left(\mathbb{H}-\mathbb{H}_n\right)\right).$$

It follows form (4.5) and (4.6) that for the symbol $\Delta\Phi_n$ of $\mathbb{H}-\mathbb{H}_n$ we have

$$\Delta\Psi_n(k)=\frac{1}{2\pi i}\int_{\mathbb{R}}\xi_{x,t}(\lambda+ih)\frac{R_n(\lambda+ih)-R(\lambda+ih)}{\lambda-k+ih}d\lambda.\tag{5.4}$$

But, by Theorem 2.4 $R_n\to R$ uniformly on compacts in $\mathbb{C}^+$ as $n\to\infty$ and we can easily conclude that $\partial_t^m\partial_x^l\left(\mathbb{H}-\mathbb{H}_n\right)$ vanishes in the trace norm as $n\to\infty$. Therefore $\partial_t^m\partial_x^l\Delta q_n\to 0, n\to\infty$ and the right hand side of (5.3) vanishes.

By Theorem 4.5 $q(x,t)$ is meromorphic in x for any $t>0$ with all poles off the real line. It remains to show that it has no poles in the domain (5.2). It follows from (4.5) and (4.6) that

$$\begin{aligned}\|\mathbb{H}(z,t)\|&\le\|\Phi_{z,t}\|_\infty\\&\le\frac{1}{2\pi h}\int|\xi_{z,t}(\lambda+ih)|\,d\lambda.\end{aligned}$$

In virtue of (4.7)

$$\int_{\mathbb{R}}|\xi_{z,t}(\lambda+ih)|\,d\lambda=\sqrt{\frac{\pi}{24ht}}\exp\left\{8h^3t-2h\operatorname{Re}z+\frac{\operatorname{Im}^2 z}{24ht}\right\}$$

and hence for any $t,h>0$.

$$\|\mathbb{H}(z,t)\|\le\sqrt{\frac{1}{24\pi h^3t}}\exp\left\{8h^3t-2h\operatorname{Re}z+\frac{\operatorname{Im}^2 z}{24ht}\right\}.\tag{5.5}$$

The right hand side of (5.5) is less than 1 if $z\in D(\delta,t)$ with $\delta=4h^2t$. Since h is arbitrary δ is also arbitrary $\square$

In a weaker form for regular initial profiles Theorem 5.2 was proven in recent [16]. We conclude our paper with some discussions and corollaries.

Remark 5.3. *Hypothesis 1.1 does not impose any decay assumption at $-\infty$ or any type of pattern of behavior. Initial data $q(x)$ could be unbounded at $-\infty$ or behave like white noise.*

Remark 5.4. *We emphasize that our proof is based on limiting arguments and avoids dealing directly with such common in the classical IST issues as the direct/inverse scattering problem, time evolution of scattering quantities under the KdV flow, etc. (see* [19] *and the literature cited therein for some results relevant to our singular initial data). This is the main advantage of our approach and we only borrow the fact that the initial condition is satisfied in $H^{-1}_{\mathrm{loc}}(\mathbb{R})$ sense* [23].

Remark 5.5. *The estimate (4.3) means that the determinant in (5.1) rapidly converges. This fact, coupled with the recent progress in computing Fredholm determinants* [4], *suggests that (5.1) could potentially be used for numerical evaluations.*

Remark 5.6. *Theorem 5.2 says that any, no matter how rough, singular initial profile $q(x)$ instantaneously evolves under the KdV flow into a meromorphic function $q(x,t)$. This effect, also called dispersive smoothing, has a long history. While being noticed long ago, its rigorous proof took quit a bit of effort even for box shaped initial data* [26] *(see also* [39] *for other integrable systems). Note that our solutions are 'dispersive', i.e. solutions which disperse in time and do not have a soliton component.*

Remark 5.7. *Since $q(x,t)$ is a meromorphic function on $\mathbb{C}$ for any $t>0$, it is completely characterized by a countable number of time dependent parameters. Viewing a pure soliton solution as a meromorphic function of x goes back to Kruskal. In* [24] *he initiated a study of pole dynamics which has been quite active since then (see also* [1], [3], [8], [15] *to mention just four). In our soliton free situation we still in general have infinitely many poles but their nature and behavior are unclear. We so far only know that all poles are double, non-real, come in complex conjugate pairs, and stay away from the time dependant domains $D(\delta,t)$ given by (5.2). Besides this, some older general results* [35] *say that poles depend continuously on t and cannot appear or disappear. We are unaware of any relevant helpful results from the theory of Hankel operators which would shed much light on the operator-valued function $(\mathbb{I}+\mathbb{H}(x,t))^{-1}$.*

