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Jacobi Maaß forms

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Abstract In this paper, we give a new definition for the space of non-holomorphic Jacobi Maaß forms (denoted by $J_{k,m}^{nh}$) of weight $k \in \mathbb{Z}$ and index $m \in \mathbb{N}$ as eigenfunctions of a degree three differential operator $\mathcal{C}^{k,m}$. We show that the three main examples of Jacobi forms known in the literature: holomorphic, skew-holomorphic and real-analytic Eisenstein series, are contained in $J_{k,m}^{nh}$. We construct new examples of cuspidal Jacobi Maaß forms F_f of weight $k \in 2\mathbb{Z}$ and index 1 from weight $k - 1/2$ Maaß forms f with respect to $\Gamma_0(4)$ and show that the map $f \mapsto F_f$ is Hecke equivariant. We also show that the above map is compatible with the well-known representation theory of the Jacobi group. In addition, we show that all of $J_{k,m}^{nh}$ can be "essentially" obtained from scalar or vector valued half integer weight Maaß forms.

Keywords Jacobi forms · Maass forms · Jacobi group · Automorphic representation

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1 Introduction

The theory of Jacobi forms has been studied extensively in the last few decades. One of the important features of Jacobi forms is that they form a bridge between the space of elliptic modular forms and Siegel modular forms. This fact is exploited to give a proof of the Saito-Kurokawa conjecture, which states that there is a lifting from elliptic modular forms to Siegel cusp forms of genus 2 (see [10]). In [13], Ikeda has used Jacobi forms of higher genus to prove a conjecture of Duke and Imamoglu (see [8]), which generalizes the Saito-Kurokawa conjecture. A very nice and systematic development of the theory of holomorphic Jacobi forms is given in the book [10] by Eichler and Zagier. In addition to the

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holomorphic Jacobi forms, there are two main examples of Jacobi forms, namely, skew-holomorphic Jacobi forms and real analytic Eisenstein series. Analogous to Maaß forms on GL_2, one would like to have a theory of non-holomorphic Jacobi forms. Such a theory should, at the least, include the above mentioned examples of Jacobi forms as subspaces and, more ambitiously, should account for all possible automorphic forms on the Jacobi group. A theory of non-holomorphic Jacobi forms is desirable for several reasons. Firstly, given an irreducible automorphic representation of the adelic points of the Jacobi group, we would be able to pick distinguished vectors in the representation and associate to them classical modular forms. Secondly, a theory of non-holomorphic Jacobi forms of higher genus will help obtain a lifting from representations of GL_2, whose archimedean component is not a holomorphic discrete series, to representations of the symplectic group of higher genus (in analogy to Ikeda's lift). Very little is known about the latter problem.

There have been a few attempts to define non-holomorphic Jacobi forms (see [5], [23]), but the theory developed so far is somewhat unsatisfactory. In this paper, we introduce a new way to define non-holomorphic Jacobi Maaß forms of weight $k \in \mathbb{Z}$ and index $m \in \mathbb{N}$. As mentioned earlier, there are three main examples of Jacobi forms with respect to $G^J(\mathbb{Z})$ available in the literature. Here G^J denotes the Jacobi group.

- In [10], Eichler and Zagier define the holomorphic Jacobi forms F^h of weight $k > 0$ and index m with respect to $G^J(\mathbb{Z})$.
- Skoruppa defines the skew-holomorphic Jacobi forms F^{sh} of weight $k > 0$ and index m in [20].
- In [1], Arakawa defines the real analytic Jacobi Eisenstein series $E_{k,m}$ of weight $k \in \mathbb{Z}$ and index m, which is a generalization of the holomorphic Jacobi Eisenstein series from [10].

If F is any one of F^h, F^{sh} or $E_{k,m}$, and is a Hecke eigenform as well, one can construct an irreducible automorphic representation π_F of $G^J(\mathbb{A})$. If F is either F^h or F^{sh}, then π_F is cuspidal and if $F = E_{k,m}$ then π_F is not cuspidal.

Let $\pi_F = \otimes \pi_{F,p}$. Then in each of the three cases the non-archimedean local representations are spherical and completely determined by the classical Hecke eigenvalues. On the other hand, the archimedean representations are completely different:

1. If $F = F^h$ we see that $\pi_{F,\infty}$ is a lowest weight discrete series representation $\pi^+_{m,k}$ (see [4]),
2. If $F = F^{sh}$ we see that $\pi_{F,\infty}$ is a highest weight discrete series representation $\pi^-_{m,k}$ (see [3]) and
3. If $F = E_{k,m}$ we see that $\pi_{F,\infty}$ is a principle series representation $\pi_{2s-1,m,\pm 1}$ (see [5]).

The notations for the archimedean representations will be explained in details in Proposition 7.3. The type of archimedean representation obtained depends on the scalar with which the Casimir operator $\mathcal{C}$ (defined in (6)) of the Jacobi group acts on the representation $\pi_{F,\infty}$. We pull back $\mathcal{C}$ to an operator $\mathcal{C}^{k,m}$ on functions on $\mathcal{H} \times \mathbb{C}$, where $\mathcal{H}$ is the complex upper half plane. Then the scalar used to determine $\pi_{F,\infty}$ is precisely the eigenvalue of the operator $\mathcal{C}^{*,m}$ acting on F, where $*$ is the weight of F. So, we see that the representation π_F is completely determined by the Hecke eigenvalues, the integers k, m and the eigenvalue of the differential operator $\mathcal{C}^{*,m}$ acting on F.

