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Lifting from $\widetilde{\mathrm{SL}}(2)$ to $\mathrm{GSpin}(1,4)$

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1 Introduction

In [15], Ikeda defines a lifting of automorphic forms from $\widetilde{\mathrm{SL}}_2$ to the symplectic group GSp_{4n} for all n. Ikeda proves his result using the theory of Fourier-Jacobi forms. For $n=1$ he shows that his definition agrees with the classical Saito-Kurokawa lift. In [7], Duke and Imamoḡlu prove the automorphy of the classical Saito-Kurokawa lift from holomorphic half-integer weight forms using the converse theorem for Sp_4 due to Imai [16].

In this paper we present the lifting of automorphic forms from $\widetilde{\mathrm{SL}}_2$, the metaplectic group, to the Spin group $\mathrm{GSpin}(1,4)$ using the Maaß converse theorem [24]. The reason we choose to study lifts to the spin group is the following observation: for all primes $p \neq 2$, we have that $\mathrm{GSp}_4(\mathbb{Q}_p)$ is isomorphic to $\mathrm{GSpin}(1,4)(\mathbb{Q}_p)$ (see Proposition 6.3). This relation between GSp_4 and $\mathrm{GSpin}(1,4)$ suggests the possibility of developing liftings for the spin group analogous to Ikeda's liftings for the symplectic group. Another motivation for considering spin groups is to give applications for the Maaß converse theorem. As far as we know, the Maaß converse theorem has been applied only once by Duke [6] to show that a noncuspidal Theta series is automorphic for the group $\mathrm{GSpin}(1,5)$. In this paper we obtain a family of cuspidal automorphic forms for $\mathrm{GSpin}(1,4)$ using the converse theorem.

To define the lifting, we start with a weight $1/2$ Maaß Hecke eigenform f on the complex upper half plane with respect to $\Gamma_0(4)$. The candidate function F on a symmetric space of $\mathrm{GSpin}(1,4)$ is defined in terms of a Fourier expansion with Fourier coefficients

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related to the Fourier coefficients of f as in the formula (3.4). The Maaß converse Theorem 2.3 states that the function F is cuspidal automorphic with respect to the integer points of $\mathrm{GSpin}(1,4)$ if and only if a family of Dirichlet series is "nice" ("nice" means analytic continuation, bounded on vertical strips, and functional equation). To obtain the "nice" properties of the Dirichlet series, we use the method of Duke and Imamoḡlu in [7] to rewrite the Dirichlet series in terms of a Rankin triple integral of the function f, a certain Theta function, and an Eisenstein series with respect to $\Gamma_0(4)$. Then essentially the "nice" properties of the Eisenstein series give us the corresponding "nice" properties of the Dirichlet series.

Another interesting result of the paper is related to the nonvanishing of the lift. In [15] it is shown that the nonvanishing of the Ikeda lift is equivalent to the nonvanishing of certain Fourier coefficients of the holomorphic half-integer weight form, which follows from a straightforward calculation in [18]. In our case, the nonvanishing of the lift F is equivalent to the nonvanishing of certain negative Fourier coefficients of f. It seems that it is not possible to get nonvanishing of specifically negative Fourier coefficients of f using elementary methods as in [18]. To achieve this we use the work of Baruch and Mao [2] to prove a Waldspurger-type (or Kohnen-Zagier-type) formula which relates the square of the negative Fourier coefficient of f to special values of L-functions (Theorem 4.3). Then we get the nonvanishing of the lift F by the results of Friedberg and Hoffstein [10] on nonvanishing of special values of L-functions.

We also analyze the adelic automorphic representation π_F of the group $\mathrm{GSpin}(1,4)(\mathbb{A})$ obtained from the automorphic function F. To do this, we first show that if f is a Hecke eigenfunction, then so is F and explicitly calculate the eigenvalues of F in terms of the eigenvalues of f. Since the p-adic component (p an odd prime) of π_F is an irreducible unramified representation of $\mathrm{GSpin}(1,4)(\mathbb{Q}_p)$ ($\simeq \mathrm{GSp}_4(\mathbb{Q}_p)$), we know that it is the unique spherical constituent of the representation obtained by induction from an unramified character on the Borel subgroup. In Theorem 6.5, we obtain the explicit description of this character in terms of the eigenvalue of f using the Hecke theory for the Spin group.

Using the precise information about the local representations, we show that the representation π_F is a CAP representation. Let us remind the reader that if G is a reductive algebraic group and P a parabolic subgroup, then an irreducible cuspidal automorphic representation π of $G(\mathbb{A})$ is called *cuspidal associated to parabolic* (CAP) subgroup P if there is an irreducible cuspidal automorphic representation σ of the Levi subgroup of P such that π is nearly equivalent to an irreducible component of $\mathrm{Ind}_P^G(\sigma)$. The notion of CAP representations was first introduced by Piatetski-Shapiro [27] for the group Sp_4. The CAP representations are very interesting because they provide counterexamples to the generalized Ramanujan conjecture. CAP representations on Sp_4 have been extensively studied by Piatetski-Shapiro in [27] and Soudry in [34, 35]. In [11, 28], families of CAP representations on the split exceptional group of type G_2 have been constructed. In [12] the authors give a criterion for an irreducible cuspidal automorphic representation of a split group of type D_4 to be a CAP representation. In these papers the method used to construct CAP representations is theta lifting of various types. In [15], Ikeda shows that the lift he obtains is a CAP representation.

The representation π_F constructed here is interesting in this context because it is CAP to a representation of $\mathrm{GSp}_4(\mathbb{A})$ instead of $\mathrm{GSpin}(1,4)(\mathbb{A})$ (Theorem 6.7). If one considers Langland's philosophy, then it is natural to extend the notion of CAP representations to the situation where the group G is replaced by two groups G_1, G_2 which satisfy $G_{1,\nu} \backsimeq G_{2,\nu}$ for almost all ν which is the case in our present setting.

2 Preliminaries

Our main tool to prove the automorphy is the Maaß converse theorem which is stated in terms of Vahlen matrices. So we will get a realization of a symmetric space for the group Spin$(1,4)$ in terms of Vahlen matrices which we define below. Then we will define automorphic functions and state the Maaß converse theorem. We end the section with the definition and basic properties of half-integer weight Maaß forms. The main references for Sections 2.1 and 2.2 are [8, 9, 24] where the authors have considered the general case of Spin$(1,n)$. Here we will specialize their notation and results to the case Spin$(1,4)$.

2.1 Vahlen matrices

Let $C_2(\mathbb{R}) := \{\alpha = \alpha_0 + \alpha_1 i_1 + \alpha_2 i_2 + \alpha_3 i_1 i_2 : \alpha_j \in \mathbb{R},\ j = 0,1,2,3 \text{ and } i_1^2 = i_2^2 = -1,\ i_1 i_2 = -i_2 i_1\}$ be the algebra of real quaternions. For α as above define $\alpha^* := \alpha_0 + \alpha_1 i_1 + \alpha_2 i_2 - \alpha_3 i_1 i_2$, $\alpha' := \alpha_0 - \alpha_1 i_1 - \alpha_2 i_2 + \alpha_3 i_1 i_2$, $\bar{\alpha} := \alpha_0 - \alpha_1 i_1 - \alpha_2 i_2 - \alpha_3 i_1 i_2$, and $\mathrm{Norm}(\alpha) = |\alpha|^2 := \alpha\bar{\alpha}$.

The Vahlen group of matrices is defined by

$$\mathrm{SV}_2(\mathbb{R}) := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2\big(C_2(\mathbb{R})\big) : \alpha\delta^* - \beta\gamma^* = 1,\ \alpha\beta^*, \delta\gamma^* \in V_2 \right\}, \tag{2.1}$$

where $V_2 := \mathbb{R} + \mathbb{R}i_1 + \mathbb{R}i_2$. It is shown in [8, page 377] that $\mathrm{SV}_2(\mathbb{R})$ is generated by the matrices $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right)$ with $\beta \in V_2$. Also, in [8, page 382] it is shown that

$$\mathrm{SV}_2(\mathbb{R}) \simeq \mathrm{Spin}(1,4). \tag{2.2}$$

Define the 4-dimensional hyperbolic upper-half space as

$$\mathbb{H}_3 := \{x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3 : x_j \in \mathbb{R} \text{ for } j = 0, \dots, 3 \text{ and } x_3 > 0\}. \tag{2.3}$$

Here i_3 satisfies $i_3^2 = -1$ and $i_3 i_j = -i_j i_3$ for $j = 1, 2$. We consider $C_2(\mathbb{R})$ and $\mathbb{H}_3$ to be contained in the Clifford algebra over $\mathbb{R}$ generated by the units i_1, i_2, i_3 with relations $i_j^2 = -1$, $i_j i_k = -i_k i_j$ for $j \neq k$. The line element $ds^2 = (dx_0^2 + \cdots + dx_3^2)/x_3^2$ defines a Riemannian metric d on $\mathbb{H}_3$. The Laplace Beltrami operator is given by

$$\Omega_3 := x_3^{3+1} \sum_{j=0}^{3} \frac{\partial}{\partial x_j}\left(x_3^{-3+1} \frac{\partial}{\partial x_j}\right) = x_3^2 \sum_{j=0}^{3} \frac{\partial^2}{\partial x_j^2} - (3-1) x_3 \frac{\partial}{\partial x_3}. \tag{2.4}$$

We now state the result on isometries of $\mathbb{H}_3$ [8, page 381].

Proposition 2.1. If $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in SV_2(\mathbb{R})$ and $z \in \mathbb{H}_3$, then $\gamma z + \delta \in \mathbb{R} + \mathbb{R}i_1 + \mathbb{R}i_2 + \mathbb{R}i_3$ and

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot z := (\alpha z + \beta)(\gamma z + \delta)^{-1} \in \mathbb{H}_3. \tag{2.5}$$

Formula (2.5) defines an action of $SV_2(\mathbb{R})$ on $\mathbb{H}_3$ by orientation preserving isometries. This action keeps the metric d and the Laplace Beltrami operator Ω_3 invariant. The resulting sequence $1 \to \{1, -1\} \to SV_2(\mathbb{R}) \to \mathrm{Iso}^+(\mathbb{H}_3) \to 1$ is exact. Here $\mathrm{Iso}^+(\mathbb{H}_3)$ is the set of orientation preserving isometries of $\mathbb{H}_3$. □

2.2 Automorphic functions and MCT

Let us now define automorphic functions for $SV_2(\mathbb{R})$. Fix the subgroup Γ_T of $SV_2(\mathbb{R})$ defined by

$$\Gamma_T := \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} : \beta \in T \right\rangle, \tag{2.6}$$

where T is a fixed lattice in V_2.

Definition 2.2 (automorphic function). A complex-valued C^∞-function F on $\mathbb{H}_3$ is called *automorphic* with respect to Γ_T if

(1) $\Omega_3 F + (r^2 + 3^2/4) F = 0$ for some $r \in \mathbb{R}$,

(2) for certain positive constants κ_1 and κ_2, $F(x) = O(x_3^{\kappa_1})$ for $x_3 \to \infty$ and $F(x) = O(x_3^{-\kappa_2})$ for $x_3 \to 0$ uniformly on x_0, x_1, x_2,

(3) $F(\gamma x) = F(x)$ for all $\gamma \in \Gamma_T$, $x \in \mathbb{H}_3$.

As given in [24], the conditions (1) and (2) together with the invariance under the translations in Γ_T are equivalent to the fact that $F(x)$ has the following Fourier expansion:

$$F(x) = u_0(x_3) + \sum_{\substack{\beta\in S\\ \beta\neq 0}} A(\beta) x_3^{3/2} K_{ir}(2\pi|\beta|x_3) e^{2\pi i \,\mathrm{Re}(\beta x)}, \tag{2.7}$$

where

$$u_0(x_3) = \begin{cases} a_1 x_3^{3/2+ir} + a_2 x_3^{3/2-ir} & \text{if } r \neq 0,\\ a_1 x_3^{3/2} + a_2 x_3^{3/2}\log x_3 & \text{if } r = 0. \end{cases} \tag{2.8}$$

Here K_{ir} is the classical K-Bessel function that satisfies $K_{ir}(y) \to 0$ as $y \to \infty$. We have $\mathrm{Re}(x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3) := x_0$ and S is the lattice in V_2 dual to lattice T defined by $S := \{\beta \in V_2 : \mathrm{Re}(\beta T) \subset \mathbb{Z}\}$. If $u_0(x_3) = 0$, then we call F a cuspidal automorphic function. In Section 3.1 we will make a choice of the lattice T such that Γ_T has only one cusp, hence the notation is justified.

For every nonnegative integer l fix a basis $\{P_{l,\nu}\}$ of spherical harmonic polynomials of degree l in 3 variables. In [24], Maaß proves the following converse theorem.

Theorem 2.3 (Maaß converse theorem)**.** The following two statements are equivalent.

(1) $F(x) = \sum_{\substack{\beta\in S\\ \beta\neq 0}} A(\beta) x_3^{3/2} K_{ir}(2\pi|\beta|x_3) e^{2\pi i\,\mathrm{Re}(\beta x)}$ is a cuspidal automorphic function with respect to Γ_T.

(2) For all $l, P_{l,\nu}$, the $\xi(s, P_{l,\nu}) := \pi^{-2s}\Gamma(s + ir/2)\Gamma(s - ir/2)\sum_{\substack{\beta\in S\\ \beta\neq 0}}(A(\beta)P_{l,\nu}(\beta)/(|\beta|^2)^s)$ satisfy the following:

- (a) $\xi(s, P_{l,\nu})$ have analytic continuation to the complex plane,
- (b) $\xi(s, P_{l,\nu})$ are bounded on vertical strips,
- (c) $\xi(s, P_{l,\nu})$ satisfy the functional equation

$$\xi\left(\frac{3}{2} + l - s, P_{l,\nu}\right) = (-1)^l \xi(s, P'_{l,\nu}), \tag{2.9}$$

where $P'_{l,\nu}(\beta) := P_{l,\nu}(\beta')$ for $\beta \in V_2$. □

In [24], Maaß proves the above theorem for the group Spin$(1, n)$, $n \geq 2$. Also, his statement is more general than the one above in the sense that he allows F to be noncuspidal and correspondingly $\xi(s, P_0)$ can have 2 simple poles. To prove Theorem 2.3, Maaß first gets the following criteria for a function on $\mathbb{H}_3$ to be identically zero.

Proposition 2.4. Let $g(x)$ be a twice continuously differentiable function on $\mathbb{H}_3$ satisfying $\Omega_3 g + (r^2 + 3^2/4)g = 0$ for some $r \in \mathbb{R}$. Since Ω_3 is an elliptic operator, $g(x)$ can be extended to a complex neighbourhood U about any point in $\mathbb{H}_3$ such that $g(x)$ represents a regular analytic function in complex variables x_0, x_1, x_2, x_3 on U. $g(x)$ vanishes identically in $\mathbb{H}_3$ if and only if for any set of complex numbers a_0, a_1, a_2 with $a_0^2 + a_1^2 + a_2^2 = 0$ and any real $x_3 > 0$, the conditions

$$\left(\frac{d^l}{dt^l} g\big((a_0 + a_1 i_1 + a_2 i_2)t + x_3 i_3\big) \right)_{t=0} = 0 \tag{2.10}$$

are satisfied for $l = 0, 1, 2, \ldots$. □

Observe that in the case of a holomorphic function f on the upper half plane, modularity with respect to $\mathrm{SL}_2(\mathbb{Z})$ is equivalent to checking the periodicity $(f(z+1) - f(z) = 0)$ and the condition $f(-z^{-1}) = (-z)^k f(z)$ only for $z = iy$. The second condition is a direct consequence of the holomorphy of f. In our case, Proposition 2.4 replaces this condition. Note that $l = 0$ corresponds to $g(i_3 x_3) = 0$.

Take $g(x) = F(x) - F(-x^{-1})$ with F as in (2.7). Observe that $F(x)$ is automorphic with respect to Γ_T if and only if $g(x) \equiv 0$. Maaß uses Proposition 2.4 along with the standard method of Mellin transform and Mellin inversion to complete the proof of Theorem 2.3.

2.3 Maaß forms on SL_2 of half-integral weight

Automorphic forms for the metaplectic group $\widetilde{\mathrm{SL}}_2$ can be realised as half-integer weight forms on the upper half plane with respect to the group $\Gamma_0(4) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 (\bmod 4) \right\}$. Let $S_{t+1/2}(4)$, $t \in \mathbb{Z}$ be the space of functions f on the upper half plane $\mathbb{H} = \{z = x + iy : x, y \in \mathbb{R},\ y > 0\}$ satisfying the following.

(1) For all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$

$$f(\gamma z) = \left(\epsilon_d^{-1} \left(\frac{c}{d} \right) \left(\frac{cz+d}{|cz+d|} \right)^{1/2} \right)^{2t+1} f(z), \tag{2.11}$$

where

$$\epsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 (\bmod 4); \\ i & \text{if } d \equiv 3 (\bmod 4); \end{cases} \tag{2.12}$$

and $(\cdot/\cdot)$ is the Legendre symbol.

(2) $\Delta_{t+1/2} f + \lambda f = 0$ for some $\lambda \in \mathbb{C}$ where $\Delta_{t+1/2}$ is the Laplace operator given by

$$\Delta_{t+1/2} := y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - \left(t + \frac{1}{2} \right) iy \frac{\partial}{\partial x}. \tag{2.13}$$

(3) f vanishes at the cusps of $\Gamma_0(4)$ (the cusps of $\Gamma_0(4)$ are given by ∞, 0, and 1/2). f has the Fourier expansion

$$f(z) = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} c(n) W_{\mathrm{sign}(n)(t+1/2)/2, ir/2}(4\pi|n|y)e^{2\pi inx}, \tag{2.14}$$

where $\lambda = 1/4+(r/2)^2$ and $W_{\nu,\mu}(y)$ is the classical Whittaker function. The numbers $\{c(n)\}$ are called the Fourier coefficients of f. For more details on half-integral weight Maaß forms we refer the reader to [17, 20].

Define the plus space

$$S^+_{t+1/2}(4) := \left\{ f \in S_{t+1/2}(4) \mid c(n) = 0 \text{ whenever } (-1)^t n \equiv 2,3 (\mathrm{mod}\, 4) \right\}. \tag{2.15}$$

This is analogous to the Kohnen plus space for holomorphic half-integral weight modular forms introduced in [18]. For $t = 0$, $S^+_{1/2}(4)$ is same as the plus space defined in [17] where it is shown to be nonempty. It will follow from Lemma 2.5 below that $S^+_{t+1/2}(4)$ is nonempty for $t \in \mathbb{Z}$.

For $t = 0$ and every odd prime p define Hecke operators $T(p^2) : S_{1/2}(4) \to S_{1/2}(4)$, $f \to (T(p^2)f)(z) := \sum_{n\neq 0} c^{(p)}(n) W_{\mathrm{sign}(n)/4, ir/2}(4\pi|n|y)e^{2\pi inx}$ by the formula

$$c^{(p)}(n) = pc(np^2) + p^{-1/2}\left(\frac{n}{p}\right)c(n) + p^{-1}c\left(\frac{n}{p^2}\right). \tag{2.16}$$

Katok and Sarnak [17] give us that the $T(p^2)$ commute with each other, commute with $\Delta_{1/2}$, and (2.16) implies that if $c(n) = 0$ whenever $(-1)^t n \equiv 2,3 (\mathrm{mod}\, 4)$, then $c^{(p)}(n)$ satisfies the same condition, that is, $T(p^2)$ keep $S^+_{1/2}(4)$ stable.

Define the following operators.

(1) Lowering operator $L_{t+1/2} := (z-\overline{z})(\partial/\partial\overline{z}) + (t+1/2)/2 = iy(\partial/\partial x) - y(\partial/\partial y) + (t+1/2)/2 : S_{t+1/2}(4) \to S_{t-2+1/2}(4)$ satisfies $\Delta_{t-2+1/2}L_{t+1/2}(f) = \lambda L_{t+1/2}(f)$ whenever $\Delta_{t+1/2}(f) = \lambda f$.

(2) Raising operator $R_{t+1/2} := -(z-\overline{z})(\partial/\partial z) - (t+1/2)/2 = -iy(\partial/\partial x) - y(\partial/\partial y) - (t+1/2)/2 : S_{t+1/2}(4) \to S_{t+2+1/2}(4)$ satisfies $\Delta_{t+2+1/2}R_{t+1/2}(f) = \lambda R_{t+1/2}(f)$ whenever $\Delta_{t+1/2}(f) = \lambda f$.

(3) Inverting operator $I_{t+1/2} : S_{t+1/2}(4) \to S_{-(t+1/2)}(4)$ defined by $(I_{t+1/2}f)(z) := \overline{f(z)}$ satisfies $\Delta_{-(t+1/2)}I_{t+1/2}(f) = \lambda I_{t+1/2}(f)$ whenever $\Delta_{t+1/2}(f) = \lambda f$.

For more on these operators see [6]. Also, note that the raising and lowering operators above are the same as those defined by Bump [3, Section 2.1] if we replace the half integers with integers. These operators along with Lemma 2.5 below give us the freedom to move $f \in S^+_{1/2}(4)$ to functions in $S^+_{t+1/2}(4)$ for $t \in \mathbb{Z}$. This will be crucial in the proof of the nonvanishing result in Section 4.