Remark 5.8. *As a meromorphic function $q(x,t)$ cannot vanish on a set of positive Lebesgue measure for any $t>0$ unless $q(x)$ is identically zero. This simple observation quickly recovers and improves on many unique continuation results. E.g., $q(x,t)$ cannot have compact support at two different moments unless it vanishes identically. This result was first proven in* [38] *assuming that $q(x)$ is absolutely continuous and short range. The techniques of* [38] *also rely on the IST and some Hardy space arguments.*

Remark 5.9. *There is a large variety of determinant formulas similar to (5.1) available in the literature. For instance, the substitution $q(x,t)=-2\partial_x^2\tau(x,t)$ (which goes back to the seminal paper* [18]*) is commonly used as an ansatz to reduce the KdV equation to the so-called bilinear KdV which is advantageous in some situations. Formulas like (5.1) are particularly convenient for describing classes of exact solutions (see, e.g.* [25]*) and $\tau(x,t)$ typically appears as a Wronskian. We also refer to* [13], [27], [29], *and* [37] *for (5.1) in the context of the Cauchy problem for the KdV. Under our conditions on the initial data (5.1) is new.*

Remark 5.10. *It can be easily shown that $q(x,t)$ decays exponentially fast in the region $x\geq Ct$ for any $C>0$.*

6. Acknowledgement

We are grateful to Rostislav Hryniv for valuable discussions.

References

[1] Airault, H; McKean, H.; Moser, J. *Rational and elliptic solutions of the Korteweg–de Vries equation and a related many-body problem*, Commun. Pure Appl. Math. 30, (1977), 95–148.

[2] Albeverio, S.; Kostenko, A.; Malamud, M. *Spectral theory of semibounded Sturm-Liouville operators with local interactions on a discrete set.* J. Math. Phys. 51 (2010), no. 10, 102102, 24 pp.

[3] Bona, J. L.; Weissler, F. B. *Pole dynamics of interacting solitons and blowup of complex-valued solutions of KdV.* Nonlinearity 22 (2009), no. 2, 311–349.

[4] Bornemann, F. *On the numerical evaluation of Fredholm determinants.* Math. Comp. 79 (2010), no. 270, 871–915.

[5] Bötcher, A.; Silbermann B. *Analysis of Toeplitz operators.* Springer-Verlag, Berlin, 2002. 665 pp.

[6] Cohen, Amy *Solutions of the Korteweg-de Vries equation with steplike initial profile.* Comm. Partial Differential Equations 9 (1984), no. 8, 751–806.

[7] Colliander, J.; Keel, M.; Staffilani, G.; Takaoka, H.; Tao, T. *Sharp global well-posedness for KdV and modified KdV on R and T.* J. Amer. Math. Soc. 16 (2003), no. 3, 705–49. Basel-Boston, Mass., 1981. x+234 pp. ISBN: 3-7643-1297-1.

[8] Deconinck, B.; Segur, H. *Pole dynamics for elliptic solutions of the Korteweg-de Vries equation.* Math. Phys. Anal. Geom. 3 (2000), no. 1, 49–74.

[9] Dybin, V.; Grudsky S. *Introduction to the theory of Toeplitz operators with infinite index.* Birkhäuser Verlag, Basel, 2002. xii+299 pp.

[10] Dyson F. J., *Fredholm determinants and inverse scattering problems*, Comm. Math. Phys., 1976, 47(2), 171–183.

[11] Eckhardt, J.; Gesztesy, F.; Nichols, R.; Teschl, G. *Weyl-Titchmarsh theory for Sturm-Liouville operators with distributional potentials.* Opuscula Math. 33 (2013), no. 3, 467–563.

[12] Egorova, I.; Grunert K.; Teschl G. *On the Cauchy problem for the Korteweg-de Vries equation with steplike finite-gap initial data I. Schwarz-type perturbations.* Nonlinearity 22 (2009), 1431–57.

[13] Ercolani, N.; McKean, H. P. *Geometry of KdV. IV. Abel sums, Jacobi variety, and theta function in the scattering case.* Invent. Math. 99 (1990), no. 3, 483–544.

[14] Garnett, J. B. *Bounded analytic functions. Revised first edition.* Graduate Texts in Mathematics, 236. Springer, New York, 2007. xiv+459 pp.

[15] Gesztesy, F.; Unterkofler, K.; Weikard, R. *An explicit characterization of Calogero-Moser systems.* Trans. Amer. Math. Soc. 358 (2006), no. 2, 603–656.

[16] Grudsky, S; Rybkin, A. *On positive type initial profiles for the KdV equation.* Proc Amer Math Soc., 142 (2014), 2079–2086.

[17] Guo, Zihua *Global Well-posedness of Korteweg-de Vries equation in $H^{-3/4}(\mathbb{R})$.* J. Math. Pures Appl. (9) 91 (2009), no. 6, 583–97.

[18] Hirota, R. *Exact solution of the Korteweg de Vries equation for multiple collisions of solitons.* Phys. Rev. Lett. 27 (1971), 1192–1194.