The above discussion leads us to the conclusion that the most general notion of a Jacobi form with respect to $G^J(\mathbb{Z})$ (which would include the three examples above) is a function, which is an eigenfunction of the degree 3 differential operator $\mathcal{C}^{k,m}$ and satisfies the automorphy condition with respect to the non-holomorphic automorphy factor defined in (2). We make this definition precise in Definition 3.2 and denote the space of Jacobi Maaß forms of weight $k \in \mathbb{Z}$ and index $m > 0$ with respect to $G^J(\mathbb{Z})$ by $J_{k,m}^{nh}$. (The nh in the superscript corresponds to non-holomorphic.)

A general notion of a Jacobi form is worthwhile only if we can have explicit examples which were not available earlier. We obtain, in Sect. 4, new examples of Jacobi Maaß forms F_f of weight $k \in 2\mathbb{Z}$ and index $m = 1$ by constructing an injective map from the space of weight $k - 1/2$ Maaß forms f with respect to $\Gamma_0(4)$ belonging to a non-holomorphic analogue of the Kohnen plus space to $J_{k,1}^{nh}$. The motivation for this construction is the analogous situation in the case of holomorphic Jacobi forms in [10]. However, contrary to the holomorphic case, the image of this map is not all of $J_{k,1}^{nh}$ but the proper subspace $\hat{J}_{k,1}^{nh}$ of functions that are holomorphic in the z variable. This is shown in Theorem 4.5. Since the space of weight $k - 1/2$ Maaß forms with respect to $\Gamma_0(4)$ in the non-holomorphic Kohnen plus space is infinite dimensional we can conclude that the space $J_{k,1}^{nh}$, for k even, is infinite dimensional. For weight $k \in \mathbb{Z}$ and index $m \geq 1$, we show in Theorem 4.6 that subspace $\hat{J}_{k,m}^{nh}$ of $J_{k,m}^{nh}$, consisting of functions that are holomorphic in the z variable, is isomorphic to a certain space of vector valued half integer weight Maaß forms. We notice that applying differential operators to functions in $\hat{J}_{k,m}^{nh}$ generates all of $J_{k,m}^{nh}$ and in this sense, one can say that the Jacobi Maaß forms are "essentially" obtained from half integer weight Maaß forms.

Note that one usually constructs Eisenstein series as the first examples of any kind of automorphic forms. But in this case, the real analytic Jacobi Eisenstein series defined by Arakawa are precisely the Eisenstein series that one would obtain from such a construction. We show this fact in Theorem 5.1 along with the result that the holomorphic and skew-holomorphic Jacobi forms defined in [10] and [20] are indeed elements of $J_{k,m}^{nh}$. This can be described by the following diagram.

$$\begin{tikzcd}
 & J^{nh}_{k,m} & J^{nh}_{1-k,m} \\
F^h : (k,m) \arrow[ur, "y^{k/2}"] & E_{k,m} : (k,m) \arrow[u, "y^{k/2}"'] & F^{sh} : (k,m) \arrow[u, "y^{k/2}"]
\end{tikzcd} \tag{1}$$

Here the arrows correspond to the maps obtained by multiplying the function F by $y^{k/2}$. This shows that the definition of Jacobi Maaß forms (Definition 3.2) is indeed a generalization of known Jacobi forms. Since the holomorphic, skew-holomorphic Jacobi forms and the Eisenstein series are holomorphic in the z variable, we get the well-known fact that these Jacobi forms are obtained from half-integer weight modular forms.

Since the definition of Jacobi Maaß forms is inspired from the representation theory of the Jacobi group, it is important to verify that the classical Jacobi Maaß forms constructed here conform with the representation theory. We ask the following question: Start with a half integer weight Maaß cusp form f and consider the corresponding irreducible cuspidal automorphic representation $\tilde{\pi}_f$ of the metaplectic group $\widetilde{\mathrm{SL}}_2(\mathbb{A})$. We get an irreducible cuspidal automorphic representation π of $G^J(\mathbb{A})$ by setting $\pi = \tilde{\pi}_f \otimes \pi_{SW}^1$ where π_{SW}^1 is the Schrödinger-Weil representation. (The map $\tilde{\pi} \mapsto \tilde{\pi} \otimes \pi_{SW}^m$ gives a $1-1$ correspondence between representations of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ and $G^J(\mathbb{A})$ with a fixed central character depending on m.) Let $F_f \in \hat{J}_{k,1}^{nh}$ be the Jacobi Maaß form corresponding to f and let π_F be the irreducible cuspidal automorphic representation of $G^J(\mathbb{A})$ corresponding to F_f. Then is it true that $\pi = \pi_F$? In other words, is the definition of Jacobi Maaß forms compatible with the expected relationships of the representations involved? In Theorem 7.5, we indeed show that the answer to this question is affirmative. To prove Theorem 7.5, the important ingredient is the Hecke equivariance of the map $f \mapsto F_f$, which we show in Theorem 6.1.

Finally, in Sect. 8, we list some comments and future steps that one can take in the theory of Jacobi Maaß forms. This follows the ideas of what is known, due to works of others, mainly, in the case of holomorphic and skew-holomorphic Jacobi forms.