The recurrence relations for the classical Whittaker functions stated in [25, page 302] give us the following lemma.

Lemma 2.5. Let $c(n)$, $c^L(n)$, $c^R(n)$, and $c^I(n)$ be the Fourier coefficients of f, $L_{t+1/2}(f)$, $R_{t+1/2}(f)$, and $I_{t+1/2}(f)$, respectively, then

$$c^L(n) = \begin{cases} c(n) & \text{if } n < 0, \\ \left(\frac{ir}{2} + \frac{1}{2} - \frac{t+\frac{1}{2}}{2}\right)\left(\frac{t+\frac{1}{2}}{2} + \frac{ir}{2} - \frac{1}{2}\right) c(n) & \text{if } n > 0, \end{cases} \tag{2.17}$$

$$c^R(n) = \begin{cases} \left(\frac{ir}{2} + \frac{1}{2} + \frac{t+\frac{1}{2}}{2}\right)\left(-\frac{t+\frac{1}{2}}{2} + \frac{ir}{2} - \frac{1}{2}\right) c(n) & \text{if } n < 0, \\ c(n) & \text{if } n > 0, \end{cases} \tag{2.18}$$

$$c^I(n) = \overline{c(-n)}. \tag{2.19}$$

□

3 Lifting from $\widetilde{\mathrm{SL}}_2$ to $\mathrm{Spin}(1,4)$

In this section we will start with a function $f \in S^+_{1/2}(4)$ and define a function F on $\mathbb{H}_3$ given by the Fourier expansion $F(x) = \sum_{\substack{\beta \in S \\ \beta \neq 0}} A(\beta) x_3^{3/2} K_{ir}(2\pi|\beta|x_3) e^{2\pi i \,\mathrm{Re}(\beta x)}$. Then we use Theorem 2.3 to prove that the function F is a cuspidal automorphic function.

3.1 Definition of $A(\beta)$

Fix the lattice $T := \mathbb{Z} + \mathbb{Z}i_1 + \mathbb{Z}i_2 \subset V_2$. This lattice is self-dual, that is, $S = T$.

Proposition 3.1. For T as above,

$$\Gamma_T = SV_2(\mathbb{Z}) := SV_2 \cap M_2(C_2(\mathbb{Z})), \tag{3.1}$$

where $C_2(\mathbb{Z})$ is the order in C_2 consisting of elements whose coefficients are in $\mathbb{Z}$, that is, generated by $\{1, i_1, i_2, i_1 i_2\}$ over $\mathbb{Z}$. In particular, Γ_T is discrete, arithmetic, and has finite covolume. □

Proof. It is clear that $\Gamma_T \subset SV_2(\mathbb{Z})$. To show the other inclusion, we will first show that given a matrix $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in SV_2(\mathbb{Z})$ we can find an element $g \in \Gamma_T$ such that gM is upper triangular. By multiplying on the left by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ if necessary, we can assume that $|\alpha|^2 \geq |\gamma|^2$. We will first show that we can find a $g_0 \in \Gamma_T$ such that the norm of the lower-left matrix entry of $g_0 M$ is strictly less than $|\gamma|^2$. From [8, page 373] we know that $\alpha\bar{\gamma} \in V_2 \Rightarrow \alpha\gamma^{-1} = \alpha\bar{\gamma}/|\gamma|^2 \in V_2$. Hence we can find a $u \in T$ such that $|\alpha\gamma^{-1} - u|^2 < 1 \Rightarrow |\alpha - u\gamma|^2 < |\gamma|^2$. Now consider

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix} M = \begin{pmatrix} \gamma & \delta \\ -(\alpha - u\gamma) & -(\beta - u\delta) \end{pmatrix}. \tag{3.2}$$

Repeat the same process (finitely many times since the norm of the lower-left matrix entry is a nonnegative integer and it decreases at each step) until you get an upper-triangular matrix. So let $g \in \Gamma_T$ be such that $gM = \left(\begin{smallmatrix} \alpha & \beta \\ 0 & \delta \end{smallmatrix}\right)$. Since $\alpha\delta^* = 1$, we conclude that α is a unit, that is, $\alpha = \pm 1, \pm i_1, \pm i_2, \pm i_1 i_2$ and $\delta = \alpha'$. One can check easily that $\left(\begin{smallmatrix} \epsilon & 0 \\ 0 & \epsilon' \end{smallmatrix}\right) \in \Gamma_T$ for any unit ϵ. Hence

$$M = g^{-1} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha' \end{pmatrix} \begin{pmatrix} 1 & \bar{\alpha}\beta \\ 0 & 1 \end{pmatrix} \in \Gamma_T \tag{3.3}$$

as required.

Notice that $SV_2(\mathbb{Z})$ is the stabilizer of the lattice $C_2(\mathbb{Z}) \times C_2(\mathbb{Z})$ in the vector space $C_2 \times C_2$. From [9, pages 261-262] we have that $SV_2(\mathbb{Z})$ and with it Γ_T are discrete, arithmetic, and have finite covolume. ■

Remark 3.2. Let us note here that if we consider Spin$(1,n)$ with $n > 4$, then the above proposition is not true, that is, Γ_T is not equal to the integer points of Spin$(1,n)$, $n > 4$. In fact, the group Γ_T does not have finite covolume and is not arithmetic. So, in a sense, Spin$(1,4)$ is the only interesting case to apply the Maaß converse theorem. For higher values of n, the converse theorem is an analog of Hecke's converse theorem for triangle subgroups in SL_2 [14].

Let $f \in S^+_{1/2}(4)$ be a nonzero Hecke eigenform with the Fourier expansion (2.14) and Fourier coefficients $\{c(n)\}$. Write $\beta = \beta_0 + \beta_1 i_1 + \beta_2 i_2 \in S$ as $\beta = 2^u d(\alpha_0 + \alpha_1 i_1 + \alpha_2 i_2)$, where $\gcd(\alpha_0, \alpha_1, \alpha_2) = 1$, $u \geq 0$, and d is odd.

Define

$$A(\beta) := 2^{3/4}|\beta| \sum_{t=0}^{u} \left(\sum_{n|d} c\left(\frac{-|\beta|^2}{(2^t n)^2} \right) n^{-1/2} \right) (-1)^t 2^{t/2}. \tag{3.4}$$

The main result is as follows.

Theorem 3.3. With $A(\beta)$ as above and r as in (2.14),

$$F(x) := \sum_{\substack{\beta \in S \\ \beta \neq 0}} A(\beta) x_3^{3/2} K_{ir}\big(2\pi|\beta|x_3\big) e^{2\pi i \operatorname{Re}(\beta x)} \tag{3.5}$$

is a cuspidal automorphic function on $\mathbb{H}_3$ with respect to Γ_T. □

(1) We observe that the definition of $A(\beta)$ is similar to the Saito-Kurokawa lift for Sp_4. In the Sp_4 case, one starts with $g(z) = \sum_{n=1}^{\infty} b(n) e^{2\pi i n z}$, a holomorphic cusp form of weight $k + 1/2$ which lies in the Kohnen plus space. The Saito-Kurokawa lift is then given by $G(Z) = \sum_{B>0} a(B) e^{2\pi i \operatorname{Tr}(BZ)} : \mathfrak{H}_2 \to \mathbb{C}$ (the sum is over all positive definite symmetric half-integral 2×2 matrices and $\mathfrak{H}_2$ is the Siegel upper half plane), where

$$a(B) := a\begin{pmatrix} m & \frac{r}{2} \\ \frac{r}{2} & n \end{pmatrix} = \sum_{d|(m,n,r)} d^k b\left(\frac{\det(2B)}{d^2}\right). \tag{3.6}$$

The above formula differs from (3.4) in the sense that in (3.4) we have to treat the prime 2 dividing $\gcd(\beta_0, \beta_1, \beta_2)$ separately. This can be explained by noting that $Sp_4(\mathbb{Q}_p) \simeq \mathrm{Spin}(1,4)(\mathbb{Q}_p)$ for every odd prime $p \neq 2$ as shown in Proposition 6.3.

(2) Note that we can find constants C_0, k_0 independent of β such that $A(\beta)$ satisfies

$$\big|A(\beta)\big| < C_0 |\beta|^{k_0}. \tag{3.7}$$

This follows easily from the definition of $A(\beta)$ and the fact that there exist positive $C_1, k_1 \in \mathbb{R}$ such that $|c(n)| < C_1 |n|^{k_1}$ for all nonzero integers n where $c(n)$ are the Fourier coefficients of $f \in S^+_{1/2}(4)$.

3.2 Rankin integral formula

Maaß converse Theorem 2.3 says that it is enough to show that

$$\xi(s, P_{l,\nu}) = \pi^{-2s} \Gamma\left(s + \frac{ir}{2}\right) \Gamma\left(s - \frac{ir}{2}\right) \sum_{\beta \in S}{}' \frac{A(\beta) P_{l,\nu}(\beta)}{\big(|\beta|^2\big)^s} \tag{3.8}$$

convergent for $\operatorname{Re}(s) \gg 0$ from (3.7), have analytic continuation, are bounded on vertical strips, and satisfy the functional equation

$$\xi\left(\frac{3}{2} + l - s, P_{l,\nu}\right) = (-1)^l \xi\big(s, P'_{l,\nu}\big) \tag{3.9}$$

for all integers $l \geq 0$ and all spherical harmonic polynomials $P_{l,\nu}$ of degree l in 3 variables. Here the prime on the summation means that we exclude $\beta = 0$. Note that $A(-\beta) = A(\beta)$ and $P_{l,\nu}(-\beta) = (-1)^l P_{l,\nu}(\beta)$ give us

$$\sum_{\beta \in S}{}' \frac{A(\beta)P_{l,\nu}(\beta)}{\left(|\beta|^2\right)^s} = (-1)^l \sum_{\beta \in S}{}' \frac{A(\beta)P_{l,\nu}(\beta)}{\left(|\beta|^2\right)^s} \tag{3.10}$$

which implies that if l is odd, then $\xi(s, P_{l,\nu}) = 0$ and the above conditions are trivially satisfied. Henceforth we assume that l is even.

The objective of this section is to get a Rankin integral formula for the Dirichlet series $\xi(s, P_{l,\nu})$ which we will use to show the required properties of Theorem 2.3. We first substitute the definition (3.4) for $A(\beta)$ in the formula for $\xi(s, P_{l,\nu})$ to get the following proposition.

Proposition 3.4.

$$\begin{aligned} &\xi\left(s + \frac{l}{2} + \frac{1}{4}, P_{l,\nu}\right) \\ &\quad = \pi^{-2s-l-1/2} 2^{3/4} \left[\frac{2^s - 2^{-s}}{2^s + 2^{1-s}}\right] \Gamma\left(s + \frac{l}{2} + \frac{1}{4} + \frac{ir}{2}\right) \Gamma\left(s + \frac{l}{2} + \frac{1}{4} - \frac{ir}{2}\right) \\ &\qquad \times \zeta(2s) \sum_{m=1}^{\infty} \frac{c(-m)b(m)}{m^{s+l/2-1/4}}, \end{aligned} \tag{3.11}$$

where $b(m) := \sum_{|\beta|^2 = m} P_{l,\nu}(\beta)$. □

The proof of this proposition involves interchanging the order of summation and careful bookkeeping of the indices.

We now need the following Theta function and Eisenstein series to formulate the Rankin integral formula. Define

$$\Theta_{l,\nu}(z) := \sum_{\beta \in \mathbb{Z}^3} P_{l,\nu}(\beta) e^{2\pi i |\beta|^2 z} = \sum_{m=1}^{\infty} b(m) e^{2\pi i m z}. \tag{3.12}$$

Here we have identified the lattice T defined in Section 3.1 with $\mathbb{Z}^3$. By [32, page 456], $\Theta_{l,\nu}(z)$ is a holomorphic modular form of weight $l + 3/2$ for the group $\Gamma_0(4)$, that is,

$$\Theta_{l,\nu}(\gamma z) = \left(\epsilon_d^{-1} \left(\frac{c}{d}\right) (cz + d)^{1/2}\right)^{2l+3} \Theta_{l,\nu}(z) \tag{3.13}$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$. It is cuspidal when $l \geq 1$. Note that the function $\Theta_{l,\nu}(z)$ is not changed if we replace the polynomial $P_{l,\nu}(\beta)$ by $P_{l,\nu}{}'(\beta)$ because $\{\beta : |\beta|^2 = m\} = \{\beta' : |\beta'|^2 = m\}$.

As in [7], define the normalized Eisenstein series of weight $-(l+2)$ for $\Gamma_0(4)$ by

$$\widetilde{E}_\infty(z,s) := (4\pi)^{l/2}\frac{\Gamma(s+l+1)}{\Gamma(s)}\left(\pi^{-s}\Gamma(s)\zeta(2s)\right)\frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)}\left(\frac{cz+d}{|cz+d|}\right)^{l+2}\mathrm{Im}(\gamma z)^s. \tag{3.14}$$

Here $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ and $\Gamma_\infty := \left\{\left(\begin{smallmatrix} 1 & b \\ 0 & 1\end{smallmatrix}\right) \mid b \in \mathbb{Z}\right\}$. The above series converges for $\mathrm{Re}(s) \gg 0$. From [7], $\widetilde{E}_\infty(z,s)$ has a meromorphic continuation to the whole complex plane with a possibility of 2 simple poles with constant residues and is bounded on vertical strips.

The Eisenstein series satisfies

$$\widetilde{E}_\infty(\gamma z,s) = \left(\frac{cz+d}{|cz+d|}\right)^{-(l+2)}\widetilde{E}_\infty(z,s) \tag{3.15}$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(4)$.

Define the integral

$$I(s) := \int_{\Gamma_0(4)\backslash\mathbb{H}} f(z)\Theta_{l,\nu}(z)\widetilde{E}_\infty(z,s)y^{(l+2)/2-1/4}\frac{dx\,dy}{y^2}. \tag{3.16}$$

One can check that $I(s)$ is well defined using the transformation property of the integrand with respect to elements of $\Gamma_0(4)$. Note that $I(s)$ is convergent for all $s \in \mathbb{C}$. This is because firstly, $I(s)$ converges for all s which are not a pole for the Eisenstein series since f is a cusp form and $\Theta_{l,\nu}$, $\widetilde{E}_\infty$ have moderate growth. At the poles of the Eisenstein series, its residue is a constant and hence the residue of $I(s)$ is given by a constant multiple of $\int_{\Gamma_0(4)\backslash\mathbb{H}} f(z)\Theta_{l,\nu}(z)y^{(l+2)/2-1/4}(dx\,dy/y^2)$. This integral is zero since f is a nonholomorphic cusp form and $\Theta_{l,\nu}$ is a holomorphic form.

Proposition 3.5. For $\mathrm{Re}(s) \gg 0$,

$$\begin{aligned} I(s) &= \pi^{-2s+1/4}2^{-2s-1/2}\Gamma\left(s+\frac{l}{2}+\frac{1}{4}+\frac{ir}{2}\right) \\ &\quad\times\Gamma\left(s+\frac{l}{2}+\frac{1}{4}-\frac{ir}{2}\right)\zeta(2s)\sum_{m=1}^{\infty}\frac{c(-m)b(m)}{m^{s+l/2-1/4}}. \end{aligned} \tag{3.17}$$

□

Proof.

$$\begin{aligned} I(s) &= \underbrace{2^{l-1}\pi^{l/2-s}\Gamma(s+l+1)\zeta(2s)}_{u(s)} \\ &\quad\times\int_{\Gamma_0(4)\backslash\mathbb{H}} f(z)\Theta_{l,\nu}(z)\sum_{\Gamma_\infty\backslash\Gamma_0(4)}\left(\frac{cz+d}{|cz+d|}\right)^{l+2}\mathrm{Im}(\gamma z)^s y^{(l+2)/2-1/4}\frac{dx\,dy}{y^2}. \end{aligned} \tag{3.18}$$

Here we have used (3.14). Now using (2.11), (3.13) we get

$$\begin{aligned} I(s) &= u(s)\int_{\Gamma_\infty\backslash\mathbb{H}} f(z)\Theta_{l,\nu}(z)y^{s+(l+2)/2-1/4}\frac{dx\,dy}{y^2} \\ &= u(s)\int_0^\infty\int_0^1\left[\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} c(m)W_{\mathrm{sign}(m)/4,ir/2}\big(4\pi|m|y\big)e^{2\pi imx}\right] \\ &\qquad\times\left[\sum_{n=1}^{\infty} b(n)e^{2\pi inz}\right]y^{s+(l+2)/2-1/4}\frac{dx\,dy}{y^2}. \end{aligned} \tag{3.19}$$

We have used (2.14), (3.12). Now we integrate with respect to x first:

$$I(s) = u(s)\sum_{m=1}^{\infty} c(-m)b(m)\int_0^\infty W_{-1/4,ir/2}(4\pi my)e^{-2\pi my}y^{s+l/2-1/4}\frac{dy}{y}. \tag{3.20}$$

The change of variable $y \to (4\pi m)^{-1}y$ gives

$$\begin{aligned} I(s) &= u(s)(4\pi)^{-(s+l/2-1/4}\sum_{m=1}^{\infty}\frac{c(-m)b(m)}{m^{s+l/2-1/4}}\int_0^\infty W_{-1/4,ir/2}(y)e^{-y/2}y^{s+l/2-1/4}\frac{dy}{y} \\ &= \pi^{-2s+1/4}2^{-2s-1/2}\Gamma\left(s+\frac{l}{2}+\frac{1}{4}+\frac{ir}{2}\right) \\ &\quad\times\Gamma\left(s+\frac{l}{2}+\frac{1}{4}-\frac{ir}{2}\right)\zeta(2s)\sum_{m=1}^{\infty}\frac{c(-m)b(m)}{m^{s+l/2-1/4}}. \end{aligned} \tag{3.21}$$

For the integral formula above involving the Whittaker function, we refer the reader to [25, page 316]. ■

Comparing (3.17) with (3.11) we get the following proposition.

Proposition 3.6.

$$\begin{aligned} &\xi\left(s+\frac{l}{2}+\frac{1}{4},P_{l,\nu}\right) \\ &\quad= \pi^{-l-3/4}2^{5/4}\left[\frac{2^{3s}-2^s}{2^s+2^{1-s}}\right]\int_{\Gamma_0(4)\backslash\mathbb{H}} f(z)\Theta_{l,\nu}(z)\widetilde{E}_\infty(z,s)y^{(l+2)/2-1/4}\frac{dx\,dy}{y^2} \\ &\quad= \pi^{-l-3/4}2^{5/4}\left[\frac{2^{3s}-2^s}{2^s+2^{1-s}}\right]I(s). \end{aligned} \tag{3.22}$$

□

From the remarks about the convergence of $I(s)$ made before Proposition 3.5, we get the analytic continuation of $\xi(s, P_{l,\nu})$ to the entire complex plane. Also, $\xi(s, P_{l,\nu})$ is bounded on vertical strips since the same is true of the Eisenstein series, f is a cusp form, and $\Theta_{l,\nu}$ has moderate growth. Now to complete the proof of Theorem 3.3, we need to prove the functional equation of $\xi(s, P_{l,\nu})$.

Observe that the functional equation from Maaß converse theorem $\xi(3/2 + l - s, P_{l,\nu}) = (-1)^l \xi(s, P'_{l,\nu})$ is equivalent to $\xi(s + l/2 + 1/4, P_{l,\nu}) = \xi(1 - s + l/2 + 1/4, P'_{l,\nu})$. This implies that we have to show the functional equation

$$\left[2^{3s} - 2^s\right] I(s) = \left[2^{3(1-s)} - 2^{(1-s)}\right] I(1 - s) \tag{3.23}$$

since the term $2^s + 2^{1-s}$ in the denominator of (3.22) is already invariant under $s \to 1 - s$. (Note that $I(s)$ is unchanged if $P_{l,\nu}$ is replaced by $P'_{l,\nu}$ since the same is true of $\Theta_{l,\nu}$.)

We remark here that in the definition (3.4) of $A(\beta)$, the terms involving prime 2 are chosen so that we get the appropriate rational function in 2^s which together with the integral $I(s)$ have a functional equation.

3.3 The functional equation

To get (3.23), we will use the functional equation for the Eisenstein series. For that, we need to define two more Eisenstein series corresponding to the cusps 0 and 1/2 of $\Gamma_0(4)$ as in [7]:

$$\widetilde{E}_0(z, s) := \left(\frac{-z}{|z|}\right)^{l+2} \widetilde{E}_\infty\left(\frac{-1}{4z}, s\right), \tag{3.24}$$

$$\widetilde{E}_{1/2}(z, s) := \left(\frac{-2z+1}{|-2z+1|}\right)^{l+2} \widetilde{E}_\infty\left(\frac{-1}{4z-2}, s\right). \tag{3.25}$$

We have the following lemma [7, page 352].