[19] Hryniv, R.; Mykytyuk, Y.; Perry, P. *Sobolev mapping properties of the scattering transform for the Schrödinger equation.* Spectral theory and geometric analysis, 79–93, Contemp. Math., 535, Amer. Math. Soc., Providence, RI, 2011.

[20] Nikolski, N. K. *Operators, functions, and systems: An easy reading. Volume 1: Hardy, Hankel and Toeplitz.* Mathematical Surveys and Monographs, vol. 92, Amer. Math. Soc., Providence, 2002. 461 pp.

[21] Kappeler, T. *Solutions to the Korteweg-de Vries equation with irregular initial data.* Comm. Partial Diff. Eq. 11 (1986), 927–45.

[22] Kappeler, T.; Topalov, P. *Global wellposedness of KdV in $H^{-1}(\mathbb{T},\mathbb{R})$.* Duke Math. J. Volume 135, Number 2 (2006), 327-36.

[23] Kappeler, T.; Perry, P.; Shubin, M.; Topalov, P. *The Miura map on the line.* Int. Math. Res. Not. (2005), no. 50, 3091–133.

[24] Kruskal, Martin D. *The Korteweg-de Vries equation and related evolution equations.* Nonlinear wave motion (Proc. AMS-SIAM Summer Sem., Clarkson Coll. Tech., Potsdam, N.Y., 1972), pp. 61–83. Lectures in Appl. Math., Vol. 15, Amer. Math. Soc., Providence, R.I., 1974.

[25] Ma, Wen-Xiu; You, Yuncheng *Solving the Korteweg-de Vries equation by its bilinear form: Wronskian solutions.* Trans. Amer. Math. Soc. 357 (2005), no. 5, 1753–1778.

[26] Murray, Amy Cohen *Solutions of the Korteweg-de Vries equation from irregular data.* Duke Math. J. 45 (1978), no. 1, 149–181.

[27] Novikov, S.; Manakov, S. V.; Pitaevskiĭ, L. P.; Zakharov, V. E *Theory of solitons. The inverse scattering method.* Translated from the Russian. Contemporary Soviet Mathematics. Consultants Bureau [Plenum], New York, 1984. xi+276 pp. ISBN: 0-306-10977-8.

[28] Peller, V. V. *Hankel operators and their applications. Springer Monographs in Mathematics.* Springer-Verlag, New York, 2003. xvi+784 pp. ISBN: 0-387-95548-8.

[29] Pöppe, Ch., *The Fredholm determinant method for the KdV equations*, Phys. D, 1984, 13(1–2), 137–160.

[30] Remling, C. *Generalized reflection coefficients*, to appear in Comm. Math. Phys.

[31] Rybkin, A. *Spatial analyticity of solutions to integrable systems. I. The KdV case* Communications in Partial Differential Equations, Volume 38 (2013), Issue 5, 802-822.

[32] Rybkin, A. *The Hirota τ-function and well-posedness of the KdV equation with an arbitrary step like initial profile decaying on the right half line*, Nonlinearity 24 (2011), 2953-2990.

[33] Rybkin, A. *Meromorphic solutions to the KdV equation with non-decaying initial data supported on a left half line*. Nonlinearity 23 (2010), no. 5, 1143–1167.

[34] Savchuk, A.M.; Shkalikov, A.A. *Sturm-Liouville operators with distribution potentials.* Trans. Moscow Math. Soc. 64 (2003), 143–92.

[35] Steinberg, S. *Meromorphic families of compact operators.* Arch. Ration. Mech. Anal. 31 (1968/19693) 72–9.

[36] Tao, T. *Nonlinear dispersive equations.* Local and global analysis. CBMS Regional Conference Series in Mathematics, 106. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. xvi+373 pp. ISBN: 0-8218-4143-2.

[37] Venakides, S. *Long time asymptotics of the Korteweg - de Vries equation.* Trans. Amer. Math. Soc. 293 (1986), no. 1, 411–419.

[38] Zhang, Bing Yu *Unique continuation for the Korteweg-de Vries equation.* SIAM J. Math. Anal. 23 (1992), no. 1, 55–71.

[39] Zhou, Xin. *Strong regularizing effect of integrable systems.* Comm. Partial Differential Equations 22 (1997), no. 3-4, 503–526.

Departamento de Matematicas, CINVESTAV del I.P.N. Aportado Postal 14-740, 07000 Mexico, D.F., Mexico

E-mail address: grudsky@math.cinvestav.mx

Department of Mathemics, University of Oklahoma, Norman, OK 73019

E-mail address: cremling@math.ou.edu

Department of Mathematics and Statistics, University of Alaska Fairbanks, PO Box 756660, Fairbanks, AK 99775

E-mail address: arybkin@alaska.edu