2 Preliminaries on the Jacobi group

Let Sp_4 be the symplectic group defined by

$$\mathrm{Sp}_4 := \{g \in \mathrm{GL}_4 : {}^t g J g = J\} \quad \text{where } J = \begin{bmatrix} & I_2 \\ -I_2 & \end{bmatrix}.$$

The Jacobi group G^J is defined as the subgroup of Sp_4 given by the matrices whose last row is $\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$. We can realize G^J as the semidirect product $G^J = \mathrm{SL}_2 \ltimes H$, where H is the Heisenberg group consisting of elements $(\lambda, \mu, \kappa) =: (X, \kappa)$, where $X = (\lambda, \mu)$. It is convenient to use the following coordinate systems on the real points of the Jacobi group $G^J(\mathbb{R})$:

1. The EZ-coordinates (due to Eichler-Zagier) $(x, y, \theta, \lambda, \mu, \kappa)$ give us the element $M(X, \kappa) \in G^J(\mathbb{R})$ where
$$M = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \quad \text{with } x \in \mathbb{R},\ y \in \mathbb{R}^+,\ 0 \le \theta < 2\pi,$$
$$X = (\lambda, \mu) \in \mathbb{R}^2 \quad \text{and} \quad \kappa \in \mathbb{R}.$$
2. The S-coordinates (due to Siegel) $(x, y, \theta, p, q, \kappa)$ give us the element $(Y, \kappa)M \in G^J(\mathbb{R})$ where M is as above and $Y = (p, q) = XM^{-1} \in \mathbb{R}^2$.

The action of $G^J(\mathbb{R})$ on $\mathcal{H} \times \mathbb{C}$, where $\mathcal{H} = \{\tau = x + iy \in \mathbb{C} : y > 0\}$ is the complex upper half plane, is given, in terms of the two coordinate systems, as follows:

1. If $g = M(X, \kappa) \in G^J(\mathbb{R})$ is given in the EZ-coordinates, then for $\tau \in \mathcal{H}, z \in \mathbb{C}$ we have
$$g(\tau, z) := \left(M\langle\tau\rangle, \frac{z + \lambda\tau + \mu}{c\tau + d} \right) \quad \text{where } M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad M\langle\tau\rangle = \frac{a\tau + b}{c\tau + d}.$$
2. In the S-coordinates we have
$$\begin{aligned} G^J(\mathbb{R})/(SO(2) \times \mathbb{R}) &\xrightarrow{\sim} \mathcal{H} \times \mathbb{C} \\ g = (p, q, \kappa)M &\mapsto g(i, 0) = (\tau, p\tau + q) \quad \text{where } \tau = M\langle i\rangle. \end{aligned}$$

For more details regarding the Jacobi group we direct the reader to [5].

3 Non-holomorphic automorphy factor and differential operators

Let us now introduce the non-holomorphic automorphy factor $j^{nh}_{k,m}$ ($k \in \mathbb{Z}$, $m \in \mathbb{N}$) for the Jacobi group in the EZ-coordinates as follows:

$$j^{nh}_{k,m}(g, (\tau, z)) := e^{2\pi i m(\kappa - \frac{c(z+\lambda\tau+\mu)^2}{c\tau+d} + \lambda^2\tau + 2\lambda z + \lambda\mu)} \left(\frac{c\tau + d}{|c\tau + d|} \right)^{-k} \qquad (2)$$

where $g = M(X,\kappa)$ with $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, $X = (\lambda, \mu)$ and $(\tau, z) \in \mathcal{H} \times \mathbb{C}$ (here the superscript "nh" corresponds to non-holomorphic). In the S-coordinates, the above automorphy factor looks much simpler. Let $g = (x, y, \theta, p, q, \kappa) \in G^J(\mathbb{R})$. Then we have

$$j^{nh}_{k,m}(g, (i, 0)) = e^{2\pi i m(\kappa + pz)} e^{ik\theta} \quad \text{where } z = p(x + iy) + q.$$

It is easy to check that the automorphy factor satisfies the following condition

$$\begin{aligned} &j^{nh}_{k,m}(g_1 g_2, (\tau, z)) = j^{nh}_{k,m}(g_1, g_2(\tau, z)) j^{nh}_{k,m}(g_2, (\tau, z)) \\ &\text{for all } g_1, g_2 \in G^J(\mathbb{R}),\ (\tau, z) \in \mathcal{H} \times \mathbb{C}. \end{aligned} \tag{3}$$

We can now define a slash operator on functions on $\mathcal{H} \times \mathbb{C}$ as follows: for $g \in G^J(\mathbb{R})$, $(\tau, z) \in \mathcal{H} \times \mathbb{C}$ and a smooth function $F : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ we set

$$(F|_{k,m} g)(\tau, z) := j^{nh}_{k,m}(g, (\tau, z)) F(g(\tau, z)). \tag{4}$$

From (3), we can see that

$$F|_{k,m}(g_1 g_2) = (F|_{k,m} g_1)|_{k,m} g_2.$$

This slash operator now allows us to construct a map $\Phi_{k,m}$ from the functions on $\mathcal{H} \times \mathbb{C}$ to the functions on the group $G^J(\mathbb{R})$ as follows. Suppose we have a smooth function $F : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ then define $\Phi_{k,m} F : G^J(\mathbb{R}) \to \mathbb{C}$ by the formula

$$(\Phi_{k,m} F)(g) := (F|_{k,m} g)(i, 0). \tag{5}$$

In particular, if $g = (p, q, \kappa)M$ with $M = \left[\begin{smallmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{smallmatrix}\right]\left[\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}\right]$ then

$$(\Phi_{k,m} F)(g) = F(\tau, z) e^{2\pi i m(\kappa + pz)} e^{ik\theta}$$

where $\tau = x + iy$ and $z = p\tau + q$.