Lemma 3.7.

$$\widetilde{E}_\infty(z, 1-s) = \underbrace{\frac{2^{4s-3}}{1-2^{2s-2}}}_{d_1(s)} \widetilde{E}_\infty(z, s) + \underbrace{\frac{2^{2s-2}\left(1-2^{2s-1}\right)}{1-2^{2s-2}}}_{d_2(s)} \left[\widetilde{E}_0(z, s) + \widetilde{E}_{1/2}(z, s)\right]. \tag{3.26}$$

□

Let us define $I_j(s) := \int_{\Gamma_0(4)\backslash\mathbb{H}} f(z)\Theta_{l,\nu}(z)\widetilde{E}_j(z,s)y^{(l+2)/2-1/4}(dx\,dy/y^2)$ for $j = 0, 1/2$. Now (3.26) gives us

$$I(1-s) = d_1(s)I(s) + d_2(s)\big[I_0(s) + I_{1/2}(s)\big]. \tag{3.27}$$

To simplify the above expression, we need the following transformation laws for f and $\Theta_{l,\nu}$.

Lemma 3.8.

$$e^{i\pi/4}\left(\frac{z}{|z|}\right)^{-1/2} f\left(\frac{-1}{4z}\right) = \sqrt{2}f_0(z), \tag{3.28}$$

$$e^{i\pi/4}\left(\frac{z}{|z|}\right)^{-1/2} f\left(\frac{-1}{4z}+\frac{1}{2}\right) = \sqrt{2}f_1(z), \tag{3.29}$$

$$\Theta_{l,\nu}\left(-\frac{1}{4z}\right) = (-i)^l 2^{3/2}(-iz)^{l+3/2} \sum_{m\equiv 0(\mathrm{mod}\ 4)} b(m)e^{2\pi i m(z/4)}, \tag{3.30}$$

$$\Theta_{l,\nu}\left(-\frac{1}{4z}+\frac{1}{2}\right) = (-i)^l 2^{3/2}(-iz)^{l+3/2} \sum_{m\equiv 3(\mathrm{mod}\ 4)} b(m)e^{2\pi i m(z/4)}, \tag{3.31}$$

where $f_j(z) := \sum_{m\equiv j(\mathrm{mod}\ 4)} c(m)W_{\mathrm{sign}(m)/4,ir/2}(4\pi|m|y/4)e^{2\pi i m(x/4)}$ for $j = 0, 1$. □

Proof. The first two equations follow from [17] or [7, page 354]. Following Shimura [32], we define the following Theta functions for each $h \in \mathbb{Z}^3$:

$$\Theta_{l,\nu}(z;h) := \sum_{\beta\equiv h(\mathrm{mod}\ 2)} P_{l,\nu}(\beta)e^{2\pi i|\beta|^2(z/4)}. \tag{3.32}$$

It is clear from the definition that $\Theta_{l,\nu}(z) = \sum_{h(\mathrm{mod}\ 2)} \Theta_{l,\nu}(4z;h)$. Shimura [32, page 454] gives us the following formulae for $\Theta_{l,\nu}(z;h)$:

$$\begin{aligned}\Theta_{l,\nu}\left(-\frac{1}{z};h\right) &= (-i)^l 2^{-3/2}(-iz)^{l+3/2} \sum_{k(\mathrm{mod}\ 2)} e^{{}^t kh/2}\Theta_{l,\nu}(z;k),\\ \Theta_{l,\nu}(z+2;h) &= e^{{}^t hh/2}\Theta_{l,\nu}(z;h),\end{aligned} \tag{3.33}$$

where ${}^t k$ stands for the transpose. Using these formulae, a straightforward calculation gives us the last two equations of the lemma. ■

Set $\Theta^0_{l,\nu}(z) := \sum_{m\equiv 0(\mathrm{mod}\ 4)} b(m)e^{2\pi i m(z/4)}$ and $\Theta^3_{l,\nu}(z) := \sum_{m\equiv 3(\mathrm{mod}\ 4)} b(m)e^{2\pi i m(z/4)}$.

Proposition 3.9.

$$I_0(s) + I_{1/2}(s) = -2^{2s} I(s). \tag{3.34}$$

□

Proof. Since $I_0(s)$, $I_{1/2}(s)$, and $I(s)$ are analytic functions, it is enough to prove (3.34) for $\mathrm{Re}(s) \gg 0$. So we will assume that $\mathrm{Re}(s) \gg 0$ and proceed as in the proof of Proposition 3.5. We have

$$I_0(s) = \int_{\Gamma_0(4)\backslash\mathbb{H}} f(z)\Theta_{l,\nu}(z)\widetilde{E}_0(z,s)y^{(l+2)/2-1/4}\frac{dx\,dy}{y^2}. \tag{3.35}$$

We make the change of variable $z \to -1/4z$. This corresponds to the action by the element $\left(\begin{smallmatrix} 0 & 1/2 \\ -2 & 0 \end{smallmatrix}\right)$ in $SL(2,\mathbb{R})$ which normalizes $\Gamma_0(4)$:

$$I_0(s) = \int_{\Gamma_0(4)\backslash\mathbb{H}} f\left(\frac{-1}{4z}\right)\Theta_{l,\nu}\left(\frac{-1}{4z}\right)\widetilde{E}_0\left(\frac{-1}{4z}, s\right)\mathrm{Im}\left(\frac{-1}{4z}\right)^{(l+2)/2-1/4}\frac{dx\,dy}{y^2}. \tag{3.36}$$

Now using (3.24), (3.28), and (3.30) we get

$$I_0(s) = -2^{-l+1/2}\int_{\Gamma_0(4)\backslash\mathbb{H}} f_0(z)\Theta^0_{l,\nu}(z)\widetilde{E}_\infty(z,s)y^{(l+2)/2-1/4}\frac{dx\,dy}{y^2}. \tag{3.37}$$

Proceeding as in the evaluation of $I(s)$

$$\begin{aligned} I_0(s) &= -2^{-l+1/2}u(s) \\ &\quad\times \sum_{m\equiv 0 (\mathrm{mod}\, 4)} c(-m)b(m)\int_0^\infty W_{-1/4,ir/2}\left(4\pi m\frac{y}{4}\right)e^{-2\pi m(y/4)}y^{s+l/2-1/4}\frac{dy}{y}. \end{aligned} \tag{3.38}$$

$u(s)$ was defined in Proposition 3.5. Change of variable $y \to (\pi m)^{-1}y$ gives

$$\begin{aligned} I_0(s) &= -2^{-l+1/2}u(s)(\pi)^{-(s+l/2-1/4)} \\ &\quad\times \sum_{m\equiv 0 (\mathrm{mod}\, 4)} \frac{c(-m)b(m)}{m^{s+l/2-1/4}}\int_0^\infty W_{-1/4,ir/2}(y)e^{-y/2}y^{s+l/2-1/4}\frac{dy}{y} \\ &= -2^{-1/2}\pi^{-2s+1/4}\Gamma\left(s+\frac{l}{2}+\frac{1}{4}+\frac{ir}{2}\right)\Gamma\left(s+\frac{l}{2}+\frac{1}{4}-\frac{ir}{2}\right)\zeta(2s) \\ &\quad\times \sum_{m\equiv 0 (\mathrm{mod}\, 4)} \frac{c(-m)b(m)}{m^{s+l/2-1/4}}. \end{aligned} \tag{3.39}$$

In the case of the integral $I_{1/2}(s)$, we make a change of variable $z \to -1/4z + 1/2$ corresponding to the action by the element $\left(\begin{smallmatrix} 1 & 1/2 \\ -2 & 0 \end{smallmatrix}\right)$ in $SL(2,\mathbb{R})$ which normalizes $\Gamma_0(4)$. Since this change of variable gives the same expression for $f(z)$, $\Theta_{l,\nu}(z)$, $\widetilde{E}_{1/2}(z,s)$, and y with f_0 and $\Theta^0_{l,\nu}$ replaced by f_1 and $\Theta^3_{l,\nu}$, respectively, the calculation for $I_{1/2}(s)$ proceeds in the exact same way as above to yield

$$\begin{aligned} I_{1/2}(s) = -2^{-1/2}\pi^{-2s+1/4}\Gamma\left(s+\frac{l}{2}+\frac{1}{4}+\frac{ir}{2}\right) \\ \times \Gamma\left(s+\frac{l}{2}+\frac{1}{4}-\frac{ir}{2}\right)\zeta(2s)\sum_{m\equiv 3 (\mathrm{mod}\, 4)} \frac{c(-m)b(m)}{m^{s+l/2-1/4}}. \end{aligned} \tag{3.40}$$

This gives us

$$\begin{aligned} &I_0(s) + I_{1/2}(s) \\ &\quad = -2^{-1/2}\pi^{-2s+1/4}\Gamma\left(s+\frac{l}{2}+\frac{1}{4}+\frac{ir}{2}\right)\Gamma\left(s+\frac{l}{2}+\frac{1}{4}-\frac{ir}{2}\right)\zeta(2s)\sum_{m=1}^{\infty}\frac{c(-m)b(m)}{m^{s+l/2-1/4}} \\ &\quad = -2^{2s}I(s). \end{aligned} \tag{3.41}$$

■

An easy calculation gives us

$$d_1(s) - 2^{2s}d_2(s) = \frac{2^{3s}-2^{s}}{2^{3-3s}-2^{1-s}}. \tag{3.42}$$

Using (3.27), (3.34), and (3.42) we finally get the following theorem.

Theorem 3.10.

$$\left(2^{3(1-s)} - 2^{(1-s)}\right)I(1-s) = \left(2^{3s}-2^{s}\right)I(s). \tag{3.43}$$

□

This gives us the required functional equation for $\xi(s, P_{l,\nu})$ and completes the proof of our main Theorem 3.3.

4 Nonvanishing of $A(\beta)$

We will first show that $A(\beta)$ is nonvanishing if and only if certain Fourier coefficients $c(n)$ are nonzero. To get the nonvanishing of $c(n)$, we derive an analogue of the Waldspurger formula for the Fourier coefficients of nonholomorphic Maaß forms for $SL_2(\mathbb{R})$ following the paper of Baruch and Mao [2]. This formula relates the value of $|c(n)|^2$ with the central value of L-functions. Finally, we use the results of Friedberg and Hoffstein [10] to get the nonvanishing of the L-values.

4.1 Criteria for nonvanishing

We have assumed that $f(z)$ is a Hecke eigenfunction. So for an odd prime p we have

$$pc(np^2) + p^{-1/2}\left(\frac{n}{p}\right)c(n) + p^{-1}c\left(\frac{n}{p^2}\right) = \lambda_p c(n), \tag{4.1}$$

where $T(p^2)f = \lambda_p f$.

Theorem 4.1. $A(\beta) = 0$ for all $\beta \in V_2(\mathbb{Z})$ if and only if $c(-n) = 0$ for all $n > 0$ and not of the form $n = 4^u(8k+7)$. □

Proof. We remind the reader of a fact in basic number theory: a positive integer n can be written as a sum of three squares if and only if n is not of the form $4^u(8k+7)$.

First let us assume that $c(-n) = 0$ unless $n = 4^u(8k+7)$. From the definition (3.4) we can immediately conclude that $A(\beta) = 0$ for all $\beta \in S$ since $|\beta|^2/(2^t n)^2$ is always of the form $4^u m$ with $m \not\equiv 7 (\mathrm{mod}\, 8)$.

To show the converse let us now assume that $A(\beta) = 0$ for all $\beta \in V_2(\mathbb{Z})$. We want to show that if $n = 4^u m$, where $m \not\equiv 7 (\mathrm{mod}\, 8)$, then $c(-n) = 0$. It is clear that we can take $4 \nmid m$. If m is square-free, then choose $\beta \in T$ such that $|\beta|^2 = m$. We get $c(-4^u m) = 0$ from the definition of $A(\beta)$ and induction on u. Now if $n = 4^u m j^2$ with m square-free and j odd, then we get $c(-n) = 0$ from the use of (4.1) and $c(-4^u m) = 0$ which is already shown above. ■

4.2 Waldspurger's formula for Maaß forms

In their paper [2], Baruch and Mao get an adelic Waldspurger's formula and, as an application, use it to derive the Kohnen-Zagier formula for holomorphic half-integral weight forms [19]. We will follow their paper and do the required calculation to get the Maaß forms case.

Let $\mathbb{A}$ denote the ring of adeles for the global field $\mathbb{Q}$. Let (π, V_π) be an irreducible cuspidal automorphic representation of $GL_2(\mathbb{A})$ and ψ a nontrivial additive character of $\mathbb{A}/\mathbb{Q}$. Then $\pi \simeq \otimes'_v \pi_v$ where (π_v, V_{π_v}) is an irreducible cuspidal automorphic representation of $GL_2(\mathbb{Q}_v)$. Let $\{W_\phi \mid \phi \in V_\pi\}$ be the Whittaker model for π with respect to the character ψ. Fix a finite set S of places v containing those where π_v is not unramified. Fix unramified vectors $\phi_{0,v}$ for $v \notin S$. Now choose local Whittaker functionals L_v such that $L_v(\phi_{0,v}) = 1$ and $\|\phi_{0,v}\|_v = 1$ for all $v \notin S$. Here $\|\cdot\|_v$ is defined by

$$\|\phi_v\|_v = (\phi_v, \phi_v) := \int_{\mathbb{Q}^*} L_v\left(\pi_v\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\phi_v\right)\overline{L_v\left(\pi_v\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\phi_v\right)}\frac{da}{|a|_v}. \tag{4.2}$$

Let $\phi = \otimes_{v\in S}\phi_v \otimes_{v\notin S} \phi_{0,v} \in V_\pi$ such that $L_v(\phi_v) \neq 0$ for $v \in S$. Here we have fixed an identification between V_π and the restricted tensor product $\otimes'_v V_{\pi_v}$. Then define

$$d_\pi(S,\psi) := \frac{|W_\phi(1)|}{\|\phi\|} \prod_{v\in S} \frac{\|\phi_v\|}{|L_v(\phi_v)|}. \tag{4.3}$$

It is shown in [2] that $d_\pi(S,\psi)$ is independent of the choice of L_v and ϕ so long as the conditions above are satisfied.

Let $D \in \mathbb{Z}$ and define $\psi^D(x) := \psi(Dx)$. Let $(\widetilde{\pi}, V_{\widetilde{\pi}})$ be an irreducible cuspidal automorphic representation of the adelic metaplectic group $\widetilde{\mathrm{SL}}_2(\mathbb{A})$. We have $\widetilde{\pi} \simeq \otimes_v \widetilde{\pi}_v$ where $(\widetilde{\pi}_v, V_{\widetilde{\pi}_v})$ is an irreducible cuspidal automorphic representation of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_v)$. Assume that $\widetilde{\pi}$ has a ψ^D-Whittaker model $\{\widetilde{W}^D_{\widetilde{\phi}} \mid \widetilde{\phi} \in V_{\widetilde{\pi}}\}$. Let S be a finite set of places v containing those where $\widetilde{\pi}_v$ is not unramified. Fix unramified vectors $\widetilde{\phi}_{0,v}$ for $v \notin S$. Choose local Whittaker functionals $\widetilde{L}^D_v$ such that $\widetilde{L}^D_v(\widetilde{\phi}_{0,v}) = 1$ and $\|\widetilde{\phi}_{0,v}\|_v = 1$ for all $v \notin S$. Here $\|\widetilde{\phi}_v\|_v$ is defined using $\widetilde{L}^D_v$ as in the linear case above. Let $\widetilde{\phi} = \otimes_{v\in S}\widetilde{\phi}_v \otimes_{v\notin S} \widetilde{\phi}_{0,v} \in V_{\widetilde{\pi}}$ such that $\widetilde{L}^D_v(\widetilde{\phi}_v) \neq 0$ for all $v \in S$. Here we have fixed an identification between $V_{\widetilde{\pi}}$ and the restricted tensor product $\otimes'_v V_{\widetilde{\pi}_v}$. Then define

$$d_{\widetilde{\pi}}(S,\psi^D) := \frac{|\widetilde{W}^D_{\widetilde{\phi}}(1)|}{\|\widetilde{\phi}\|} \prod_{v\in S} \frac{\|\widetilde{\phi}_v\|}{|\widetilde{L}^D_v(\widetilde{\phi}_v)|}. \tag{4.4}$$

It is shown in [2] that $d_{\widetilde{\pi}}(S,\psi^D)$ is independent of the choice of $\widetilde{L}^D_v$ and $\widetilde{\phi}$ so long as the conditions above are satisfied.

We now state [2, Theorem 4.3] for a representation π of $\mathrm{GL}_2(\mathbb{A})$ with trivial central character such that π_p is unramified for all finite places p and π_∞ is a unitary principal series representation. This is the case needed for our application.

Theorem 4.2 (Baruch-Mao)**.** Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ as above. Let $\widetilde{\pi} = \Theta(\pi,\psi)$ be the irreducible automorphic cuspidal representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ related to π by the theta correspondence [2, Section 3]. Let S be a finite set of places including ∞. Then for a fundamental discriminant D,

$$\left|d_{\widetilde{\pi}}(S,\psi^D)\right|^2 = \left|d_\pi(S,\psi)\right|^2 L^S\left(\frac{1}{2}, \pi\otimes\chi_D\right) \prod_{v\in S} |D|_v^{-1}, \tag{4.5}$$

where χ_D is the quadratic idele class character of $\mathbb{A}^*/\mathbb{Q}^*$ corresponding to D and $L^S(\cdot, \pi\otimes\chi_D)$ is the partial L-function. □

Let $g(z) \in S^+_{t+1/2}$ with Fourier coefficients $\{b(n)\}$ and let $h(z)$ be the corresponding Maaß form of weight $2t$ with respect to $\mathrm{SL}_2(\mathbb{Z})$ with Fourier coefficients $\{a(n)\}$ given by the Shimura correspondence for Maaß forms (see [17, 20]). Following [2] we obtain an automorphic cusp form ϕ on $\mathrm{GL}_2(\mathbb{A})$ from h and an automorphic cusp form $\widetilde{\phi}$ on $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ from g. Then $\phi \in V_\pi$ for some irreducible cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A})$ with trivial central character and $\widetilde{\phi} \in V_{\widetilde{\pi}}$ for some irreducible cuspidal automorphic representation $\widetilde{\pi}$ of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$. We can show that $\widetilde{\pi} = \Theta(\pi, \psi)$ [2]. We can apply Theorem 4.2 to π and $\widetilde{\pi}$ with $S = \{\infty, 2\}$ and $\widetilde{\psi}(x) = \psi((-1)^t x)$ where $\psi(x) = e^{2\pi i x}$ if $x \in \mathbb{R}$ and for $x \in \mathbb{Q}_p$ set $\psi(x) = e^{-2\pi i \widehat{x}}$ where we choose $\widehat{x} \in \mathbb{Q}$ satisfying $|x - \widehat{x}|_p \leq 1$.

Choose D such that $(-1)^t D > 0$. Let us evaluate $d_\pi(S, \widetilde{\psi})$ (which is equal to $d_\pi(S, \psi)$ by [2, Lemma 2.3]) and $d_{\widetilde{\pi}}(S, \widetilde{\psi}^D)$ (which is equal to $d_{\widetilde{\pi}}(S, \psi^{|D|})$):

$$\begin{aligned}
\left|d_\pi(S,\psi)\right|^2 &= \frac{\left|W_\phi(1)\right|^2}{\|\phi\|^2} \prod_{v\in\{\infty,2\}} \frac{\|\phi_v\|^2}{\left|L_v(\phi_v)\right|^2} \\
&= \frac{\left|W_{t,ir/2}(4\pi)a(1)\right|^2}{\|\phi\|^2} \frac{\|\phi_\infty\|^2}{\left|c_1 W_{t,ir/2}(4\pi)\right|^2} \frac{\|\phi_2\|^2}{\left|L_2(\phi_2)\right|^2} \\
&= \frac{|a(1)|^2}{|c_1|^2} \frac{\|\phi_\infty\|^2}{\|\phi\|^2} \frac{3}{2} \left|1 - 2^{-1-2ir}\right|^{-2}.
\end{aligned} \tag{4.6}$$

The calculation at $v = 2$ is the same as in [2]. We have also used that $L_\infty(\phi_\infty) = c_1 W_{t,ir/2}(4\pi)$ for some nonzero constant c_1 due to the uniqueness of local Whittaker models,

$$\begin{aligned}
&\left|d_{\widetilde{\pi}}(S,\psi^{|D|})\right|^2 \\
&\quad= \frac{\left|\widetilde{W}_{\widetilde{\phi}}^{|D|}(1)\right|^2}{\|\widetilde{\phi}\|^2} \prod_{v\in\{\infty,2\}} \frac{\|\widetilde{\phi}_v\|^2}{\left|\widetilde{L}_v^{|D|}(\widetilde{\phi}_v)\right|^2} \\
&\quad= \frac{\left|W_{(t+1/2)/2,ir/2}(4\pi|D|)b(|D|)\right|^2}{\|\widetilde{\phi}\|^2} \frac{\|\widetilde{\phi}_\infty\|^2}{\left|c_D W_{(t+1/2)/2,ir/2}(4\pi|D|)\right|^2} \frac{\|\widetilde{\phi}_2\|^2}{\left|\widetilde{L}_2^{|D|}(\widetilde{\phi}_2)\right|^2} \\
&\quad= \frac{\left|b(|D|)\right|^2}{|c_D|^2} \frac{\|\widetilde{\phi}_\infty\|^2}{\|\widetilde{\phi}\|^2} \epsilon_2(D),
\end{aligned} \tag{4.7}$$

where again from [2], we have

$$\epsilon_2(D) = \begin{cases} \dfrac{3}{4}\left|1 + 2^{-1/2-ir}\right|^{-2} & \text{when } D \equiv 1 (\mathrm{mod}\, 4), \\ \dfrac{3}{4}\left|1 - 2^{-1-2ir}\right|^{-2} |D|_2^{-1} & \text{when } \dfrac{D}{4} \equiv 2, 3 (\mathrm{mod}\, 4). \end{cases} \tag{4.8}$$

We have also used that $\widetilde{L}_\infty^{|D|}(\widetilde{\phi}_\infty) = c_D W_{(t+1/2)/2,ir/2}(4\pi|D|)$ for some nonzero constant c_D due to the uniqueness of local Whittaker models.