Let us recall the following differential operators (in S-coordinates) on the functions on $G^J(\mathbb{R})$ coming from the Lie algebra of the Jacobi group, as given in [5, p. 12, 38]. Fix $m \in \mathbb{N}$.

$$\begin{aligned} &Y_+ = \frac{1}{2} y^{-1/2} e^{i\theta}(\partial_p - \bar{\tau}\partial_q - \bar{z}\partial_\kappa), \qquad Y_- = \frac{1}{2} y^{-1/2} e^{-i\theta}(\partial_p - \tau\partial_q - z\partial_\kappa), \\ &X_+ = \frac{i}{2} e^{2i\theta}(4y\partial_\tau - \partial_\theta), \qquad X_- = -\frac{i}{2} e^{-2i\theta}(4y\partial_{\bar{\tau}} - \partial_\theta), \\ &Z = -i\partial_\theta, \qquad D_\pm = X_\pm \pm \frac{1}{4\pi m} Y_\pm^2, \qquad \Delta_1 = Z + \frac{1}{4\pi m}(Y_+ Y_- + Y_- Y_+). \end{aligned}$$

Let us now define the following differential operators on smooth functions F on $\mathcal{H} \times \mathbb{C}$. By abuse of notation we will write F as $F(\tau, \bar{\tau}, z, \bar{z})$. Fix $k \in \mathbb{Z}, m \in \mathbb{N}$.

$$\begin{aligned} &Y^{k,m}_+ F := i\left(\frac{\tau - \bar{\tau}}{2i}\right)^{1/2}\left(F_z + 4\pi i m \frac{z - \bar{z}}{\tau - \bar{\tau}} F\right), \qquad Y^{k,m}_- F := -i\left(\frac{\tau - \bar{\tau}}{2i}\right)^{1/2} F_{\bar{z}}, \\ &X^{k,m}_+ F := (\tau - \bar{\tau}) F_\tau + (z - \bar{z}) F_z + 2\pi i m \frac{(z - \bar{z})^2}{\tau - \bar{\tau}} F + \frac{k}{2} F, \end{aligned}$$

$$X_-^{k,m}F := -(\tau-\bar\tau)F_{\bar\tau} - (z-\bar z)F_{\bar z} - \frac{k}{2}F,$$

$$D_+^{k,m}F := \left(X_+^{k,m} + \frac{1}{4\pi m}(Y_+^{k+1,m}Y_+^{k,m})\right)F = (\tau-\bar\tau)F_\tau - \frac{1}{4\pi m}\left(\frac{\tau-\bar\tau}{2i}\right)F_{zz} + \frac{k-1}{2}F,$$

$$D_-^{k,m}F := \left(X_-^{k,m} - \frac{1}{4\pi m}(Y_-^{k-1,m}Y_-^{k,m})\right)F$$

$$= -(\tau-\bar\tau)F_{\bar\tau} - (z-\bar z)F_{\bar z} - \frac{k}{2}F + \frac{1}{4\pi m}\left(\frac{\tau-\bar\tau}{2i}\right)F_{\bar z\bar z},$$

$$\Delta_1^{k,m}F := \left(k + \frac{1}{4\pi m}(Y_+^{k-1,m}Y_-^{k,m} + Y_-^{k+1,m}Y_+^{k,m})\right)F.$$

The relation between the differential operators acting on the functions on the group and those acting on the functions on $\mathcal{H}\times\mathbb{C}$ is given in the following proposition.

Proposition 3.1 *Let $X_\pm, Y_\pm, D_\pm, \Delta_1, X_\pm^{k,m}, Y_\pm^{k,m}, D_\pm^{k,m}, \Delta_1^{k,m}$ be as defined above. Let $F : \mathcal{H}\times\mathbb{C}\to\mathbb{C}$ be a smooth function. Then we have*

$$X_\pm(\Phi_{k,m}F) = \Phi_{k\pm2,m}(X_\pm^{k,m}F), \qquad Y_\pm(\Phi_{k,m}F) = \Phi_{k\pm1,m}(Y_\pm^{k,m}F),$$
$$D_\pm(\Phi_{k,m}F) = \Phi_{k\pm2,m}(D_\pm^{k,m}F), \qquad \Delta_1(\Phi_{k,m}F) = \Phi_{k,m}(\Delta_1^{k,m}F).$$

Proof The proposition is proved by direct computation. □

Recall the Casimir operator defined in [5, p. 38] by the formula

$$\mathcal{C} = D_+D_- + D_-D_+ + \frac{1}{2}\Delta_1^2. \tag{6}$$

It is shown in [5, p. 38] that $\mathcal{C}$ lies in the center of $U(\mathfrak{g}_\mathbb{C}^J)/(Z_0-4\pi m)$, where $U(\mathfrak{g}_\mathbb{C}^J)$ is the universal enveloping algebra of the complexified Lie algebra $\mathfrak{g}_\mathbb{C}^J$ of the Jacobi group and Z_0, in the S-coordinates, is given by the differential operator $-i\partial_\kappa$. For more on the operator $\mathcal{C}$, also see [6].

Using Proposition 3.1, we see that the differential operator

$$\mathcal{C}^{k,m} := D_+^{k-2,m}D_-^{k,m} + D_-^{k+2,m}D_+^{k,m} + \frac{1}{2}(\Delta_1^{k,m})^2$$

acting on the functions on $\mathcal{H}\times\mathbb{C}$, satisfies

$$\mathcal{C}(\Phi_{k,m}F) = \Phi_{k,m}(\mathcal{C}^{k,m}F), \tag{7}$$

i.e., $\mathcal{C}^{k,m}$ is the pullback of $\mathcal{C}$ using the non-holomorphic automorphy factor $j_{k,m}^{nh}$. Substituting the definition of $D_\pm^{k,m}$ and $\Delta_1^{k,m}$ we get

$$\mathcal{C}^{k,m}F = \frac{5}{8}F - 2(\tau-\bar\tau)^2F_{\tau\bar\tau} - (k-1)(\tau-\bar\tau)F_{\bar\tau} - k(\tau-\bar\tau)F_\tau$$
$$+ \frac{k(\tau-\bar\tau)}{8\pi im}F_{zz} + \frac{(\tau-\bar\tau)^2}{4\pi im}F_{\bar\tau zz}$$

$$+\frac{k(\tau-\bar\tau)}{4\pi i m}F_{z\bar z}+\frac{(\tau-\bar\tau)(z-\bar z)}{4\pi i m}F_{zz\bar z}-2(\tau-\bar\tau)(z-\bar z)F_{\tau\bar z}+\frac{(\tau-\bar\tau)^2}{4\pi i m}F_{\tau\bar z\bar z}$$
$$+\left(\frac{(z-\bar z)^2}{2}+\frac{k(\tau-\bar\tau)}{8\pi i m}\right)F_{\bar z\bar z}+\frac{(\tau-\bar\tau)(z-\bar z)}{4\pi i m}F_{z\bar z\bar z}. \tag{8}$$

Let us note here that the above operator simplifies significantly if one applies it to functions F with some extra properties. For example, if F is holomorphic in the z variable then the last two lines of (8) give us zero. In addition, if F is also holomorphic in the τ variable then $\mathcal{C}^{k,m}$ reduces to $\frac{5}{8}F-k(\tau-\bar\tau)F_\tau+\frac{k(\tau-\bar\tau)}{8\pi i m}F_{zz}$, which is closely related to the heat operator $8\pi i m\partial_\tau-\partial_z^2$.

Consider the discrete subgroup $\Gamma^J:=\mathrm{SL}_2(\mathbb{Z})\ltimes H(\mathbb{Z})$ of $G^J(\mathbb{R})$.

Definition 3.2 A smooth function $F:\mathcal{H}\times\mathbb{C}\to\mathbb{C}$ is called a **Jacobi Maaß form** of weight k $(k\in\mathbb{Z})$ and index m $(m\in\mathbb{N})$ with respect to Γ^J if

1. $(F|_{k,m}\gamma)(\tau,z)=F(\tau,z)$ for all $\gamma\in\Gamma^J$ and $(\tau,z)\in\mathcal{H}\times\mathbb{C}$,
2. $\mathcal{C}^{k,m}F=\lambda F$ for some $\lambda\in\mathbb{C}$ and
3. $F(\tau,z)=O(y^N)$ as $y\to\infty$ for some $N>0$.

If, in addition, F satisfies the condition

$$\int_0^1\int_0^1 F\left(\begin{bmatrix}1&x\\0&1\end{bmatrix}(0,u,0)(\tau,z)\right)e^{-2\pi i(nx+ru)}dx\,du=0\quad\text{for all } n,r\in\mathbb{Z}$$
$$\text{such that } 4nm-r^2=0 \tag{9}$$

then we say that F is a **Jacobi Maaß cusp form**.

Let us denote the vector space of all Jacobi Maaß forms of weight k and index m with respect to Γ^J by $J_{k,m}^{nh}$ and the subspace of cusp forms by $J_{k,m}^{nh,cusp}$.

4 Jacobi Maaß forms and half integer weight Maaß forms

In the holomorphic setting, it is shown by Eichler-Zagier [10] that the space of holomorphic Jacobi forms of even weight k and index 1 is isomorphic to the space of weight $k-1/2$ holomorphic modular forms in the Kohnen plus space. For higher index, they have obtained a theta expansion in terms of vector valued half integral weight modular forms. These facts are the motivation for the construction of Jacobi Maaß forms in this section. We will first give some background and useful results regarding Maaß forms in the plus space. Then we will introduce the non-holomorphic theta series and obtain the desired family of Jacobi Maaß forms.

4.1 Half-integer weight Maaß forms

Since the motivation for the construction we want to present here is the holomorphic setting, we first have to derive a few results about half -integer weight Maaß forms analogous to the holomorphic half-integer weight forms. In particular, we need a non-holomorphic analogue of Proposition 2 from [15, p. 255] and Lemma 2.1 from [13, p. 648]. This is achieved in Propositions 4.1 and 4.2 below.

Let $\mathfrak{G}$ be the group which consists of all pairs $(\gamma, \phi(\tau))$, where $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}_2^+(\mathbb{R})$ and $\phi(\tau)$ is a function on the upper half plane $\mathcal{H}$ such that $\phi(\tau) = t \det(\gamma)^{-1/4}(\frac{c\tau+d}{|c\tau+d|})^{1/2}$ with $t \in \mathbb{C},\ |t| = 1$. The group law is given by

$$(\gamma_1, \phi_1(\tau)) \cdot (\gamma_2, \phi_2(\tau)) = (\gamma_1\gamma_2, \phi_1(\gamma_2\langle\tau\rangle)\phi_2(\tau)),$$
$$\text{where } \gamma\langle\tau\rangle := \frac{a\tau+b}{c\tau+d} \quad \text{with } \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{10}$$