Note that the calculations above are valid only for choices of t, r, and D such that $W_{(t+1/2)/2,ir/2}(4\pi|D|) \neq 0$ and $W_{t,ir/2}(4\pi) \neq 0$. The asymptotics for the classical Whittaker functions given in [25, page 319] allow us to conclude that for fixed r, we can choose t a positive odd integer such that $W_{t,ir/2}(4\pi) \neq 0$ and for fixed r and t we can choose a positive integer $M_{t,r}$ such that whenever $|D| > M_{t,r}$ we have $W_{(t+1/2)/2,ir/2}(4\pi|D|) \neq 0$. Now, from Theorem 4.2, (4.6), and (4.7) we get the following theorem.

Theorem 4.3. Let r, t, and D be such that the above conditions for classical Whittaker functions are satisfied. Let $g(z) \in S^+_{t+1/2}$ with Fourier coefficients $\{b(n)\}$ and eigenvalue $1/4 + (r/2)^2$ with respect to the Laplace operator. Then

$$\left|b\left(|D|\right)\right|^2 = CL^{\{\infty,2\}}\left(\frac{1}{2}, \pi \otimes \chi_D\right), \tag{4.9}$$

where

$$C = \frac{c_D^2}{c_1^2}\frac{\left\|\phi_\infty\right\|^2}{\|\phi\|^2}\frac{\|\widetilde{\phi}\|^2}{\left\|\widetilde{\phi}_\infty\right\|^2}\left|a(1)\right|^2\epsilon_2(D)^{-1}\frac{3}{2}\left|1-2^{-1-2ir}\right|^{-2}\prod_{v\in\{\infty,2\}}|D|_v^{-1} \neq 0, \tag{4.10}$$

and $D < -M_{t,r}$ is a fundamental discriminant. □

4.3 The nonvanishing of the special values of L-functions

Let S be a finite set of places of $\mathbb{Q}$ and ξ a fixed quadratic character of $\mathbb{A}^*/\mathbb{Q}^*$. Set $\Psi(S,\xi) := \{\chi$ quadratic character of $\mathbb{A}^*/\mathbb{Q}^*$ such that $\chi_v = \xi_v$ for all $v \in S\}$.

Theorem 4.4 (Friedberg and Hoffstein [10])**.** Let π be an irreducible cuspidal automorphic representation of $GL_2(\mathbb{A})$ which is self-contragradient. Assume that there exists some $\chi \in \Psi(S,\xi)$ such that $\epsilon(1/2, \pi\otimes\chi) = 1$. Then there are infinitely many $\chi \in \Psi(S,\xi)$ such that $L(1/2, \pi\otimes\chi) \neq 0$. □

Note that any quadratic character of $\mathbb{A}^*/\mathbb{Q}^*$ is of the form χ_D for some $D \in \mathbb{Q}^*$. We now state the result for the nonvanishing of Fourier coefficients of Maaß forms.

Theorem 4.5. Let $f \in S^+_{1/2}(4)$ be a Hecke eigenform with Fourier coefficients $\{c(n)\}$. Then for infinitely many $D < 0$ with $D = -n$ and $n \equiv 3 \pmod 8$, it holds that $c(D) \neq 0$. □

Proof. Let $\lambda = 1/4+(r/2)^2$ be the eigenvalue of f with respect to the Laplace operator $\Delta_{1/2}$. Choose an odd integer t such that $W_{t,ir/2}(4\pi) \neq 0$. This is possible from the remark made before Theorem 4.3. Now let $g \in S^+_{t+1/2}(4)$ be defined by $g := R_{t+1/2-2} R_{t+1/2-4} \cdots R_{-1/2} I_{1/2} f$. Let $\{b(n)\}$ be the Fourier coefficients of g. From (2.18) and (2.19), we know that if $\{c(n)\}$ are the Fourier coefficients of f, then for $n > 0$ we have $c(-n) = \overline{b(n)}$. As in the previous section, let h be the Maaß form of weight $2t$ with respect to $SL_2(\mathbb{Z})$ corresponding to g and let ϕ be the adelic cuspidal automorphic form corresponding to h. Then ϕ lies in an irreducible cuspidal automorphic representation π of $GL_2(\mathbb{A})$ with trivial central character, hence π is self-contragradient.

Since π_p is unramified at all finite places p, we obtain $\epsilon(1/2, \pi_p) = 1$ for all p. π_∞ is the unitary principal series representation with even K-type. If $\pi_\infty = \pi(\mu_1, \mu_2)$ where μ_1, μ_2 are unitary unramified characters, then from [13] we have $\epsilon(1/2, \pi_\infty) = \epsilon(1/2, \mu_1)\epsilon(1/2, \mu_2) = 1$ since $\epsilon(1/2, \mu) = 1$ for an unramified character μ. From [2, (3.1)], we have $\epsilon(1/2, \pi \otimes \chi_D) = \epsilon(1/2, \pi)$ for all $D \in \mathbb{Q}^*$. So $\epsilon(1/2, \pi \otimes \chi_D) = 1$ for all $D \in \mathbb{Q}^*$ and hence the hypothesis of Theorem 4.4 is satisfied by any choice of S and ξ. Let us choose $S = \{\infty, 2\}$ and $\xi = \chi_{-3}$. Then by definition $\chi_D \in \Psi(S, \xi) \Leftrightarrow D \equiv 5 (\mathrm{mod}\, 8)$ and $D < 0$. We can take D to be square-free since D and Dm^2 give the same character, that is, $\chi_D = \chi_{Dm^2}$. Theorem 4.4 then gives us $L(1/2, \pi \otimes \chi_D) \neq 0$ for infinitely many D satisfying $D \equiv 5 (\mathrm{mod}\, 8)$ and $D < 0$.

The function g that we constructed above satisfies the hypothesis of Theorem 4.3 and hence we get that $b(|D|) \neq 0$ for infinitely many D such that $D < -M_{t,r}$, $D \equiv 5 (\mathrm{mod}\, 8)$ and square-free. (Note that a number D satisfying the above conditions is a fundamental discriminant.) This gives us the required result. ■

Note that we can get similar nonvanishing results for different D's by choosing appropriate $\xi = \chi_D$. Finally, from Theorems 4.1 and 4.5 we get the nonvanishing of $A(\beta)$.

Theorem 4.6. $A(\beta) \neq 0$ for infinitely many $\beta \in V_2(\mathbb{Z})$. In particular, the map $f(z) \mapsto F(x)$ is injective. □

5 Hecke theory

We now present the Hecke theory for the Spin(1, 4) case. In Section 5.1 we get the generators for the Hecke algebra of the Spin group. For this we will follow Krieg's work [21, 22] in which he obtains the generators of the Hecke algebra in a setting similar to ours. In Section 5.2 we define the Hecke operator T_p on automorphic functions on Spin(1, 4) with respect to $\Gamma = SV_2(\mathbb{Z})$ by its action on the Fourier coefficients $A(\beta)$. We show that if $f \in S^+_{1/2}(\Gamma_0(4))$ is a Hecke eigenfunction, then F defined in Section 3.2 is an eigenfunction of T_p with eigenvalue $p^{3/2}\lambda_p + p(p+1)$, where λ_p is the pth eigenvalue of f. In Section 5.3 we define the Hecke operator T_{p^2} on automorphic functions on Spin(1, 4) with respect to Γ by its action on the Fourier coefficients $A(\beta)$. We show that if $f \in S^+_{1/2}(\Gamma_0(4))$ is a Hecke eigenfunction, then F defined in Section 3.2 is an eigenfunction of T_{p^2} with eigenvalue $(p+1)p^{3/2}\lambda_p + (p-1)(p+1)$, where λ_p is the pth eigenvalue of f.

5.1 Hecke algebra H_p

As mentioned above, we will use the results of Krieg. In his papers [21, 22], Krieg works with the Hurwitz order $O := \mathbb{Z} + \mathbb{Z}i_1 + \mathbb{Z}i_2 + \mathbb{Z}(1/2)(1 + i_1 + i_2 + i_1 i_2)$ instead of the integer quaternions $C_2(\mathbb{Z}) = \mathbb{Z} + \mathbb{Z}i_1 + \mathbb{Z}i_2 + \mathbb{Z}i_1 i_2$. We will use Krieg's results whenever directly applicable or adopt his proofs in our case otherwise.

Define the similitude group $\mathrm{GSV}_2^+(\mathbb{Q}) := \{M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{Mat}_2(C_2(\mathbb{Q})) : \alpha\delta^* - \beta\gamma^* = \mu(M) \in \mathbb{Q}^+,\ \alpha\beta^*, \delta\gamma^* \in V\}$. Here $V = \mathbb{R} + \mathbb{R}i_1 + \mathbb{R}_2$ as defined in (2.1). For $M_1, M_2 \in \mathrm{GSV}_2^+(\mathbb{Q})$ with single-coset decompositions $\Gamma M_i \Gamma = \bigcup_\mu \Gamma M_{i,\mu}$, $i = 1, 2$ define the product

$$(\Gamma M_1 \Gamma) \cdot (\Gamma M_2 \Gamma) := \sum_{C \in \Gamma \backslash \mathrm{GSV}_2^+(\mathbb{Q})/\Gamma} t(C)\Gamma C \Gamma \quad \text{with } t(C) := \sharp\{(\mu, \nu) : \Gamma M_{1,\mu} M_{2,\nu} = \Gamma C\}. \tag{5.1}$$

The Hecke algebra of GSV_2^+ is the algebra over integers generated by the set of double cosets $\{\Gamma M \Gamma : M \in \mathrm{GSV}_2^+(\mathbb{Q})\}$ with product as defined in (5.1). Let H_p be the subalgebra of double cosets generated by $\{\Gamma M \Gamma : M \in \mathrm{GSV}_2^+(\mathbb{Q}) \cap \mathrm{Mat}_2(C_2(\mathbb{Z}[p^{-1}])),\ \mu(M) = p^k \text{ with } k \in \mathbb{Z}\}$. Here $\mathbb{Z}[p^{-1}]$ is the ring of rational numbers with only powers of p in the denominator. For $M \in \mathrm{GSV}_2^+(\mathbb{Q}) \cap \mathrm{Mat}_2(C_2(\mathbb{Z}[p^{-1}]))$ let k_0 be the smallest nonnegative integer such that $p^{k_0} M =: M' \in \mathrm{Mat}_2(C_2(\mathbb{Z}))$. Then we have

$$\Gamma M \Gamma = \left(\Gamma \begin{pmatrix} p^{-1} & 0 \\ 0 & p^{-1} \end{pmatrix} \Gamma\right)^{k_0} (\Gamma M' \Gamma). \tag{5.2}$$

Let $\widetilde{H}_p$ be the subalgebra of H_p generated by $\Gamma M \Gamma \in H_p$ with $M \in \mathrm{Mat}_2(C_2(\mathbb{Z}))$. Equation (5.2) implies that H_p is generated by $\widetilde{H}_p$ and the double coset $\Gamma \left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & p^{-1} \end{smallmatrix}\right) \Gamma$. The main result of this section is the explicit calculation of the generators of $\widetilde{H}_p$.

Set $\mathrm{GSV}_2^+(\mathbb{Z}) = \mathrm{GSV}_2^+(\mathbb{Q}) \cap \mathrm{Mat}_2(C_2(\mathbb{Z}))$. Fix an $\widehat{\alpha} \in C_2(\mathbb{Z})$ such that $|\widehat{\alpha}|^2 = p$ and $p \nmid \widehat{\alpha}^n$ for every $n \geq 1$. (This condition is satisfied as long as $\widehat{\alpha} \neq -\bar{\widehat{\alpha}}$ and $|\widehat{\alpha}|^2 = p$. To see this use the fact that for any $x \in C_2(\mathbb{R})$ we have $x^2 = 2\,\mathrm{Re}(x)x - |x|^2$.) The following lemma shows that each double coset in $\widetilde{H}_p$ contains a diagonal matrix of a given type.

Lemma 5.1. Let $M \in \mathrm{GSV}_2^+(\mathbb{Z})$ with $\mu(M) = p^m$ where p is an odd prime and $m \geq 1$. Then $\Gamma M \Gamma$ contains an element of the form $\left(\begin{smallmatrix} \widehat{\alpha}^l p^{k_1} & 0 \\ 0 & (\widehat{\alpha}')^l p^{k_2} \end{smallmatrix}\right)$ with $\widehat{\alpha}$ as above and $l, k_1 \geq 0$, $l + k_1 + k_2 = m$, and $l + k_1 \leq k_2$. □

Proof. It is easy to show that we can find a diagonal element $\left(\begin{smallmatrix}\eta & 0\\ 0 & \delta\end{smallmatrix}\right) \in \Gamma M \Gamma$. Let $\eta = p^{k_1}\tau$ with $p \nmid \tau$ and hence $\delta = p^{k_2}\tau'$ where $p^m = p^{k_1+k_2}|\tau|^2$. Let $|\tau|^2 = p^l$. Now we get the result of the lemma since by methods as in the proof of [22, Theorem 6(b)], we can find two elements $g_1, g_2 \in \Gamma$ such that $g_1 \left(\begin{smallmatrix}\tau p^{k_1} & 0\\ 0 & \tau' p^{k_2}\end{smallmatrix}\right) g_2 = \left(\begin{smallmatrix}\widehat{\alpha}^l p^{k_1} & 0\\ 0 & (\widehat{\alpha}')^l p^{k_2}\end{smallmatrix}\right)$. ■

For the proof of the main theorem of this section we need the single-coset decomposition of the two double cosets $\Gamma\left(\begin{smallmatrix}1 & 0\\ 0 & p\end{smallmatrix}\right)\Gamma$ and $\Gamma\left(\begin{smallmatrix}\widehat{\alpha} & 0\\ 0 & p\widehat{\alpha}'\end{smallmatrix}\right)\Gamma$.

Proposition 5.2.

$$\Gamma\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Gamma = \left(\bigcup_{\{\widehat{\beta}\}}\Gamma\begin{pmatrix}1 & \widehat{\beta}\\ 0 & p\end{pmatrix}\right) \cup \Gamma\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix} \cup \left(\bigcup_{\{\alpha,\delta\}}\Gamma\begin{pmatrix}\alpha & \delta\\ 0 & \alpha'\end{pmatrix}\right), \tag{5.3}$$

where $\widehat{\beta}$ in the first term runs through the set $\{\beta_0 + \beta_1 i_1 + \beta_2 i_2 : 0 \le \beta_j < p,\ j = 0, 1, 2\}$ and $\{\alpha, \delta\}$ in the third term runs through the set of equivalence classes of the set $C := \{\{\alpha, \delta\} := \left(\begin{smallmatrix}\alpha & \delta\\ 0 & \alpha'\end{smallmatrix}\right) : \mu(\{\alpha, \delta\}) = p\}$ under the action of Γ by left multiplication. There are exactly $p(p+1)$ distinct elements in the third term. Furthermore, the union above is a disjoint union. □

Proof. It is clear that the right-hand side of (5.3) is contained in the left-hand side since $\Gamma\left(\begin{smallmatrix}1 & 0\\ 0 & p\end{smallmatrix}\right)\Gamma = \{M \in \mathrm{GSV}_2^+(\mathbb{Z}) : \mu(M) = p\}$ by Lemma 5.1. Now we show the opposite inclusion. Let M be such that $\mu(M) = p$. As in the proof of Proposition 3.1, we can find an element $g \in \Gamma$ such that $gM = \left(\begin{smallmatrix}\alpha & \beta\\ 0 & \delta\end{smallmatrix}\right)$. Since $\alpha\delta^* = p$, we have $|\alpha|^2|\delta|^2 = p^2$ and $\delta = (\alpha'/|\alpha|^2)p$. We have three cases depending on the value of $|\alpha|^2$. If $|\alpha|^2 = 1$, p^2, and p, respectively, then M belongs to the first, second, and third term, respectively, of (5.3). We illustrate the calculation for the case $|\alpha|^2 = 1$. We have $\alpha \in \{\pm 1, \pm i_1, \pm i_2, \pm i_1 i_2\}$ and $\delta = \alpha' p$. Multiply gM on the left by $\left(\begin{smallmatrix}\bar{\alpha} & 0\\ 0 & \alpha^*\end{smallmatrix}\right) \in \Gamma$ to get

$$\begin{pmatrix}\bar{\alpha} & 0\\ 0 & \alpha^*\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ 0 & \alpha' p\end{pmatrix} = \begin{pmatrix}1 & \bar{\alpha}\beta\\ 0 & p\end{pmatrix}. \tag{5.4}$$

Write $\bar{\alpha}\beta = x_0 + x_1 i_1 + x_2 i_2 = p(y_0 + y_1 i_1 + y_2 i_2) + (\beta_0 + \beta_1 i_1 + \beta_2 i_2)$ with $0 \le \beta_j \le p-1$. So

$$\begin{pmatrix}1 & -(y_0 + y_1 i_1 + y_2 i_2)\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & \bar{\alpha}\beta\\ 0 & p\end{pmatrix} = \begin{pmatrix}1 & \beta_0 + \beta_1 i_1 + \beta_2 i_2\\ 0 & p\end{pmatrix}. \tag{5.5}$$

This gives us $M \in \bigcup_{\{\widehat{\beta}\}}\Gamma\left(\begin{smallmatrix}1 & \widehat{\beta}\\ 0 & p\end{smallmatrix}\right)$ as required.

It remains to show that there are $p(p+1)$ distinct single cosets in the third term of (5.3). Observe that $|\{\alpha \in C_2(\mathbb{Z}) : |\alpha|^2 = p\}| = 8(p+1)$. There are exactly $p+1$ orbits of 8 elements each when we consider the action of units in $C_2(\mathbb{Z})$ by left multiplication on the above set. Now let α and α_1 be two elements such that $|\alpha|^2 = |\alpha_1|^2 = p$. Suppose $\left(\begin{smallmatrix} \alpha & \delta_1 \\ 0 & \alpha' \end{smallmatrix}\right) = g\left(\begin{smallmatrix} \alpha_1 & \delta_2 \\ 0 & \alpha_1' \end{smallmatrix}\right)$ for some element $g \in \Gamma$. Then α and α_1 differ only by a unit. Hence we can assume that $\alpha = \alpha_1$ and g is of the form $\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$ where $u \in \mathbb{Z} + \mathbb{Z}i_1 + \mathbb{Z}i_2$. From the definition of SV_2 we know that $\delta_j = \nu_j\alpha'$ with $\nu_j \in V_2$ for $j = 1, 2$ and hence for a fixed α two choices of δ give the same left coset if $\nu_1 = \nu_2 + u$. So for a fixed α the distinct left cosets are given by the choices of $\delta = \nu\alpha'$ where $\nu = a + bi_1 + ci_2$ satisfies

$$0 \le a, b, c < 1, \qquad \nu\alpha' \in C_2(\mathbb{Z}). \tag{5.6}$$

Let $\alpha = \alpha_0 + \alpha_1 i_1 + \alpha_2 i_2 + \alpha_3 i_1 i_2$ and set $a_1 := pa$, $b_1 := pb$, $c_1 := pc$. One can check that $a_1, b_1, c_1 \in \{0, 1, \ldots, p-1\}$. Then (5.6) is equivalent to

$$\underbrace{\begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ -\alpha_1 & \alpha_0 & \alpha_3 \\ -\alpha_2 & -\alpha_3 & \alpha_0 \end{pmatrix}}_{A} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \equiv 0 (\operatorname{mod} p). \tag{5.7}$$

Consider A a linear transformation on a 3-dimensional vector space over $\mathbb{Z}/p\mathbb{Z}$. $\operatorname{Det}(A) = \alpha_0|\alpha|^2 = \alpha_0 p \equiv 0 (\operatorname{mod} p)$. Also $A \not\equiv 0 (\operatorname{mod} p)$. One can check that rank of A is equal to 2. This implies that $\dim(\operatorname{Ker}(A)) = 1$. Hence the number of solutions to (5.6) is exactly p. This completes the proof of Proposition 5.2. ■

Next we give the Peirce decomposition for quaternions. Let p be an odd prime. Fix $r, s \in \mathbb{Z}$ such that $1 + r^2 + s^2 \equiv 0 (\operatorname{mod} p)$. This is always possible, in fact, one can show that the number of solutions to the equation $x^2 + y^2 \equiv -1 (\operatorname{mod} p)$ is $p+1$ if $p \equiv 3 (\operatorname{mod} 4)$ and $p - 1$ if $p \equiv 1 (\operatorname{mod} 4)$. Given $\alpha = \alpha_0 + \alpha_1 i_1 + \alpha_2 i_2 + \alpha_3 i_1 i_2 \in C_2(\mathbb{Z})$ define

$$\psi_p(\alpha) := \begin{pmatrix} \alpha_0 - \alpha_2 r - \alpha_3 s & \alpha_1 - \alpha_2 s + \alpha_3 r \\ -\alpha_1 - \alpha_2 s + \alpha_3 r & \alpha_0 + \alpha_2 r + \alpha_3 s \end{pmatrix} \in \operatorname{Mat}_2(\mathbb{Z}). \tag{5.8}$$

Lemma 5.3. For an odd prime p, the map ψ_p defined above gives an isomorphism of rings between $\operatorname{Mat}_2(\mathbb{Z}/p\mathbb{Z})$ and $C_2(\mathbb{Z})/pC_2(\mathbb{Z})$. ψ_p satisfies $|\alpha|^2 \equiv \det(\psi_p(\alpha)) (\operatorname{mod} p)$. □

This lemma follows directly from [21, Lemma 1, page 329] when one observes that for an odd prime p we have $C_2(\mathbb{Z})/pC_2(\mathbb{Z}) \simeq O/pO$ where O is the Hurwitz order.