Let $\Gamma_0(4) := \left\{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod 4\right\}$. There exists an injective homomorphism $\Gamma_0(4) \mapsto \mathfrak{G}$ given by $\gamma \mapsto \gamma^* := (\gamma, j(\gamma, \tau))$, where

$$j(\gamma, \tau) := \left(\frac{c}{d}\right)\epsilon_d^{-1}\left(\frac{c\tau+d}{|c\tau+d|}\right)^{1/2} = \frac{\theta(\gamma\langle\tau\rangle)}{\theta(\tau)},$$
$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(4),\ \theta(\tau) := y^{1/4}\sum_{n=-\infty}^{\infty} e^{2\pi i n^2 \tau}$$

with

$$\epsilon_d = \begin{cases} 1, & \text{if } d \equiv 1 \pmod 4) \\ i, & \text{if } d \equiv 3 \pmod 4) \end{cases}$$

and $(\frac{c}{d})$ is defined as in [19, p. 442].

For an integer k define the slash operator $||_{k-1/2}$ (to distinguish it from the slash operator defined in (4)) on functions on the upper half plane as follows:

$$(f\|_{k-1/2}(\gamma, \phi))(\tau) := f(\gamma\langle\tau\rangle)\phi(\tau)^{-(2k-1)}. \tag{11}$$

We say that a smooth function $f : \mathcal{H} \to \mathbb{C}$ is a Maaß form of weight $k - 1/2$ with respect to $\Gamma_0(4)$ if the following conditions are satisfied:

1. For every $\gamma \in \Gamma_0(4)$ we have $f\|_{k-1/2}\gamma^* = f$.
2. $\Delta_{k-1/2} f = \Lambda f$ for some $\Lambda \in \mathbb{C}$ where $\Delta_{k-1/2}$ is the Laplace Beltrami operator given by

$$\Delta_{k-1/2} := y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - \left(k - \frac{1}{2}\right) iy\frac{\partial}{\partial x}. \tag{12}$$

3. $f(\tau) = O(y^N)$ as $y \to \infty$ for some $N > 0$.

If, in addition, f vanishes at all the cusps of $\Gamma_0(4)$, then we say that f is a Maaß cusp form.

Let us denote the space of Maaß forms of weight $k - 1/2$ with respect to $\Gamma_0(4)$ by $M_{k-1/2}(4)$ and the subspace of Maaß cusp forms by $S_{k-1/2}(4)$. As shown in [18] or [14], if $f \in M_{k-1/2}(4)$ then f has the following Fourier expansion

$$f(\tau) = \sum_{n\in\mathbb{Z}} c(n) W_{\mathrm{sgn}(n)\frac{k-1/2}{2}, \frac{il}{2}}(4\pi|n|y)e^{2\pi i n x} \tag{13}$$

where $\Lambda = -(1/4 + (l/2)^2)$ and $W_{\nu,\mu}(y)$ is the classical Whittaker function (see [16]), which is normalized so that $W_{\nu,\mu}(y) \sim e^{-y/2}y^\nu$ as $y \to \infty$. If $f \in S_{k-1/2}(4)$ then we have $c(0) = 0$ in the Fourier expansion above. Define the plus space by

$$M^+_{k-1/2}(4) := \{f \in M_{k-1/2}(4) : c(n) = 0 \text{ whenever } (-1)^{k-1}n \equiv 2, 3 \pmod 4\}. \tag{14}$$

This is the non-holomorphic analogue of the Kohnen plus space defined in [15]. Set $S^{+}_{k-1/2}(4) = M^{+}_{k-1/2}(4) \cap S_{k-1/2}(4)$. Using the non-holomorphic Shimura correspondence established in [14] and the fact that the Laplacian Δ_0 has infinitely many eigenvalues (see Theorem 2.11 in [12]), we know that the space $S^{+}_{1/2}(4)$ is infinite dimensional. Applying the raising, lowering and inverting operators (see [18, p. 3925]) one can see that $S^{+}_{k-1/2}(4)$ is infinite dimensional for any integer k.

For every odd prime p we have the Hecke operator T_{p^2} acting on $M^{+}_{k-1/2}(4)$. If $\{c^{(p^2)}(n)\}$ denote the Fourier coefficients of $T_{p^2} f$ then we have the following relation

$$c^{(p^2)}(n) = p\, c(p^2 n) + p^{-\frac{1}{2}} \left(\frac{(-1)^{k-1} n}{p} \right) c(n) + p^{-1} c\left(\frac{n}{p^2} \right) \tag{15}$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the Legendre symbol and $c(n) = 0$ if $n \notin \mathbb{Z}$. From [14], we know that the space $S^{+}_{k-1/2}(4)$ has a basis consisting of simultaneous eigenfunctions of T_{p^2} for all odd primes p.