We can extend the map ψ_p naturally to a ring isomorphism between $\operatorname{Mat}_2(C_2(\mathbb{Z}))/p\operatorname{Mat}_2(C_2(\mathbb{Z}))$ and $\operatorname{Mat}_4(\mathbb{Z}/p\mathbb{Z})$. For an element $\alpha \in C_2(\mathbb{Z})$ (resp., $g \in \operatorname{Mat}_2(C_2(\mathbb{Z}))$), define

$\mathrm{rk}(\alpha)$ (resp., $\mathrm{rk}(g)$) as the rank of the matrix $\psi_p(\alpha)$ as an element of $\mathrm{Mat}_2(\mathbb{Z}/p\mathbb{Z})$ (resp., as the rank of the matrix $\psi_p(g)$ as an element of $\mathrm{Mat}_4(\mathbb{Z}/p\mathbb{Z})$). Note that $\mathrm{rk}(g_1 g g_2) = \mathrm{rk}(g)$ for any $g_1, g_2 \in \Gamma$. The notion of rank will play a crucial role in the proof of many results.

Proposition 5.4.

$$\Gamma\begin{pmatrix}\widehat{\alpha} & 0\\ 0 & p\widehat{\alpha}'\end{pmatrix}\Gamma = \left(\bigcup_{\alpha}\Gamma\begin{pmatrix}p\alpha & 0\\ 0 & \alpha'\end{pmatrix}\right)\cup\left(\bigcup_{\alpha,\delta}\Gamma\begin{pmatrix}\alpha & \delta\\ 0 & p\alpha'\end{pmatrix}\right)\cup\left(\bigcup_{\nu}\Gamma\begin{pmatrix}p & \nu\\ 0 & p\end{pmatrix}\right). \tag{5.9}$$

Here the union in the first term runs over the orbits of the action of the group of units in $C_2(\mathbb{Z})$ on the set $\{\alpha \in C_2(\mathbb{Z}) : |\alpha|^2 = p\}$. The union in the second term runs over the same set of equivalence classes for α and for each α the set δ is determined in Lemma 5.5 below. The union in the third term is over the set $\{\nu = \nu_0 + \nu_1 i_1 + \nu_2 i_2 : \nu_0, \nu_1, \nu_2 \in \mathbb{Z},\ 0 \le \nu_0, \nu_1, \nu_2 \le p-1,\ \nu \ne 0,\ |\nu|^2 \equiv 0 (\mathrm{mod}\, p)\}$. □

Proof. An element M with $\mu(M) = p^2$ lies in the double coset $\Gamma\left(\begin{smallmatrix}\widehat{\alpha} & 0\\ 0 & p\widehat{\alpha}'\end{smallmatrix}\right)\Gamma$ if and only if $\mathrm{rk}(M) = 1$. Take such a matrix M. We can find an element $g \in \Gamma$ such that $gM = \left(\begin{smallmatrix}\mu & \delta\\ 0 & (\mu'/|\mu|^2)p^2\end{smallmatrix}\right)$. As in the proof of Proposition 5.2 we have $|\mu|^2 = 1, p, p^2, p^3$, or p^4. Notice that $|\mu|^2 = 1$ or p^4 implies that one of the diagonal entries above is a unit and hence we get $\mathrm{rk}(M) \ge 2$, a contradiction. Again arguing as in Proposition 5.2 the cases $|\mu|^2 = p^3, p, p^2$ imply that M lies in the first, second, and third term on the right-hand side of (5.9), respectively. Finally, the precise description of the distinct cosets in the second term of (5.9) is given by the following lemma. ■

Lemma 5.5. For each α such that $|\alpha|^2 = p$, the distinct cosets in the second term of (5.9) are given by those δ that satisfy the following two conditions:

(1) $\delta \in C_2(\mathbb{Z})$ has to be of the form $u\alpha'$ where $u = a + bi_1 + ci_2 \in V_2(\mathbb{R})$ with $0 \le a, b, c < p$,

(2) $\alpha^*\delta' \equiv 0 (\mathrm{mod}\, p)$.

The number of δ satisfying the above properties is p^3. □

Proof. The first condition is obtained by a similar argument as given in the proof of Proposition 5.2. To get the second condition we will use the fact that $\mathrm{rk}(M) = 1$. From (5.8) we have

$$\psi_p\left(\begin{pmatrix}\alpha & \delta\\ 0 & p\alpha'\end{pmatrix}\right) = \begin{pmatrix}\psi_p(\alpha) & \psi_p(\delta)\\ 0 & 0\end{pmatrix}. \tag{5.10}$$

Let C_1 and C_2 (resp., C_3 and C_4) be the columns of the matrices $\psi_p(\alpha)$ (resp., of $\psi_p(\delta)$.)

The condition $\mathrm{rk}(M) = 1$ implies that determinant of every 2×2 matrix obtained from any two of the four columns C_1, C_2, C_3, or C_4 is equivalent to zero mod p. There are 6 such matrices and we get

$$|\alpha|^2 \equiv |\delta|^2 \equiv \mathrm{Re}\left(\alpha^*\delta'\right) \equiv \mathrm{Re}\left(i_1\alpha^*\delta'\right) \equiv \mathrm{Re}\left(i_2\alpha^*\delta'\right) \equiv \mathrm{Re}\left(i_1 i_2\alpha^*\delta'\right) \equiv 0 (\mathrm{mod}\, p). \tag{5.11}$$

This gives us the second condition in the statement of the lemma.

Now we prove that the number of δ satisfying the above conditions is p^3. We first remind the reader that from Proposition 5.2 there are exactly p choices of $\delta \in C_2(\mathbb{Z})$ of the form $u\alpha'$ where the coordinates of u lie between 0 and 1. Let us denote them by $0 = u_1\alpha', u_2\alpha', \ldots, u_p\alpha'$. (Note that again from Proposition 5.2, these p elements form a line $\{tu_2\alpha' : t = 0, 1, \ldots, p-1\}$ mod p.) One can check that all the possible choices of δ satisfying the first condition above are contained in the set $\{(u+u_j)\alpha' : u = x+yi_1+zi_2,\ x, y, z \in \{0, 1, \ldots, p-1\},\ j = 1, 2, \ldots, p\}$.

It is now enough to show that for every $u = x + yi_1 + zi_2,\ x, y, z \in \{0, 1, \ldots, p-1\}$ there is a unique $t = 0, 1, \ldots, p-1$ such that $\alpha^*((u + tu_2)\alpha')' \equiv 0 (\mathrm{mod}\, p)$. To prove this we define a linear transformation N_α on the 3-dimensional space $(\mathbb{Z}/p\mathbb{Z})^3$ by $N_\alpha(u(\mathrm{mod}\, p)) := \alpha^* u'\alpha(\mathrm{mod}\, p)$. Then the map N_α satisfies the following properties.

(1) Rank of N_α is equal to 1.

(2) The eigenvalues of N_α are given by 0 and $4\alpha_3^2$, where $\alpha = \alpha_0 + \alpha_1 i_1 + \alpha_2 i_2 + \alpha_3 i_1 i_2$. (Note that we can assume that $\alpha_3 \neq 0$ by multiplying α by a suitable unit, if necessary. We are allowed to do this since α is fixed only up to a unit.)

(3) For every $t = 0, 1, \ldots, p-1$ the vectors $tu_2'\alpha(\mathrm{mod}\, p)$ are eigenvectors of N_α with eigenvalue $4\alpha_3^2$.

All of the above three properties can be proved by writing down N_α in the form of a 3×3 matrix and doing the required calculations. From these properties we can conclude that the image of the map N_α is precisely the line $\{tu_2'\alpha : t = 0, 1, \ldots, p-1\}$. Hence given $u = x + yi_1 + zi_2,\ x, y, z \in \{0, 1, \ldots, p-1\}$ choose the unique $t_0 \in \{0, 1, \ldots, p-1\}$ such that $N_\alpha(u) = -t_0 u_2'\alpha$. Then $(u + t_0 u_2)\alpha'$ satisfies the second condition in the statement of the lemma. This proves Lemma 5.5 and with it Proposition 5.4. ■

Now we state the main theorem on generators of the Hecke algebra.

Theorem 5.6. The double cosets $X_1 := \Gamma\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)\Gamma$, $X_2 := \Gamma\left(\begin{smallmatrix} \widehat{\alpha} & 0 \\ 0 & \widehat{\alpha}' p \end{smallmatrix}\right)\Gamma$, and $X_3 := \Gamma\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)\Gamma$ generate the algebra $\widetilde{H}_p$ and they are algebraically independent. As in the remarks before Lemma 5.1, $\widehat{\alpha} \in C_2(\mathbb{Z})$ is chosen so that it satisfies $|\widehat{\alpha}|^2 = p$ and $p \nmid \widehat{\alpha}^n$ for every $n \geq 1$. □

Proof. We will prove that $\Gamma M\Gamma$, where $\mu(M)=p^m$, is given by an algebraic expression in terms of X_1, X_2, and X_3 by induction on m. If $m=1$, then the result follows from Lemma 5.1. If $m=2$, then again from Lemma 5.1 we have that $\Gamma M\Gamma=\Gamma\left(\begin{smallmatrix}1&0\\0&p^2\end{smallmatrix}\right)\Gamma$ or X_2 or X_3. Write

$$\left(\Gamma\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma\right)\left(\Gamma\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma\right)=t_0\Gamma\begin{pmatrix}1&0\\0&p^2\end{pmatrix}\Gamma+t_1\Gamma\begin{pmatrix}p&0\\0&p\end{pmatrix}\Gamma+t_2\Gamma\begin{pmatrix}\widehat{\alpha}&0\\0&\widehat{\alpha}'p\end{pmatrix}\Gamma, \tag{5.12}$$

where $t_0, t_1, t_2 \in \mathbb{Z}$. We can do this because every double coset arising in the product above has similitude p^2. If $\Gamma\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)\Gamma = \bigcup_{r=1}^{e}\Gamma M_r$ is the decomposition into disjoint single cosets, then by (5.1) we have $t_0 = \operatorname{card}\left\{(i,j) : \Gamma M_i M_j = \Gamma\left(\begin{smallmatrix}1&0\\0&p^2\end{smallmatrix}\right)\right\}$. Now one can use Proposition 5.2 to check that $t_0 = 1$. Hence we get the required result for $m = 2$.

Now let us assume that the theorem is true for all M with $\mu(M) = p^k$, $k \le m-1$. Let $\mu(M) = p^m$ and from Lemma 5.1 write $\Gamma M\Gamma = \Gamma\left(\begin{smallmatrix}\widehat{\alpha}^l p^{k_1}&0\\0&(\widehat{\alpha}')^l p^{k_2}\end{smallmatrix}\right)\Gamma$ with l, k_1, k_2 satisfying the conditions stated in Lemma 5.1. From Lemma 5.3 and comments following it we know that

$$\operatorname{rk}(M)=\begin{cases}0 & \text{if } k_1>0;\\ 1 & \text{if } k_1=0,\ l>0;\\ 2 & \text{if } k_1=l=0.\end{cases} \tag{5.13}$$

We cannot have $\operatorname{rk}(M) = 3$ or 4. For $\operatorname{rk}(M) = 0$ the theorem follows from the induction hypothesis and the relation $\Gamma M\Gamma = p\Gamma\left(\begin{smallmatrix}\widehat{\alpha}^l p^{k_1-1}&0\\0&(\widehat{\alpha}')^l p^{k_2-1}\end{smallmatrix}\right)\Gamma$.

Now let $\operatorname{rk}(M) = 1$. Denote by B the matrix $\left(\begin{smallmatrix}\widehat{\alpha}^{l-1}&0\\0&(\widehat{\alpha}')^{l-1}p^{k_2-1}\end{smallmatrix}\right)$ and consider the product

$$(\Gamma B\Gamma)\left(\Gamma\begin{pmatrix}\widehat{\alpha}&0\\0&p\widehat{\alpha}'\end{pmatrix}\Gamma\right)=\sum_{C\in\Gamma\backslash\Gamma^{(p^m)}/\Gamma}t(C)\Gamma C\Gamma, \tag{5.14}$$

where $t(C) \in \mathbb{Z}$ and $\Gamma^{(p^m)} := \{M \in GSV_2^+(\mathbb{Z}) : \mu(M) = p^m\}$. Since the rank of every single-coset representative of X_2 is 1, the formula for $t(C)$ in (5.1) gives us that $t(C) \ne 0$ implies that $\operatorname{rk}(C) \le 1$. Fix $C = \left(\begin{smallmatrix}\widehat{\alpha}^{l_1}&0\\0&(\widehat{\alpha}')^{l_1}p^{k_3}\end{smallmatrix}\right)$ with $l_1 + k_3 = m$, $l_1 > 0$, $k_3 \ge l_1$ with $\operatorname{rk}(C) = 1$ and $t(C) \ne 0$. Let $\Gamma B\Gamma = \bigcup \Gamma B_i$ be the decomposition into disjoint single cosets where the B_i are chosen to be upper triangular. Let $\Gamma\left(\begin{smallmatrix}\widehat{\alpha}&0\\0&p\widehat{\alpha}'\end{smallmatrix}\right)\Gamma = \bigcup \Gamma A_i$ be the decomposition into disjoint single cosets as in Proposition 5.4. Choose B_i and A_j such that $B_iA_j = gC$ for some $g \in \Gamma$. Since the left side is upper triangular, we get that g has to be of the form $\left(\begin{smallmatrix}1&u\\0&1\end{smallmatrix}\right)$ with $u \in C_2(\mathbb{Z}) \cap V$. (We are allowed to have units as the diagonal entries but we can multiply both sides with suitable matrix from Γ and get g in the above form.) Again using **Proposition 5.4**, we can conclude that $A_j = \left(\begin{smallmatrix} \widehat{\alpha} & 0 \\ 0 & p\widehat{\alpha}' \end{smallmatrix}\right)$ and hence B_i is forced to be equal to $\left(\begin{smallmatrix} \widehat{\alpha}^{l_1-1} & 0 \\ 0 & (\widehat{\alpha}')^{l_1-1}p^{k_3-1} \end{smallmatrix}\right)$. This is possible if and only if $l_1 = l$ and $k_3 = k_2$, that is, $C = M$. Hence we can conclude that $t(C) \neq 0$ and $\mathrm{rk}(C) = 1 \Leftrightarrow C = M$ and in this case $t(M) = 1$. Now (5.14) gives us

$$(\Gamma B \Gamma)\left(\Gamma \begin{pmatrix} \widehat{\alpha} & 0 \\ 0 & p\widehat{\alpha}' \end{pmatrix} \Gamma\right) = \Gamma M \Gamma + \sum_{\substack{C \in \Gamma \backslash \Gamma^{(p^m)}/\Gamma \\ \mathrm{rk}(C)=0}} t(C) \Gamma C \Gamma \tag{5.15}$$

which gives us the result for $\mathrm{rk}(M) = 1$ using the induction hypothesis and the result for rank 0.

Finally, if $\mathrm{rk}(M) = 2$, we can assume that $M = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^m \end{smallmatrix}\right)$. Using arguments as above one can conclude that

$$\left(\Gamma \begin{pmatrix} 1 & 0 \\ 0 & p^{m-1} \end{pmatrix} \Gamma\right)\left(\Gamma \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma\right) = \Gamma M \Gamma + \sum_{\substack{C \in \Gamma \backslash \Gamma^{(p^m)}/\Gamma \\ \mathrm{rk}(C) \leq 1}} t(C) \Gamma C \Gamma \tag{5.16}$$

with $t(C) \in \mathbb{Z}$. Since we have already shown that the theorem is valid for elements with rank ≤ 1, the above equation completes the proof of the theorem. ■

5.2 Hecke operator T_p

In this section we define the Hecke operator T_p and get a formula for T_p in terms of the Fourier coefficients $A(\beta)$ in **Proposition 5.8**. In **Theorem 5.9** we show that if $f \in S^+_{1/2}(\Gamma_0(4))$ is a Hecke eigenfunction, then F defined in **Section 3.2** is an eigenfunction of T_p with eigenvalue $p^{3/2}\lambda_p + p(p+1)$, where λ_p is the pth eigenvalue of f.

Definition 5.7. Let F be a function on $\mathbb{H}_3$. For any odd prime p, define the Hecke operator T_p by

$$(T_p F)(x) := \sum_{\{\widehat{\beta}\}} F\left(\frac{x + \widehat{\beta}}{p}\right) + F(px) + \sum_{\alpha, \delta} F\big((\alpha x + \delta)(\alpha')^{-1}\big). \tag{5.17}$$

Here $\{\widehat{\beta}\} = \{\beta_0 + \beta_1 i_1 + \beta_2 i_2 : 0 \leq \beta_j \leq p-1,\ j = 0, 1, 2\}$, α, δ are as in **Proposition 5.2**, and $x \in \mathbb{H}_3$.

Note that the Hecke operator is defined by evaluating F at the points obtained by acting the single-coset representatives of **Proposition 5.2** on x and then adding them up. If we multiply the single-coset representatives by an element of Γ, we again get a set of single-coset representatives. Hence if F is an automorphic form with respect to Γ, that is, $F(gx) = F(x)$ for all $g \in \Gamma$, then so is $T_p F$.

Let $\alpha^1, \ldots, \alpha^{p+1}$ be representatives for the orbits of $\{\alpha \in C_2(\mathbb{Z}) : |\alpha|^2 = p\}$ under the left action by the units in $C_2(\mathbb{Z})$. We will identify the lattice $T = V_2(\mathbb{Z})$ with $\mathbb{Z}^3$ and the multiplication of two elements will be as elements of $V_2(\mathbb{Z})$.

Proposition 5.8. (1) For any $\beta \in \mathbb{Z}^3$

$$\left(T_p F\right)_{(\beta)} = p^{3/2} A(p\beta) + p^{3/2} A(\beta/p) + p \sum_{i=1}^{p+1} A\left(\frac{\alpha^{(i)'} \beta \overline{\alpha^{(i)}}}{p}\right). \tag{5.18}$$

Here $(T_p F)_{(\beta)}$ is the βth Fourier coefficient of $T_p F$. If β/p or $\alpha^{(i)'} \beta \overline{\alpha^{(i)}}/p \notin C_2(\mathbb{Z})$, then the corresponding terms are assumed to be zero.

(2) For any prime q define $\nu_q(\beta)$ as the highest power of q dividing all the coordinates of β.

(a) If $q \neq p$, then for every $i = 1, \ldots, p+1$ it holds that $\nu_q(\beta) = \nu_q(\alpha^{(i)'} \beta \overline{\alpha^{(i)}}/p)$.

(b) Write $\beta = \gcd(\beta)(\beta_0 + \beta_1 i_1 + \beta_2 i_2)$ with $\gcd(\beta_0, \beta_1, \beta_2) = 1$.

(i) If $-(\beta_0^2 + \beta_1^2 + \beta_2^2)$ is not a square mod p, then for all $i = 1, \ldots, p+1$ it holds that $\nu_p(\alpha^{(i)'} \beta \overline{\alpha^{(i)}}/p) = \nu_p(\beta) - 1$.

(ii) If $\beta_0^2 + \beta_1^2 + \beta_2^2 \equiv 0 (\bmod p)$ but $\beta_0^2 + \beta_1^2 + \beta_2^2 \not\equiv 0 (\bmod p^2)$, then there is exactly one i such that $\nu_p(\alpha^{(i)'} \beta \overline{\alpha^{(i)}}/p) = \nu_p(\beta)$ and for all $j \neq i$ it holds that $\nu_p(\alpha^{(j)'} \beta \overline{\alpha^{(j)}}/p) = \nu_p(\beta) - 1$.