Define the following operators on $M_{k-1/2}(4)$.

$$(f|U)(\tau) = \frac{2^{k-1/2}}{4} \sum_{\nu=0}^{3} f\left(\frac{\tau+\nu}{4} \right) = \frac{1}{4} \sum_{\nu=0}^{3} \left(f||_{k-1/2} \left[\begin{pmatrix} 1 & \nu \\ 0 & 4 \end{pmatrix}, 2^{-1/2} \right] \right)(\tau), \tag{16}$$

$$\begin{aligned}(f|W)(\tau) &= \left(\frac{-i\tau}{|\tau|} \right)^{-(k-1/2)} f\left(\frac{-1}{4\tau} \right) \\ &= 2^{-(k-1/2)} \left(f||_{k-1/2} \left[\begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}, 2^{-1/2} \left(\frac{-i\tau}{|\tau|} \right)^{1/2} \right] \right)(\tau).\end{aligned} \tag{17}$$

Note that $|U$ and $|W$ are the non-holomorphic analogues of the operators $|U_4$ and $|W_4$ from [15, p. 250]. As in the holomorphic case, it can be checked that $f|U$ and $f|W$ are indeed in $M_{k-1/2}(4)$. Also, note that the operators $|U$ and $|W$ commute with $\Delta_{k-1/2}$. If f has the Fourier expansion given in (13) then a straightforward computation shows that $f|U$ has the Fourier expansion

$$(f|U)(\tau) = 2^{k-1/2} \sum_{n\in\mathbb{Z}} c(4n) W_{\operatorname{sgn}(n)\frac{k-1/2}{2}, \frac{il}{2}}(4\pi|n|y) e^{2\pi i n x}. \tag{18}$$

In addition, if $f \in M^{+}_{k-1/2}(4)$ then we have the identity

$$\frac{2^{k-1/2}}{4} \left[f\left(\frac{\tau}{4} \right) + f\left(\frac{\tau+2}{4} \right) \right] = \frac{1}{2}(f|U)(\tau). \tag{19}$$

Next, we show that, as in the holomorphic case, elements of $M^{+}_{k-1/2}(4)$ are eigenfunctions of the operator $|U|W$.

Proposition 4.1 *If $f \in M^{+}_{k-1/2}(4)$ then we have*

$$f|U|W = 2^{k-1} i^{k^2-k} f. \tag{20}$$

Proof Write $f|U|W = f_1 + f_2$ where

$$f_1 = \frac{1}{4} \left(f||_{k-1/2} \left[\begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}, 2^{-1/2} \right] + f||_{k-1/2} \left[\begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}, 2^{-1/2} \right] \right) \Big| W$$

and

$$f_2 = \frac{2^{k-1/2}}{4}\left[f\left(\frac{\tau}{4}\right) + f\left(\frac{\tau+2}{4}\right)\right]\Big|W.$$

We have

$$\begin{aligned}
2^{k-\frac{1}{2}}4f_1 &= f||_{k-1/2}\left[\begin{pmatrix}4 & -1\\ 16 & 0\end{pmatrix}, 2^{-1}\left(\frac{-i\tau}{|\tau|}\right)^{1/2}\right]\\
&\quad + f||_{k-1/2}\left[\begin{pmatrix}12 & -1\\ 16 & 0\end{pmatrix}, 2^{-1}\left(\frac{-i\tau}{|\tau|}\right)^{1/2}\right]\\
&= f||_{k-1/2}\begin{pmatrix}1 & 0\\ -4 & 1\end{pmatrix}^{*}||_{k-1/2}\left[\begin{pmatrix}4 & -1\\ 16 & 0\end{pmatrix}, 2^{-1}\left(\frac{-i\tau}{|\tau|}\right)^{1/2}\right]\\
&\quad + f||_{k-1/2}\begin{pmatrix}-1 & 1\\ -4 & 3\end{pmatrix}^{*}||_{k-1/2}\left[\begin{pmatrix}12 & -1\\ 16 & 0\end{pmatrix}, 2^{-1}\left(\frac{-i\tau}{|\tau|}\right)^{1/2}\right]\\
&= f||_{k-1/2}\left[\begin{pmatrix}4 & -1\\ 0 & 4\end{pmatrix}, 2^{-1}e^{-\frac{i\pi}{4}}\right] + f||_{k-1/2}\left[\begin{pmatrix}4 & 1\\ 0 & 4\end{pmatrix}, 2^{-1}e^{\frac{i\pi}{4}}\right]\\
&= 2^{2k-3/2}\big(i^{k-1}(1+i)f(\tau-1/4) + i^{1-k}(1-i)f(\tau+1/4)\big) = 2^{2k-1/2}i^{k^2-k}f(\tau).
\end{aligned}$$

We get the last equality because $c(n)=0$ if $(-1)^{k-1}n \equiv 2,3 \pmod 4$. Now putting this together with $f_2 = \frac{1}{2}f|U|W$ (see 19) we get the result. $\square$

For $f(\tau) = \sum_{n\in\mathbb{Z}} c(n) W_{\mathrm{sgn}(n)\frac{k-1/2}{2},\frac{il}{2}}(4\pi|n|y)e^{2\pi inx} \in M^{+}_{k-1/2}(4)$, define the functions

$$f^{(0)}(\tau) = \sum_{n\in\mathbb{Z}} c(4n) W_{\mathrm{sgn}(n)\frac{k-1/2}{2},\frac{il}{2}}(4\pi|n|y)e^{2\pi inx}, \tag{21}$$

$$f^{(1)}(\tau) = \sum_{n\in\mathbb{Z}} c(4n+\epsilon) W_{\mathrm{sgn}(n)\frac{k-1/2}{2},\frac{il}{2}}\left(4\pi\left|n+\frac{\epsilon}{4}\right|y\right)e^{2\pi i(n+\frac{\epsilon}{4})x} \tag{22}$$

where $\epsilon = (-1)^{k-1}$. We have by explicit computation

$$\begin{aligned}
&f^{(0)}(\tau) = \frac{1}{4}\sum_{\nu=0}^{3} f\left(\frac{\tau+\nu}{4}\right), \qquad f^{(1)}(\tau) = \frac{1}{4}\sum_{\nu=0}^{3}(-\epsilon i)^{\nu} f\left(\frac{\tau+\nu}{4}\right),\\
&f(\tau) = (f^{(0)} + f^{(1)})(4\tau).
\end{aligned} \tag{23}$$

It is clear from the Fourier expansion (21), (22) or the relations (23) that both $f^{(0)}$ and $f^{(1)}$ are eigenfunctions of $\Delta_{k-1/2}$ with the same eigenvalue as f.