(iii) If $\beta_0^2 + \beta_1^2 + \beta_2^2 \equiv 0 (\bmod p^2)$, then there is exactly one i such that $\nu_p(\alpha^{(i)'} \beta \overline{\alpha^{(i)}}/p) = \nu_p(\beta) + 1$ and for all $j \neq i$ it holds that $\nu_p(\alpha^{(j)'} \beta \overline{\alpha^{(j)}}/p) = \nu_p(\beta) - 1$.

(iv) If $-(\beta_0^2 + \beta_1^2 + \beta_2^2)$ is a square mod p, then there are exactly two distinct $i_{\kappa_1}, i_{\kappa_2}$ such that $\nu_p(\alpha^{(i_\mu)'} \beta \overline{\alpha^{(i_\mu)}}/p) = \nu_p(\beta)$ for $\mu = \kappa_1, \kappa_2$ and for all $j \neq i_{\kappa_1}, i_{\kappa_2}$ it holds that $\nu_p(\alpha^{(j)'} \beta \overline{\alpha^{(j)}}/p) = \nu_p(\beta) - 1$. □

Proof. Let $\beta = \beta_0 + \beta_1 i_1 + \beta_2 i_2$ and $\alpha = \alpha_0 + \alpha_1 i_1 + \alpha_2 i_2 + \alpha_3 i_1 i_2$. (We will drop the superscript i for α whenever there is no confusion.) Write $\alpha' \beta \bar{\alpha} = \widehat{\beta_0} + \widehat{\beta_1} i_1 + \widehat{\beta_2} i_2$. Writing in terms of the coordinates of β and α we have

(I)

$$\begin{aligned}\widehat{\beta_0} &= -2\alpha_3\big(\beta_2\alpha_1-\beta_1\alpha_2-\beta_0\alpha_3\big)+2\alpha_0\big(\beta_0\alpha_0+\beta_1\alpha_1+\beta_2\alpha_2\big)-p\beta_0,\\ \widehat{\beta_1} &= -2\alpha_2\big(\beta_2\alpha_1-\beta_1\alpha_2-\beta_0\alpha_3\big)+2\alpha_0\big(-\beta_0\alpha_1+\beta_1\alpha_0-\beta_2\alpha_3\big)-p\beta_1,\\ \widehat{\beta_2} &= 2\alpha_1\big(\beta_2\alpha_1-\beta_1\alpha_2-\beta_0\alpha_3\big)+2\alpha_0\big(-\beta_0\alpha_2+\beta_1\alpha_3+\beta_2\alpha_0\big)-p\beta_2.\end{aligned} \tag{5.19}$$

Using these formulae we have

(II)

$$\begin{aligned}\beta_0\widehat{\beta_0}+\beta_1\widehat{\beta_1}+\beta_2\widehat{\beta_2} &= -p|\beta|^2+2\alpha_0^2|\beta|^2+2\big(\beta_2\alpha_1-\beta_1\alpha_2-\beta_0\alpha_3\big)^2,\\ -\beta_0\widehat{\beta_0}+\beta_1\widehat{\beta_1}-\beta_2\widehat{\beta_2} &= -p|\beta|^2+2\alpha_2^2|\beta|^2+2\big(-\beta_0\alpha_1+\beta_1\alpha_0-\beta_2\alpha_3\big)^2,\\ \beta_0\widehat{\beta_0}-\beta_1\widehat{\beta_1}-\beta_2\widehat{\beta_2} &= -p|\beta|^2+2\alpha_3^2|\beta|^2+2\big(\beta_0\alpha_0+\beta_1\alpha_1+\beta_2\alpha_2\big)^2,\\ -\beta_0\widehat{\beta_0}-\beta_1\widehat{\beta_1}+\beta_2\widehat{\beta_2} &= -p|\beta|^2+2\alpha_1^2|\beta|^2+2\big(-\beta_0\alpha_2+\beta_1\alpha_3+\beta_2\alpha_0\big)^2.\end{aligned} \tag{5.20}$$

These equations essentially give us the proof of part (2) of the proposition. Let us illustrate the proof of (2)(b)(i). To prove (2)(b)(i) we have to show that for all the $p+1$ values of α we have $\alpha'\beta\bar{\alpha} \not\equiv 0(\operatorname{mod} p)$ assuming that $-|\beta|^2$ is not a square mod p. Suppose not, then $\widehat{\beta_0}, \widehat{\beta_1}, \widehat{\beta_2} \equiv 0(\operatorname{mod} p)$. We can assume that $\alpha_0 \neq 0$ (we can always multiply α by a unit to ensure this). Then (5.20) above gives us

$$\begin{aligned}&\big(\beta_2\alpha_1-\beta_1\alpha_2-\beta_0\alpha_3\big)^2+\alpha_0^2|\beta|^2 \equiv 0(\operatorname{mod} p)\\ &\Longrightarrow \big(\alpha_0^{-1}\big(\beta_2\alpha_1-\beta_1\alpha_2-\beta_0\alpha_3\big)\big)^2 \equiv -|\beta|^2(\operatorname{mod} p).\end{aligned} \tag{5.21}$$

This contradicts the assumption that $-|\beta|^2$ is not a square mod p. This proves (2)(b)(i). The proofs of the other statements involve manipulation of equations in (5.20) and we do not present them here.

Now we will prove (1). We have

$$F(x) = \sum_{\substack{\beta\in\mathbb{Z}^3\\ \beta\neq 0}} A(\beta)x_3^{3/2}K_{ir}\big(2\pi|\beta|x_3\big)e^{2\pi i\,\mathrm{Re}(\beta x)}. \tag{5.22}$$

To evaluate the first term on the right-hand side of (5.17), we have to sum over $\widehat{\beta}$ the following expression:

$$F\left(\frac{x+\widehat{\beta}}{p}\right) = \sum_{\substack{\beta\in\mathbb{Z}^3\\ \beta\neq 0}} A(\beta)\big(x_3p^{-1}\big)^{3/2}K_{ir}\big(2\pi|\beta|p^{-1}x_3\big)e^{2\pi i\,\mathrm{Re}(\beta p^{-1}x)}e^{2\pi i\,\mathrm{Re}(\beta\widehat{\beta}/p)}. \tag{5.23}$$

One can check that

$$\sum_{\{\widehat{\beta}\}} e^{2\pi i \operatorname{Re}(\beta\widehat{\beta}/p)} = \begin{cases} p^3 & \text{if } p \mid \beta, \\ 0 & \text{otherwise.} \end{cases} \tag{5.24}$$

We first sum over $\widehat{\beta}$ and then (5.24) implies that we can replace β by $p\beta$ since the elements in the lattice $\mathbb{Z}^3$ that are not divisible by p vanish:

$$\sum_{\{\widehat{\beta}\}} F\left(\frac{x+\widehat{\beta}}{p}\right) = \sum_{\substack{\beta\in\mathbb{Z}^3\\ \beta\neq 0}} p^{3/2} A(p\beta) x_3^{3/2} K_{ir}\left(2\pi|\beta|x_3\right) e^{2\pi i \operatorname{Re}(\beta x)}. \tag{5.25}$$

Next let us calculate

$$F(px) = \sum_{\substack{\beta\in\mathbb{Z}^3\\ \beta\neq 0}} A(\beta)\left(x_3 p\right)^{3/2} K_{ir}\left(2\pi|\beta|p x_3\right) e^{2\pi i \operatorname{Re}(\beta p x)}. \tag{5.26}$$

Replacing β by β/p we get

$$F(px) = \sum_{\substack{\beta\in\mathbb{Z}^3\\ \beta\neq 0}} p^{3/2} A\left(\frac{\beta}{p}\right) x_3^{3/2} K_{ir}\left(2\pi|\beta|x_3\right) e^{2\pi i \operatorname{Re}(\beta x)}. \tag{5.27}$$

The indexing set above should consist of those β such that $\beta/p \in \mathbb{Z}^3$. But since we assume that $A(\beta/p) = 0$ if p does not divide β, we can write the indexing set as $\beta \in \mathbb{Z}^3$. Now let us consider

$$F\left((\alpha x+\delta)\alpha'^{-1}\right) = F\left((\alpha x+\delta)\frac{\alpha^*}{p}\right) = F\left(\frac{\alpha x \alpha^*}{p} + \frac{\delta\alpha^*}{p}\right). \tag{5.28}$$

Notice that the x_3 component remains unchanged when x is replaced by $\alpha x \alpha^*/p$. We have

$$F\left((\alpha x+\delta)\alpha'^{-1}\right) = \sum_{\substack{\beta\in\mathbb{Z}^3\\ \beta\neq 0}} A(\beta) x_3^{3/2} K_{ir}\left(2\pi|\beta|x_3\right) e^{2\pi i \operatorname{Re}(\beta\alpha x\alpha^*/p)} e^{2\pi i \operatorname{Re}(\beta\delta\alpha^*/p)}. \tag{5.29}$$

Replace β by $\alpha'\beta\bar{\alpha}/p$. Using $\operatorname{Re}(ab) = \operatorname{Re}(ba)$ and $|\beta|^2 = |\alpha'\beta\bar{\alpha}/p|^2$, we get

$$F\left((\alpha x+\delta)\alpha'^{-1}\right) = \sum_{\substack{\beta\in\mathbb{Z}^3\\ \beta\neq 0}} A\left(\frac{\alpha'\beta\bar{\alpha}}{p}\right) x_3^{3/2} K_{ir}\left(2\pi|\beta|x_3\right) e^{2\pi i \operatorname{Re}(\beta x)} e^{2\pi i \operatorname{Re}(\beta\bar{\alpha}\delta/p)}. \tag{5.30}$$

We assume that $A(\alpha'\beta\bar{\alpha}/p) = 0$ if p does not divide $\alpha'\beta\bar{\alpha}$. The indexing set in (5.30) can be written as $\beta \in \mathbb{Z}^3$ by the same arguments as in (5.27). One can check that for every α such that $\alpha'\beta\bar{\alpha} \equiv 0 (\mathrm{mod}\, p)$, we have

$$\sum_{\delta} e^{2\pi i \operatorname{Re}(\beta\bar{\alpha}\delta/p)} = p. \tag{5.31}$$

Finally, putting together (5.25), (5.27), (5.30), and (5.31) we get (5.18) as required. This completes the proof of Proposition 5.8. ■

We are now ready to state and prove the main theorem of this section. Let us remind the reader of the definition of the Fourier coefficients $A(\beta)$ of F given in (3.4): let $\beta = p^s 2^u d(\beta_0 + \beta_1 i_1 + \beta_2 i_2)$ with $s, u \geq 0$, d odd such that $p \nmid d$ and $\gcd(\beta_0, \beta_1, \beta_2) = 1$. Then

$$A(\beta) := 2^{3/4}|\beta| \sum_{m=0}^{s} \sum_{t=0}^{u} \left(\sum_{n|d} c\left(\frac{-|\beta|^2}{(2^t p^m n)^2} \right) (p^m n)^{-1/2} \right) (-1)^t 2^{t/2}. \tag{5.32}$$

Theorem 5.9. Let $f(z) = \sum_{n \neq 0} c(n) W_{\mathrm{sign}(n)/4, ir/2}(4\pi|n|y) e^{2\pi i n x}$ be a Hecke eigenform in $S^{+}_{1/2}(4)$ with eigenvalue λ_p for every odd prime p. Then F defined by Fourier coefficients $A(\beta)$ as above satisfies

$$T_p F = \big(p^{3/2}\lambda_p + p(p+1)\big) F \tag{5.33}$$

for all odd prime numbers p. □

Proof. For every β we have to show that $(T_p F)_{(\beta)} = (p^{3/2}\lambda_p + p(p+1))A(\beta)$. From Proposition 5.8(2)(a) we know that the power of $q \neq p$ dividing β does not change when β is replaced by $\alpha'\beta\bar{\alpha}/p$. It will be clear from the proof that the computations only involve the prime p. Hence it is enough to show the result for the case $\nu_q(\beta) = 0$ for all $q \neq p$. Hence let $\beta = p^s(\beta_0 + \beta_1 i_1 + \beta_2 i_2) =: p^s \tau$ with $\mathrm{gcb}(\beta_0, \beta_1, \beta_2) = 1$. Then we have

$$\begin{aligned} A(p\beta) = 2^{3/4}|\beta| \Bigg\{ &\sum_{m=0}^{s-1} pc\left(\frac{-p^{2s}|\tau|^2 p^2}{p^{2m}} \right) p^{-m/2} \\ &+ pc\big(-|\tau|^2 p^2\big) p^{-s/2} + pc\big(-|\tau|^2\big) p^{-s/2} p^{-1/2} \Bigg\}, \end{aligned}$$

$$A\left(\frac{\beta}{p}\right) = 2^{3/4}|\beta| \sum_{r=0}^{s-1} p^{-1} c\left(\frac{-p^{2s}|\tau|^2}{p^{2r}p^2}\right) p^{-r/2},$$

$$A\left(\frac{\alpha'\beta\bar{\alpha}}{p}\right) = \begin{cases} A(\beta) & \text{if } \nu_p\left(\frac{\alpha'\beta\bar{\alpha}}{p}\right) = \nu_p(\beta) = s; \\ A(\beta) - 2^{3/4}|\beta| c\left(-|\tau|^2\right) p^{-s/2} & \text{if } \nu_p\left(\frac{\alpha'\beta\bar{\alpha}}{p}\right) = \nu_p(\beta) - 1 = s - 1 \geq 0; \\ A(\beta) + 2^{3/4}|\beta| c\left(\frac{-|\tau|^2}{p^2}\right) p^{-s/2} p^{-1/2} & \text{if } \nu_p\left(\frac{\alpha'\beta\bar{\alpha}}{p}\right) = \nu_p(\beta) + 1 = s + 1; \\ 0 & \text{if } \nu_p\left(\frac{\alpha'\beta\bar{\alpha}}{p}\right) < 0. \end{cases} \tag{5.34}$$

Here we have used the fact that $|\alpha'\beta\bar{\alpha}/p| = |\beta|$. Using the following relation satisfied by the coefficients $c(n)$ of f:

$$pc\left(np^2\right) + p^{-1/2}\left(\frac{n}{p}\right)c(n) + p^{-1}c\left(\frac{n}{p^2}\right) = \lambda_p c(n) \tag{5.35}$$

and Proposition 5.8(2)(b), we get the result of Theorem 5.9. Notice that we get four different equations from (5.35) according to whether $(n/p) = -1$, $n \equiv 0 (\mathrm{mod}\, p)$ but $n \not\equiv 0 (\mathrm{mod}\, p^2)$, $n \equiv 0 (\mathrm{mod}\, p^2)$, or $(n/p) = +1$. The four cases in Proposition 5.8(2)(b) correspond to these when we set $n = -(\beta_0^2 + \beta_1^2 + \beta_2^2)$. ■

5.3 Hecke operator T_{p^2}

In this section we define the Hecke operator T_{p^2} and get a formula for T_{p^2} in terms of the Fourier coefficients $A(\beta)$ in Proposition 5.11. In Theorem 5.12 we show that if $f \in S^+_{1/2}(\Gamma_0(4))$ is a Hecke eigenfunction, then F defined in Section 3.2 is an eigenfunction of T_{p^2} with eigenvalue $(p+1)p^{3/2}\lambda_p + (p-1)(p+1)$, where λ_p is the pth eigenvalue of f.

We will now give the definition of the Hecke operator T_{p^2}.

Definition 5.10. Let F be a function on $\mathbb{H}_3$. For any odd prime p, define the Hecke operator T_{p^2} by

$$\left(T_{p^2}F\right)(x) := \sum_{\alpha} \left(F\left(\alpha x \alpha^*\right) + \sum_{\delta} F\left(\frac{\alpha x \alpha^*}{p^2} + \frac{\delta\alpha^*}{p^2}\right)\right) + \sum_{\nu} F\left(x + \frac{\nu}{p}\right). \tag{5.36}$$

Here the sum over α, δ, and ν is the same as in Proposition 5.4 and $x \in \mathbb{H}_3$.

Note that just as in the case of the Hecke operator T_p in the previous section, T_{p^2} maps an automorphic function F to an automorphic function.

Let $(T_{p^2}F)_{(\beta)}$ be the βth Fourier coefficient of $T_{p^2}F$. The following proposition describes $(T_{p^2}F)_{(\beta)}$ in terms of $A(\beta)$.

Proposition 5.11. With notations as above, it holds that

$$(T_{p^2}F)_{(\beta)} = p^{3/2}\sum_{\alpha}\left(A(\alpha'\beta\bar{\alpha}) + A\left(\frac{\alpha'\beta\bar{\alpha}}{p^2}\right)\right) + \left(\sum_{\nu} e^{2\pi i\,\mathrm{Re}(\beta\nu/p)}\right)A(\beta), \tag{5.37}$$

where α runs through the orbits of the action of the group of units in $C_2(\mathbb{Z})$ on the set $\{\alpha \in C_2(\mathbb{Z}) : |\alpha|^2 = p\}$ and ν runs through the set $\{\nu = \nu_0 + \nu_1 i_1 + \nu_2 i_2 : \nu_0, \nu_1, \nu_2 \in \mathbb{Z},\ 0 \le \nu_0, \nu_1, \nu_2 \le p-1,\ \nu \ne 0,\ |\nu|^2 \equiv 0 (\mathrm{mod}\, p)\}$. Here, assume that $A(\alpha'\beta\bar{\alpha}/p^2) = 0$ if p^2 does not divide $\alpha'\beta\bar{\alpha}$. $\square$

Proof. We will evaluate (5.36) by substituting the Fourier expansion of F. Notice that the third term on the right-hand side of (5.37) follows directly from the third term on the right-hand side of (5.36). Fix α such that $|\alpha|^2 = p$,

$$F(\alpha x\alpha^*) = \sum_{\substack{\beta\in\mathbb{Z}^3\\ \beta\ne 0}} A(\beta)(px_3)^{3/2}K_{ir}(2\pi|\beta|px_3)e^{2\pi i\,\mathrm{Re}(\beta\alpha x\alpha^*)}. \tag{5.38}$$

Here we have used that the coefficient of i_3 in $\alpha x\alpha^*$ is equal to px_3. Now replace β by $(\alpha'\beta\bar{\alpha})/p^2$ to get

$$F(\alpha x\alpha^*) = \sum_{\substack{\beta\in\mathbb{Z}^3\\ \beta\ne 0}} p^{3/2}A\left(\frac{\alpha'\beta\bar{\alpha}}{p^2}\right)x_3^{3/2}K_{ir}(2\pi|\beta|x_3)e^{2\pi i\,\mathrm{Re}(\beta x)}. \tag{5.39}$$

Next, we have

$$\begin{aligned} &F\left(\frac{\alpha x\alpha^*}{p^2} + \frac{\delta\alpha^*}{p^2}\right) \\ &\quad = \sum_{\substack{\beta\in\mathbb{Z}^3\\ \beta\ne 0}} A(\beta)\left(\frac{x_3}{p}\right)^{3/2} K_{ir}\left(\frac{2\pi|\beta|x_3}{p}\right)e^{2\pi i\,\mathrm{Re}(\beta\alpha x\alpha^*/p^2)}e^{2\pi i\,\mathrm{Re}(\beta\delta\alpha^*/p^2)}. \end{aligned} \tag{5.40}$$

Here we have used that the coefficient of i_3 in $\alpha x\alpha^*/p^2$ is equal to x_3/p. Now replace β by $\alpha'\beta\bar{\alpha}$ to get

$$F\left(\frac{\alpha x\alpha^*}{p^2} + \frac{\delta\alpha^*}{p^2}\right) = \sum_{\substack{\beta\in\mathbb{Z}^3\\ \beta\ne 0}} p^{-3/2}A(\alpha'\beta\bar{\alpha})x_3^{3/2}K_{ir}(2\pi|\beta|x_3)e^{2\pi i\,\mathrm{Re}(\beta x)}. \tag{5.41}$$

The indexing set in (5.39) and (5.41) can be taken as $\mathbb{Z}^3$ by arguments similar to those in the proof of Proposition 5.8. In (5.41) we have used that $e^{2\pi i\,\mathrm{Re}(\alpha'\beta\bar{\alpha}\delta\alpha^*/p^2)} = e^{2\pi i\,\mathrm{Re}(\alpha^*\alpha'\beta\bar{\alpha}\delta/p^2)} = e^{2\pi i\,\mathrm{Re}(\beta\bar{\alpha}\delta/p)} = 1$ because $\bar{\alpha}\delta \equiv 0 (\mathrm{mod}\, p)$ by the second condition in Lemma 5.5. Since there are exactly p^3 values of δ for each α, (5.39) and (5.41) complete the proof of Proposition 5.11. ■

Now we state the main theorem of this section.

Theorem 5.12. Let $f \in S^+_{1/2}(\Gamma_0(4))$ be a Hecke eigenfunction with eigenvalue λ_p for every odd prime p. Let F be the function on $\mathbb{H}_3$ defined in Section 3.2. Then

$$T_{p^2}F = \big((p+1)p^{3/2}\lambda_p + (p-1)(p+1)\big)F. \tag{5.42}$$

□

Proof. Let $\beta = 2^u dp^s\tau$ with $u, s \geq 0$, d odd not divisible by p, and $\tau = \tau_0 + \tau_1 i_1 + \tau_2 i_2$ with $\gcd(\tau_0, \tau_1, \tau_2) = 1$. From Proposition 5.8(2)(a) we can assume without loss of generality as in the proof of Theorem 5.9 that $u = 0$ and $d = 1$. Hence $\beta = p^s\tau$. We will first evaluate $R_\beta := p^{3/2}\sum_\alpha (A(\alpha'\beta\bar{\alpha}) + A(\alpha'\beta\bar{\alpha}/p^2))$ for different possibilities of s and $|\tau|^2$. We claim that

$$R_\beta = \begin{cases} (p+1)p^{3/2}\lambda_p A(\beta) & \text{if } p \mid \beta; \\ (p+1)p^{3/2}\lambda_p A(\beta) + p^2 A(\beta) & \text{if } p \mid |\beta|^2,\ p \nmid \beta; \\ (p+1)p^{3/2}\lambda_p A(\beta) + p(p-1)A(\beta) & \text{if } -|\beta|^2 \text{ is a square } \mathrm{mod}\, p; \\ (p+1)p^{3/2}\lambda_p A(\beta) + p(p+1)A(\beta) & \text{if } -|\beta|^2 \text{ is not a square mod}\, p. \end{cases} \tag{5.43}$$

Let us illustrate the computation of the fourth case above. We have $s = 0$ and from Proposition 5.8(2)(b)(i) we have $\nu_p(\alpha'\beta\bar{\alpha}) = \nu_p(\beta)$ for all α:

$$\begin{aligned} R_\beta &= (p+1)p^{3/2}2^{3/4}|\beta| p c\big(-p^2|\tau|^2\big) \\ &= (p+1)p^{3/2}\lambda_p A(\beta) + (p+1)pA(\beta). \end{aligned} \tag{5.44}$$

We get the last equality by using the following formula for $c(n)$ which is obtained by substituting $n = -|\tau|^2$ in (5.35):

$$pc\big(-p^2|\tau|^2\big) = \lambda_p c\big(-|\tau|^2\big) + p^{-1/2}c\big(-|\tau|^2\big). \tag{5.45}$$

■

We get the remaining cases by similar calculation. Now we get the theorem from the following lemma.

Lemma 5.13.

$$\sum_{\nu} e^{2\pi i\,\mathrm{Re}(\beta\nu/p)} = \begin{cases} p^2-1 & \text{if } p \mid \beta; \\ -1 & \text{if } p \mid |\beta|^2,\ p \nmid \beta; \\ p-1 & \text{if } -|\beta|^2 \text{ is a square } \bmod p; \\ -(p+1) & \text{if } -|\beta|^2 \text{ is not a square } \bmod p. \end{cases} \tag{5.46}$$

Here the sum on the left is over the set $\{\nu = \nu_0 + \nu_1 i_1 + \nu_2 i_2 : \nu_0, \nu_1, \nu_2 \in \mathbb{Z},\ 0 \le \nu_0, \nu_1, \nu_2 \le p-1,\ \nu \neq 0,\ |\nu|^2 \equiv 0 (\bmod p)\}$. □

Proof. The first case is obvious since the number of elements in the set $K := \{\nu \in \mathbb{Z}/p\mathbb{Z} + \mathbb{Z}/p\mathbb{Z}i_1 + \mathbb{Z}/p\mathbb{Z}i_2 : \nu \neq 0,\ |\nu|^2 = 0\}$ is p^2-1. Denote by $U = \mathbb{Z}/p\mathbb{Z} + \mathbb{Z}/p\mathbb{Z}i_1 + \mathbb{Z}/p\mathbb{Z}i_2$ the 3-dimensional space over $\mathbb{Z}/p\mathbb{Z}$. Since $\nu \in K \Rightarrow t\nu \in K$ for $t = 1, \ldots, p-1$, we can see that the set K is the union of $p+1$ lines without the origin. Let us fix $p+1$ vectors $\nu_1, \ldots, \nu_{p+1}$ such that $K = \{t\nu_j : t = 1, \ldots, p-1 \text{ and } j = 1, \ldots, p+1\}$. Define a map I_β from U to $\mathbb{Z}/p\mathbb{Z}$ by $I_\beta(u) := \mathrm{Re}(\beta u)(\bmod p)$ for $u \in U$. Since $\beta \neq 0$, the dimension of $\mathrm{Ker}(I_\beta)$ is 2.

Case 1. Let $|\beta|^2 \equiv 0 (\bmod p)$ but $p \nmid \beta$. We can show that if $u \in \mathrm{Ker}(I_\beta) \cap K$, then $u = t\bar{\beta}(\bmod p)$ for some $t = 1, \ldots, p-1$. From this we get

$$\begin{aligned} \sum_{\nu} e^{2\pi i\,\mathrm{Re}(\beta\nu/p)} &= \sum_{t=1}^{p-1} e^{2\pi i t\,\mathrm{Re}(\beta\bar{\beta}/p)} + \sum_{\nu_j \neq \bar{\beta}} \sum_{t=1}^{p-1} e^{2\pi i t\,\mathrm{Re}(\beta\nu_j/p)} \\ &= (p-1) + p(-1) = -1 \end{aligned} \tag{5.47}$$

as required.

Case 2. Let $-|\beta|^2 \equiv x^2 (\bmod p)$ for some $x \not\equiv 0 (\bmod p)$. We can show that there are exactly two linearly independent elements ν_x and ν_{-x} in K which are in $\mathrm{Ker}(I_\beta)$. Hence we have

$$\begin{aligned} \sum_{\nu} e^{2\pi i\,\mathrm{Re}(\beta\nu/p)} &= \sum_{t=1}^{p-1} e^{2\pi i t\,\mathrm{Re}(\beta\nu_x/p)} + \sum_{t=1}^{p-1} e^{2\pi i t\,\mathrm{Re}(\beta\nu_{-x}/p)} \\ &\quad + \sum_{\nu_j \neq \nu_x, \nu_{-x}} \sum_{t=1}^{p-1} e^{2\pi i t\,\mathrm{Re}(\beta\nu_j/p)} \\ &= 2(p-1) + (p-1)(-1) = p-1 \end{aligned} \tag{5.48}$$

as required.

Case 3. Let us assume that $-|\beta|^2$ is not a square mod p. We can show that $\mathrm{Ker}(I_\beta)\bigcap K$ is empty. So we have

$$\begin{aligned}\sum_{\nu} e^{2\pi i\,\mathrm{Re}(\beta\nu/p)} &= \sum_{j=1}^{p+1}\sum_{t=1}^{p-1} e^{2\pi i t\,\mathrm{Re}(\beta\nu_j/p)} \\ &= (p+1)(-1) = -(p+1).\end{aligned} \tag{5.49}$$

This completes the proof of Lemma 5.13 and hence the proof of Theorem 5.12. ■

6 Automorphic representation corresponding to F

In this section we will give the classical to adelic calculation. Starting from the automorphic cuspidal function F defined in (3.4), which is an eigenfunction for the Hecke operators T_p and T_{p^2} for every odd prime p, we define a cuspidal automorphic form on the adelic group. This form gives an irreducible automorphic representation and we will explicitly calculate its p-adic component for $p \neq 2$.

In Sections 2–5 we have considered the function $F : \mathbb{H}_3 \to \mathbb{C}$ as an automorphic function for the group $SV_2(\mathbb{R}) \simeq \mathrm{Spin}(1,4)(\mathbb{R})$. Since $\mathbb{H}_3$ is also a symmetric space for $GSV_2^+(\mathbb{R}) \simeq \mathrm{GSpin}^+(1,4)(\mathbb{R})$, the similitude group, we can consider F to be an automorphic function for the group $\mathrm{GSpin}^+(1,4)(\mathbb{R})$ with trivial central character.

Let us denote by $G := \mathrm{GSpin}(1,4)$ the similitude group. Let $\mathbb{A}$ be the ring of adeles for the global field $\mathbb{Q}$. We have the following strong approximation for G:

$$G(\mathbb{A}) \simeq G(\mathbb{Q})G^+(\mathbb{R})K_0, \quad \text{where } K_0 := \prod_{p<\infty} G(\mathbb{Z}_p). \tag{6.1}$$

We refer the reader to [26, Theorem 104:4] for details on the above strong approximation result. Note that the hyperbolic upper-half space $\mathbb{H}_3$ is also a symmetric space for $GSV_2^+(\mathbb{R}) \simeq G^+(\mathbb{R})$. Given a cuspidal automorphic Hecke eigenform $F : \mathbb{H}_3 \to \mathbb{C}$ write $g = g_{\mathbb{Q}} g_\infty k_0$ where $g \in G(\mathbb{A})$, $g_{\mathbb{Q}} \in G(\mathbb{Q})$, $g_\infty \in G^+(\mathbb{R})$, and $k_0 \in K_0$ and define $\Phi_F : G(\mathbb{A}) \to \mathbb{C}$ as $\Phi_F(g) := F(g_\infty(i_3))$. Φ_F satisfies the following properties:

(1) $\Phi_F(\rho g) = \Phi_F(g)$ for $\rho \in G(\mathbb{Q})$,

(2) $\Phi_F(gk_0) = \Phi_F(g)$ for $k_0 \in K_0$,

(3) $\Phi_F(gz) = \Phi_F(g)$ for $z \in \mathbb{Z}(\mathbb{A}) \simeq GL_1(\mathbb{A})$.

Since F is a cusp form, we can show that Φ_F is cuspidal as in [1, Lemma 5]. Now we consider the representation of $G(\mathbb{A})$ obtained from Φ_F by right translations. Denote by (π_F, V_F) an irreducible component containing Φ_F. (We will drop the subscript F whenever there is no confusion.)

Write $\pi \simeq \otimes'_p \pi_p$ where π_p is a representation of $G_p := G(\mathbb{Q}_p)$. Note that for p, an odd prime, π_p is an irreducible unramified representation of G_p. From [5] we know that there exists an unramified character χ of the Borel subgroup of G_p, unique up to the Weyl group orbit, such that π_p is isomorphic to the unique spherical constituent π_χ of the normalized induced representation $\mathrm{Ind}_B^{G_p}(\chi)$. Again from [5] we have a one-to-one correspondence between unramified representations of G_p and algebra homomorphisms of the Hecke algebra $H(G_p, K_p)$ to $\mathbb{C}$. Here $K_p = G(\mathbb{Z}_p)$. We will use the calculation involving the classical Hecke algebra in Section 5 to obtain the Hecke algebra homomorphism corresponding to π_p. Using this homomorphism we will get an explicit formula for the unramified character χ.

6.1 Unramified calculation

For the rest of the section, p is an odd prime. As seen in Section 5, the classical Hecke algebra H_p is generated by double cosets $\Gamma M \Gamma$ where $M \in \mathrm{GSpin}(1,4)^+(\mathbb{Z}[p^{-1}]) =: G^+(\mathbb{Z}[p^{-1}])$. Here $\mathbb{Z}[p^{-1}]$ is the set of rational numbers with only powers of p in the denominator. In Theorem 5.6 we gave the four algebraically independent generators of this algebra. $H(G_p, K_p)$ is the convolution algebra of K_p-biinvariant compactly supported functions on G_p. This is generated by the characteristic functions of double cosets $K_p g K_p$ with $g \in G_p$.

Proposition 6.1. With notations as above

$$H_p \simeq H\big(G_p, K_p\big). \tag{6.2}$$

□

Proof. We have the following bijections from the natural inclusions:

$$\Gamma \backslash G^+\big(\mathbb{Z}\big[p^{-1}\big]\big)/\Gamma \simeq G(\mathbb{Z}) \backslash G\big(\mathbb{Z}\big[p^{-1}\big]\big)/G(\mathbb{Z}) \simeq K_p \backslash G_p / K_p. \tag{6.3}$$

The first bijection is obvious since $G(\mathbb{Z}) = \Gamma \cup \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\Gamma$ and $G(\mathbb{Z}[p^{-1}]) = G^+(\mathbb{Z}[p^{-1}]) \cup \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \times G^+(\mathbb{Z}[p^{-1}])$. To get the second one, note that we get injectivity from the fact that $G(\mathbb{Z}[p^{-1}]) \cap K_p = G(\mathbb{Z})$. We get surjectivity from the isomorphism $G_p \simeq G(\mathbb{Q})K_p \simeq G(\mathbb{Z}[p^{-1}])K_p$. This tells us that the two Hecke algebras are isomorphic as vector spaces. We get similar bijections for single cosets. This is used to check that the classical double-coset multiplication coincides with the convolution product on the p-adic Hecke algebra.

■

Hence $H(G_p, K_p)$ is generated by the functions $\phi_1 := (1/\mathrm{Vol}(K_p))\,\mathrm{char}(K_p \left(\begin{smallmatrix} 1 & \\ & p \end{smallmatrix}\right) K_p)$, $\phi_2 := (1/\mathrm{Vol}(K_p))\,\mathrm{char}(K_p \left(\begin{smallmatrix} \widehat{\alpha} & \\ & p\widehat{\alpha}' \end{smallmatrix}\right) K_p)$, $\phi_3 := (1/\mathrm{Vol}(K_p))\,\mathrm{char}\big(K_p \left(\begin{smallmatrix} p & \\ & p \end{smallmatrix}\right) K_p\big)$, and $\phi_4 := (1/\mathrm{Vol}(K_p))\,\mathrm{char}\big(K_p \left(\begin{smallmatrix} p^{-1} & \\ & p^{-1} \end{smallmatrix}\right) K_p\big)$. Since the representation π_p is unramified, we have a unique (up to a scalar) K_p fixed unramified vector $F_p^0 \in V_{\pi_p}$. Now Proposition 6.1, Theorems 5.9 and 5.12 give us the following proposition.

Proposition 6.2. With notations as above, $\phi_1 F_p^0 = (p^{3/2}\lambda_p + p(p+1))F_p^0$, $\phi_2 F_p^0 = ((p+1)p^{3/2}\lambda_p + (p^2-1))F_p^0$, $\phi_3 F_p^0 = F_p^0$, and $\phi_4 F_p^0 = F_p^0$. □

Since the ϕ_i generate the Hecke algebra freely, Proposition 6.2 gives us an algebra homomorphism of $H(G_p, K_p)$. Our next step is to find the unramified character which corresponds to this homomorphism. For this it is convenient to work in the setting of the symplectic group. Define $GSp_4(\mathbb{Q}_p) := \{M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Mat_4(\mathbb{Q}_p) : A^tD - B^tC = \mu(M)I_2,\ \mu(M) \in \mathbb{Q}_p^\times,\ B^tD = D^tB,\ A^tC = C^tA\}$.

Proposition 6.3. Let p be an odd prime. Then with notations as above, $G_p \simeq GSp_4(\mathbb{Q}_p)$ and $K_p \simeq GSp_4(\mathbb{Z}_p)$. □

Proof. Fix $r, s \in \mathbb{Z}_p$ such that $r^2 + s^2 = -1$. (We can always choose r, s as above since p does not divide the discriminant of $\mathbb{Z}_p[\sqrt{-1}]$ and hence every unit in $\mathbb{Z}_p[\sqrt{-1}]$ is a norm.) For $\alpha = \alpha_0 + \alpha_1 i_1 + \alpha_2 i_2 + \alpha_3 i_1 i_2 \in C_2(\mathbb{Q}_p)$, define $\psi_p : C_2(\mathbb{Q}_p) \to Mat_2(\mathbb{Q}_p)$ by the formula

$$\psi_p(\alpha) := \begin{pmatrix} \alpha_0 - \alpha_1 r - \alpha_2 s & -\alpha_3 - \alpha_1 s + \alpha_2 r \\ \alpha_3 - \alpha_1 s + \alpha_2 r & \alpha_0 + \alpha_1 r + \alpha_2 s \end{pmatrix}. \tag{6.4}$$

ψ_p satisfies the following properties:

(1) ψ_p is an isomorphism of $\mathbb{Q}_p$-algebras,
(2) $\det(\psi_p(\alpha)) = |\alpha|^2$,
(3) $\psi_p(\alpha') = p\,{}^t\psi_p(\alpha)^{-1}$, $\psi_p(\alpha^*) = {}^t\psi_p(\alpha)$, and $\psi_p(\bar{\alpha}) = p\psi_p(\alpha)^{-1}$.

Extend this map to $\psi_p : G_p \to GSp_4(\mathbb{Q}_p)$:

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \begin{pmatrix} \psi_p(\alpha) & \psi_p(\beta) \\ \psi_p(\gamma) & \psi_p(\delta) \end{pmatrix}. \tag{6.5}$$

This is well defined since according to the definition of G_p, we have $\alpha\delta^* - \beta\gamma^* = \mu(M)$, $\alpha\gamma^*$ and $\beta\delta^*$ are vectors which gives us the precise conditions of the definition of $GSp_4(\mathbb{Q}_p)$ above. Hence we get the isomorphism $G_p \simeq GSp_4(\mathbb{Q}_p)$. One can also check that ψ_p maps K_p onto $GSp_4(\mathbb{Z}_p)$. ■

From now on, we will use the notation G_p for $GSp_4(\mathbb{Q}_p)$ and K_p for $GSp_4(\mathbb{Z}_p)$. Following Asgari-Schmidt [1], $G_p = BK_p$ where B is the Borel subgroup. Let $N := \left\{\left(\begin{smallmatrix} {}^tD^{-1} & X \\ 0 & D \end{smallmatrix}\right) :\right.$ D upper triangular with 1 on the diagonal, X symmetric$\left.\right\}$ be the unipotent radical and let $A := \{a = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_1^{-1}a_0 & \\ & & & a_2^{-1}a_0 \end{pmatrix} : a_0, a_1, a_2 \in \mathbb{Q}_p^\times\}$ be the torus so that $B = NA$. Then

$\delta_B(a) = |a_0^{-3}a_1^2a_2^4|_p$ is the modular function coming from the Haar measure. Given unramified characters χ_0, χ_1, χ_2 on $\mathbb{Q}_p^\times$ (trivial on $\mathbb{Z}_p^\times$), define the character χ on A by $\chi(a) := \chi_0(a_0)\chi_1(a_1)\chi_2(a_2)$. Extend χ to a character of $B = NA$ by setting it to be trivial on N. Define $I(\chi) := \mathrm{Ind}_B^{G_p}(\chi) = \{f : G_p \to \mathbb{C} :$ locally constant function such that $f(nag) = \delta_B^{1/2}(a)\chi(a)f(g)$ for $n \in N,\ a \in A,\ g \in G_p\}$. G_p acts on this space by right translation and we get the normalized induced representation $I(\chi)$.

We will now find the unramified character χ such that π_p is isomorphic to the unique spherical constituent π_χ of $I(\chi)$. The strategy is to apply the generators of the Hecke algebra $H(G_p, K_p)$ to the unramified vector by convolution and evaluate at identity. The values will be polynomials in $\chi_0(p), \chi_1(p), \chi_2(p)$ and then from Proposition 6.2 we get a relation between the character values and the eigenvalues of our lift F. We solve these equations to get the character χ.

Note that χ is unique up to the action of the Weyl group W. The Weyl group of G_p is of order 8 and is generated by the matrices

$$w_1 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad w_2 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad w_3 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{6.6}$$

The Weyl group acts on the character χ by the formula $\chi^w(a) := \chi(w^{-1}aw)$. If $\chi(a) = \chi_0(a_0)\chi_1(a_1)\chi_2(a_2)$, then $\chi^{w_1}(a) = \chi_0(a_0)\chi_2(a_1)\chi_1(a_2)$, $\chi^{w_2}(a) = (\chi_0\chi_1)(a_0)\chi_1^{-1}(a_1)\chi_2(a_2)$, and $\chi^{w_3}(a) = (\chi_0\chi_2)(a_0)\chi_1(a_1)\chi_2^{-1}(a_2)$.

Let F_p^0 be the unramified vector in the space of π_χ satisfying $F_p^0(1) = 1$. Then $F_p^0(nak) = \delta_B^{1/2}(a)\chi(a)$ where $n \in N$, $a \in A$, $k \in K_p$. Any $\phi \in H(G_p, K_p)$ acts on F_p^0 by convolution according to the formula

$$(\phi * F_p^0)(h) := \int_{G_p} \phi(g)F_p^0(gh)dg \quad \text{for } h \in G_p. \tag{6.7}$$

We will use Proposition 6.2 and the action of $H(G_p, K_p)$ on F_p^0 above to get equations satisfied by χ. For this we need the following right coset decomposition.

Proposition 6.4.

$$K_p \begin{pmatrix} 1 &&& \\ & 1 && \\ && p & \\ &&& p \end{pmatrix} K_p = \bigcup_{\substack{b_i \, (\mathrm{mod}\, p) \\ i=1,2,3}} \begin{pmatrix} p && b_1 & b_2 \\ & p & b_2 & b_3 \\ && 1 & \\ &&& 1 \end{pmatrix} K_p \bigcup \begin{pmatrix} 1 &&& \\ & 1 && \\ && p & \\ &&& p \end{pmatrix} K_p$$
$$\bigcup_{b=0}^{p-1} \begin{pmatrix} 1 && * & * \\ -b & p & * & * \\ && p & b \\ &&& 1 \end{pmatrix} K_p \bigcup \begin{pmatrix} p && * & * \\ & 1 & * & * \\ && 1 & \\ &&& p \end{pmatrix} K_p. \tag{6.8}$$

In the third and fourth terms, there are exactly p choices for the upper right-hand corner.

Moreover,

$$K_p \begin{pmatrix} \psi_p(\widehat{\alpha}) & \\ & p^t \psi_p(\widehat{\alpha})^{-1} \end{pmatrix} K_p$$
$$= \bigcup_{b=0}^{p-1} \begin{pmatrix} 1 &&& \\ -b & p && \\ && p^2 & bp \\ &&& p \end{pmatrix} K_p \bigcup \begin{pmatrix} p &&& \\ & 1 && \\ && p & \\ &&& p^2 \end{pmatrix} K_p$$
$$\bigcup \begin{pmatrix} p^2 && * & * \\ & p & * & * \\ && 1 & \\ &&& p \end{pmatrix} K_p \bigcup_{b=0}^{p-1} \begin{pmatrix} p &&* & * \\ -bp & p^2 & * & * \\ && p & b \\ &&& 1 \end{pmatrix} K_p \bigcup \begin{pmatrix} p && * & * \\ & p & * & * \\ && p & \\ &&& p \end{pmatrix} K_p. \tag{6.9}$$

In the third and fourth terms, there are exactly p^3 choices for the upper right-hand corner and in the fifth term there are exactly $p^2 - 1$ choices. □

Proof. The proposition follows from Propositions 5.2, 5.4, and 6.3. ■

Now let $\phi = \mathrm{char}(K_p M K_p) \in H(G_p, K_p)$ with $K_p M K_p = \bigcup_i M_i K_p$. Let $M_i = n_i a_i$. Then

$$\begin{aligned}(\phi * F_P^0)(e) &= \int_{G_p} \phi(g) F_p^0(g)\, dg = \int_{K_p M K_p} F_P^0(g)\, dg = \sum_i \int_{M_i K_p} F_p^0(g)\, dg \\ &= \mathrm{Vol}\,(K_p) \sum_i F_p^0(n_i a_i) = \mathrm{Vol}\,(K_p) \sum_i \delta_B^{1/2}(a_i) \chi(a_i).\end{aligned} \tag{6.10}$$

Apply (6.10) with $\phi = \phi_1 = (1/\mathrm{Vol}(K_p))\,\mathrm{char}\left(K_p\begin{pmatrix}1&&&\\&1&&\\&&p&\\&&&p\end{pmatrix}K_p\right)$. Then by the single-coset decomposition obtained in (6.8) and Proposition 6.2 we get

$$p^{3/2}\lambda_p + p(p+1) = p^{3/2}\chi_0(p)\left[\chi_1(p)\chi_2(p) + 1 + \chi_2(p) + \chi_1(p)\right]. \tag{6.11}$$

Now take $\phi = \phi_2 = (1/\mathrm{Vol}(K_p))\,\mathrm{char}\left(K_p\begin{pmatrix}\psi_p(\alpha)&\\&p\,{}^t\psi_p(\alpha)^{-1}\end{pmatrix}K_p\right)$ in (6.10). Then by the single-coset decomposition obtained in (6.9) and Proposition 6.2 we get

$$\begin{aligned}(p+1)p^{3/2}\lambda_p + \left(p^2-1\right) &= p^2\chi_0^2(p)\left[\chi_2(p) + \chi_1(p) + \chi_1^2(p)\chi_2(p) + \chi_1(p)\chi_2^2(p)\right]\\ &\quad + \left(p^2-1\right)\chi_0^2(p)\chi_1(p)\chi_2(p).\end{aligned} \tag{6.12}$$

Finally take $\phi = \phi_3 = (1/\mathrm{Vol}(K_p))\,\mathrm{char}\left(K_p\begin{pmatrix}p&&&\\&p&&\\&&p&\\&&&p\end{pmatrix}K_p\right)$ in (6.10) to get

$$1 = \chi_0^2(p)\chi_1(p)\chi_2(p). \tag{6.13}$$

This tells us that π_χ has trivial central character. Equation (6.13) implies $\chi_0(p) = (\chi_1(p)\chi_2(p))^{-1/2}$. Using this we get

$$\begin{aligned}&p^{3/2}\lambda_p + p(p+1)\\ &\quad = p^{3/2}\left(\left(\chi_1(p)\chi_2(p)\right)^{1/2} + \frac{1}{\left(\chi_1(p)\chi_2(p)\right)^{1/2}} + \left(\frac{\chi_1(p)}{\chi_2(p)}\right)^{1/2} + \left(\frac{\chi_2(p)}{\chi_1(p)}\right)^{1/2}\right),\\ &(p+1)p^{3/2}\lambda_p = p^2\left(\frac{1}{\chi_1(p)} + \frac{1}{\chi_2(p)} + \chi_1(p) + \chi_2(p)\right).\end{aligned} \tag{6.14}$$

From the earlier remark on the Weyl group of G_p, it is clear that the above equations are not changed if we replace the character χ with χ^w for any $w \in W$ the Weyl group. Hence the solutions to the above equations will be in the same Weyl group orbit as expected.

Theorem 6.5. Up to the action of the Weyl group the character χ is given by

$$\chi_1(p) = p^{1/2}\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}, \qquad \chi_2(p) = p^{1/2}\frac{\lambda_p - \sqrt{\lambda_p^2 - 4}}{2}, \qquad \chi_0(p) = p^{-1/2}. \tag{6.15}$$

□

Proof. Denote $a = (\chi_1(p)\chi_2(p))^{1/2} + (\chi_1(p)\chi_2(p))^{-1/2}$ and $b = (\chi_1(p)/\chi_2(p))^{1/2} + (\chi_2(p)/\chi_1(p))^{1/2}$. Then we have $\chi_1(p) + \chi_2(p) + \chi_1(p)^{-1} + \chi_2(p)^{-1} = ab$. Hence from (6.14) we get

$$\begin{aligned} &p^{3/2}\lambda_p + p(p+1) = p^{3/2}(a+b), \\ &(p+1)p^{3/2}\lambda_p = p^2 ab. \end{aligned} \tag{6.16}$$

Hence we get the equation

$$\frac{p^2 ab}{p+1} - p^{3/2}(a+b) + p(p+1) = \left(\frac{p^2 a}{p+1} - p^{3/2}\right)\left(b - p^{-1/2}(p+1)\right) = 0. \tag{6.17}$$

If $a = p^{-1/2}(p+1) = p^{1/2} + p^{-1/2}$, then $b = \lambda_p$ and this implies $(\chi_1(p)\chi_2(p))^{1/2} = p^{\pm 1/2}$ and $(\chi_1(p)/\chi_2(p))^{1/2} = (\lambda_p \pm \sqrt{\lambda_p^2 - 4})/2$. This gives us four solutions:

$$\begin{aligned} &\chi_1(p) = p^{\pm 1/2}\frac{\lambda_p \pm \sqrt{\lambda_p^2 - 4}}{2}, \\ &\chi_2(p) = p^{\pm 1/2}\frac{\lambda_p \mp \sqrt{\lambda_p^2 - 4}}{2}. \end{aligned} \tag{6.18}$$

If $b = p^{-1/2}(p+1) = p^{1/2} + p^{-1/2}$, then $a = \lambda_p$ and this implies $(\chi_1(p)\chi_2(p))^{1/2} = (\lambda_p \pm \sqrt{\lambda_p^2 - 4})/2$ and $(\chi_1(p)/\chi_2(p))^{1/2} = p^{\pm 1/2}$. This gives us four more solutions:

$$\begin{aligned} &\chi_1(p) = p^{\pm 1/2}\frac{\lambda_p \pm \sqrt{\lambda_p^2 - 4}}{2}, \\ &\chi_2(p) = p^{\mp 1/2}\frac{\lambda_p \pm \sqrt{\lambda_p^2 - 4}}{2}. \end{aligned} \tag{6.19}$$

From the comments before Proposition 6.4 regarding the Weyl group action on characters, one can check that the 8 choices for the character χ obtained above are in the same Weyl group orbit. This completes the proof of Theorem 6.5. ■

Since $\chi_1\chi_2 = |\cdot|^{-1}$, it follows from [29, Lemma 3.2] that the induced representation $I(\chi)$ obtained above is not irreducible. From the classification of automorphic representations of GSp(4) given in [30], we can conclude that the representation π_χ is a representation of type IIb.

6.2 CAP representation

Definition 6.6. Let G_1 and G_2 be two groups such that $G_{1,\nu} \simeq G_{2,\nu}$ for almost all places ν and P_2 be a parabolic subgroup of G_2. Then an irreducible cuspidal automorphic representation π of G_1 is called a CAP representation associated to P_2 if there is an irreducible cuspidal automorphic representation σ of M_2, the Levi component of P_2, such that $\pi_\nu \simeq \pi'_\nu$ for almost all places ν, where π' is an irreducible component of $\mathrm{Ind}_{P_2}^{G_2}(\sigma)$.

To define the normalized induction, extend (σ, V_σ) to a representation of $P_2 = M_2N_2$ by setting it to be trivial on the unipotent radical N_2 and let δ_{P_2} be the modular function obtained from the Haar measure. Then

$$\mathrm{Ind}_{P_2}^{G_2} := \left\{f : G_2 \longrightarrow V_\sigma \mid f \text{ smooth},\ f(pg) = \delta_{P_2}^{1/2}(p)\sigma(p)f(g) \text{ for } p \in P_2,\ g \in G_2\right\}. \tag{6.20}$$

Let $G_1 := \mathrm{GSpin}(1,4)$ and $G_2 := \mathrm{GSp}_4$. Then from Proposition 6.3, we have $G_{1,p} \simeq G_{2,p}$ for every odd prime p. Let

$$P := \left\{ \begin{pmatrix} g & B \\ & \mu\, {}^t g^{-1} \end{pmatrix} : g \in \mathrm{GL}_2,\ B \text{ symmetric matrix, } \mu \text{ the similitude} \right\} \tag{6.21}$$

be the Siegel parabolic subgroup of G_2. We will now construct an irreducible cuspidal automorphic representation σ of the Levi subgroup of P. Consider $f \in S^+_{1/2}(\Gamma_0(4))$ which is a Hecke eigenform with eigenvalue λ_p for every odd prime p. Let h be the weight 0 Maaß form with respect to $\mathrm{SL}_2(\mathbb{Z})$ corresponding to f by the Shimura correspondence given in [17]. If we define the Hecke operator $T(p)$ on weight 0 Maaß forms by

$$\big(T(p)h\big)(z) := \sum_{j=0}^{p-1} h\left(\frac{z+j}{p}\right) + h(pz), \tag{6.22}$$

then from [17, pages 199 and 223] we know that $T(p)h = p^{1/2}\lambda_p$. Let σ be the irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ obtained from h by the strong approximation of GL_2. Write $\sigma \simeq \otimes'_p \sigma_p$, where σ_p is given by $\sigma_p \simeq I(\eta_1, \eta_2) := \{\widehat{f} : \mathrm{GL}_2(\mathbb{Q}_p) \to \mathbb{C} \mid \widehat{f} \text{ smooth}, \widehat{f}(bg) = \delta_{B_2}(b)^{1/2}\eta(b)\widehat{f}(g)\ \forall b \in B_2,\ g \in \mathrm{GL}_2\}$. Here $B_2 := \left\{b = \left(\begin{smallmatrix} a_1 & 0 \\ a_3 & a_2 \end{smallmatrix}\right)\right\}$ is the standard lower-triangular Borel subgroup of GL_2, $\delta_{B_2}\big(\left(\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}\right)\big) = |a_1^{-1}a_2|$ is the modular function obtained from the Haar measure on GL_2, and $\eta\big(\left(\begin{smallmatrix} a_1 & 0 \\ a_3 & a_2 \end{smallmatrix}\right)\big) := \eta_1(a_1)\eta_2(a_2)$ where η_1 and η_2 are unramified unitary characters of $\mathbb{Q}_p^*$. (We consider the lower-triangular Borel subgroup here instead of the upper triangular since we have used the notations of [1] for our calculations regarding the symplectic group.) Following calculations similar to (6.10), (6.11), (6.12) we get $\eta_1(p) + \eta_2(p) = \lambda_p$ and $\eta_1(p)\eta_2(p) = 1$.

Theorem 6.7. Let π_F be the representation of G_1 from the previous section. Then π_F is CAP to an irreducible component of $\mathrm{Ind}_P^{G_2}(\eta_0 \times \sigma \times |\det|^{-1/2})$ where σ is as above and $\eta_0(\mu) := |\mu|^{1/2}$ is an unramified character that acts on the similitude. □

Proof. By transitivity of induction, we can show that for every odd prime p we have $\mathrm{Ind}_B^{G_{2,p}}(\chi) \simeq \mathrm{Ind}_P^{G_{2,p}}(\eta_0 \times \sigma_p \times |\det|^{-1/2})$ with χ as in Theorem 6.5. We can then take π' to be the irreducible automorphic constituent of $\mathrm{Ind}_P^{G_2}(\eta_0 \times \sigma \times |\det|^{-1/2})$ such that the p-adic component of π' is the spherical constituent of $\mathrm{Ind}_P^{G_{2,p}}(\eta_0 \times \sigma_p \times |\det|^{-1/2})$. It follows from [23, Lemma 1] that we can always find a π' with the above property. Then we have $\pi_{F,p} \simeq \pi'_p$ for every odd prime p and hence we get the result of the theorem. ■

We note that from [31, Lemma 2.2] and Theorem 6.7 above we can conclude that the representation $\pi_{F,p}$ is precisely the local Saito-Kurokawa lift of σ.

We want to point out that Definition 6.6 is not the same as the definition of CAP representation found in the literature in the sense that we allow two different groups G_1 and G_2 satisfying $G_{1,v} \simeq G_{2,v}$ for almost all places v instead of considering just one group. To the best of our knowledge this is the first example where such CAP representations are constructed.

One can explain why we get CAP representation involving two different groups if we consider Langlands functoriality. The two groups $\mathrm{GSpin}(1,4)$ and GSp_4 are inner forms of each other. Hence they have the same L-groups. Langlands functoriality tells us that corresponding to the identity L-homomorphism, we should get a lifting of automorphic representations from the inner form $\mathrm{GSpin}(1,4)$ to the split group GSp_4. Locally, when $\mathrm{GSpin}(1,4)_v \simeq \mathrm{GSp}_{4,v}$, the lifting is given by an isomorphism which is the content of Theorem 6.7. Hence one can say that Theorem 6.7 is a special case of the Langlands functoriality expected in this situation.

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References

[1] M. Asgari and R. Schmidt, *Siegel modular forms and representations*, Manuscripta Math. **104** (2001), no. 2, 173–200.

[2] E. M. Baruch and Z. Mao, *Central value of automorphic* $L-$*functions*, preprint, 2003.

[3] D. Bump, *Automorphic Forms and Representations*, Cambridge Studies in Advanced Mathematics, vol. 55, Cambridge University Press, Cambridge, 1997.

[4] D. Bump, J. W. Cogdell, E. de Shalit, D. Gaitsgory, E. Kowalski, and S. S. Kudla, *An Introduction to the Langlands Program*, edited by J. Bernstein and S. Gelbart, Birkhäuser Boston, Massachusetts, 2003, Lectures presented at the Hebrew University of Jerusalem, Jerusalem, March 12–16, 2001.

[5] P. Cartier, *Representations of* p*-adic groups: a survey*, Automorphic Forms, Representations and L-Functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore, 1977), Part 1, Proc. Sympos. Pure Math., vol. 33, American Mathematical Society, Rhode Island, 1979, pp. 111–155.

[6] W. Duke, *Hyperbolic distribution problems and half-integral weight Maaß forms*, Invent. Math. **92** (1988), no. 1, 73–90.

[7] W. Duke and Ö. Imamoḡlu, *A converse theorem and the Saito-Kurokawa lift*, Int. Math. Res. Not. **1996** (1996), no. 7, 347–355.

[8] J. Elstrodt, F. Grunewald, and J. Mennicke, *Vahlen's group of Clifford matrices and spin-groups*, Math. Z. **196** (1987), no. 3, 369–390.

[9] ———, *Arithmetic applications of the hyperbolic lattice point theorem*, Proc. London Math. Soc. (3) **57** (1988), no. 2, 239–283.

[10] S. Friedberg and J. Hoffstein, *Nonvanishing theorems for automorphic* L*-functions on* $\mathrm{GL}(2)$, Ann. of Math. (2) **142** (1995), no. 2, 385–423.

[11] W. T. Gan, N. Gurevich, and D. Jiang, *Cubic unipotent Arthur parameters and multiplicities of square integrable automorphic forms*, Invent. Math. **149** (2002), no. 2, 225–265.

[12] D. Ginzburg, D. Jiang, and S. Rallis, *On CAP automorphic representations of a split group of type* D_4, J. reine angew. Math. **552** (2002), 179–211.

[13] R. Godement, *Notes on Jacquet-Langlands' theory*, The Institute for Advanced Study, 1970.

[14] E. Hecke, *Lectures on Dirichlet Series*, *Modular Functions and Quadratic Forms*, Vandenhoeck & Ruprecht, Göttingen, 1983.

[15] T. Ikeda, *On the lifting of elliptic cusp forms to Siegel cusp forms of degree* $2n$, Ann. of Math. (2) **154** (2001), no. 3, 641–681.

[16] K. Imai, *Generalization of Hecke's correspondence to Siegel modular forms*, Amer. J. Math. **102** (1980), no. 5, 903–936.

[17] S. Katok and P. Sarnak, *Heegner points*, *cycles and Maaß forms*, Israel J. Math. **84** (1993), no. 1-2, 193–227.

[18] W. Kohnen, *Modular forms of half-integral weight on* $\Gamma_0(4)$, Math. Ann. **248** (1980), no. 3, 249–266.

[19] W. Kohnen and D. Zagier, *Values of* L*-series of modular forms at the center of the critical strip*, Invent. Math. **64** (1981), no. 2, 175–198.

[20] H. Kojima, *Shimura correspondence of Maaß wave forms of half integral weight*, Acta Arith. **69** (1995), no. 4, 367–385.

[21] A. Krieg, *The elementary divisor theory over the Hurwitz order of integral quaternions*, Linear and Multilinear Algebra **21** (1987), no. 4, 325–344.

[22] ———, *Eisenstein-series on the four-dimensional hyperbolic space*, J. Number Theory **30** (1988), no. 2, 177–197.

[23] R. P. Langlands, *On the notion of an automorphic representation. A supplement to the preceding paper*, Automorphic Forms, Representations and L-Functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore, 1977), Part 1, Proc. Sympos. Pure Math., vol. 33, American Mathematical Society, Rhode Island, 1979, pp. 203–207.

[24] H. Maaß, *Automorphe Funktionen von meheren Veränderlichen und Dirichletsche Reihen*, Abh. Math. Sem. Univ. Hamburg **16** (1949), no. 3-4, 72–100.

[25] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, 3rd ed., Die Grundlehren der mathematischen Wissenschaften, vol. 52, Springer, New York, 1966.

[26] O. T. O'Meara, *Introduction to Quadratic Forms*, Die Grundlehren der mathematischen Wissenschaften, vol. 117, Springer, Berlin, 1973.

[27] I. I. Piatetski-Shapiro, *On the Saito-Kurokawa lifting*, Invent. Math. **71** (1983), no. 2, 309–338.

[28] S. Rallis and G. Schiffmann, *Theta correspondence associated to* G_2, Amer. J. Math. **111** (1989), no. 5, 801–849.

[29] P. J. Sally Jr. and M. Tadić, *Induced representations and classifications for* $\mathrm{GSp}(2,F)$ *and* $\mathrm{Sp}(2,F)$, Mém. Soc. Math. France (N.S.) **52** (1993), 75–133.

[30] R. Schmidt, *Iwahori-spherical representations of* $\mathrm{GSp}(4)$ *and Siegel modular forms of degree 2 with square-free level*, J. Math. Soc. Japan **57** (2005), no. 1, 259–293.

[31] ———, *The Saito-Kurokawa lifting and functoriality*, Amer. J. Math. **127** (2005), no. 1, 209–240.

[32] G. Shimura, *On modular forms of half integral weight*, Ann. of Math. (2) **97** (1973), no. 3, 440–481.

[33] ———, *Arithmetic and Analytic Theories of Quadratic Forms and Clifford Groups*, Mathematical Surveys and Monographs, vol. 109, American Mathematical Society, Rhode Island, 2004.

[34] D. Soudry, *The CAP representations of* $\mathrm{GSp}(4,\mathbb{A})$, J. reine angew. Math. **383** (1988), 87–108.

[35] ———, *Automorphic forms on* $\mathrm{GSp}(4)$, Festschrift in Honor of I. I. Piatetski-Shapiro on the Occasion of His Sixtieth Birthday, Part II (Ramat Aviv, 1989), Israel Math. Conf. Proc., vol. 3, Weizmann, Jerusalem, 1990, pp. 291–303.

[36] J.-L. Waldspurger, *Sur les coefficients de Fourier des formes modulaires de poids demi-entier* [*On the Fourier coefficients of modular forms of half-integral weight*], J. Math. Pures Appl. (9) **60** (1981), no. 4, 375–484.

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