Proposition 4.2 *With notations as above, we have*

$$\begin{aligned}
&\begin{bmatrix}f^{(0)}(\tau+1)\\ f^{(1)}(\tau+1)\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & \epsilon i\end{bmatrix}\begin{bmatrix}f^{(0)}(\tau)\\ f^{(1)}(\tau)\end{bmatrix},\\
&\begin{bmatrix}f^{(0)}(-\tau^{-1})\\ f^{(1)}(-\tau^{-1})\end{bmatrix} = \left(\frac{\tau}{|\tau|}\right)^{k-1/2}\left(\frac{1-\epsilon i}{2}\right)\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}f^{(0)}(\tau)\\ f^{(1)}(\tau)\end{bmatrix}.
\end{aligned} \tag{24}$$

Proof It follows from the definition that $f^{(0)}(\tau+1)=f^{(0)}(\tau)$ and $f^{(1)}(\tau+1)=e^{2\pi i\frac{\epsilon}{4}}f^{(1)}(\tau)=\epsilon i f^{(1)}(\tau)$, which gives us the first equation above. To obtain the second equation, first note that $f^{(0)}=2^{-(k-1/2)}f|U$. Hence

$$f^{(0)}\left(\frac{-1}{4\tau}\right)=2^{-(k-\frac{1}{2})}(f|U)\left(\frac{-1}{4\tau}\right)=2^{-(k-\frac{1}{2})}\left(\frac{-i\tau}{|\tau|}\right)^{k-\frac{1}{2}}(f|U|W)(\tau)$$
$$=2^{-(k-\frac{1}{2})}\left(\frac{-i\tau}{|\tau|}\right)^{k-\frac{1}{2}}2^{k-1}i^{k^2-k}f(\tau).$$

Here we have used the definition of the operator $|W$ and Proposition 4.1. Note that $(-i)^{k-1/2}i^{k^2-k}=\frac{1-\epsilon i}{\sqrt{2}}$. Now substituting $\tau/4$ for τ in the above equation and using (23) we get

$$f^{(0)}(-\tau^{-1})=\left(\frac{\tau}{|\tau|}\right)^{k-1/2}\left(\frac{1-\epsilon i}{2}\right)(f^{(0)}(\tau)+f^{(1)}(\tau)). \tag{25}$$

This gives us the first part. Now substituting $-\tau^{-1}$ for τ in (25) we get

$$f^{(0)}(\tau)=\left(\frac{-\tau}{|\tau|}\right)^{-(k-1/2)}\left(\frac{1-\epsilon i}{2}\right)\left(f^{(0)}(-\tau^{-1})+f^{(1)}(-\tau^{-1})\right).$$

Solving for $f^{(1)}(-\tau^{-1})$ and using (25), we get

$$f^{(1)}(-\tau^{-1})=\left(\frac{\tau}{|\tau|}\right)^{k-1/2}\left(\frac{1-\epsilon i}{2}\right)\left(f^{(0)}(\tau)-f^{(1)}(\tau)\right)$$

which gives us (24), as required. □

4.2 Jacobi Maaß forms of even weight k and index $m=1$

Define the following theta series for $\tau=x+iy\in\mathcal{H}$ and $z\in\mathbb{C}$ and $j=0,1$:

$$\widetilde{\Theta}^{(j)}(\tau,z):=y^{\frac{1}{4}}\sum_{\substack{r\in\mathbb{Z}\\ r\equiv j \pmod 2}}e^{2\pi i\tau\frac{r^2}{4}}e^{2\pi i z r}. \tag{26}$$

From [10, p. 58–59], we have the following transformation formulae for the theta series.

$$\begin{aligned}\begin{bmatrix}\widetilde{\Theta}^{(0)}(\tau+1,z)\\ \widetilde{\Theta}^{(1)}(\tau+1,z)\end{bmatrix}&=\begin{bmatrix}1&0\\0&i\end{bmatrix}\begin{bmatrix}\widetilde{\Theta}^{(0)}(\tau,z)\\ \widetilde{\Theta}^{(1)}(\tau,z)\end{bmatrix},\\ \begin{bmatrix}\widetilde{\Theta}^{(0)}(\frac{-1}{\tau},\frac{z}{\tau})\\ \widetilde{\Theta}^{(1)}(\frac{-1}{\tau},\frac{z}{\tau})\end{bmatrix}&=2^{-1/2}\left(\frac{-i\tau}{|\tau|}\right)^{1/2}e^{2\pi i\frac{z^2}{\tau}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}\widetilde{\Theta}^{(0)}(\tau,z)\\ \widetilde{\Theta}^{(1)}(\tau,z)\end{bmatrix}.\end{aligned} \tag{27}$$

In the next lemma we state the differential equations satisfied by $\widetilde{\Theta}^{(j)}$, which will be used in the proof of Theorem 4.4.

Lemma 4.3 *For* $j=0,1$ *we have*

$$\frac{5}{8}\widetilde{\Theta}^{(j)}-2(\tau-\bar{\tau})^2\widetilde{\Theta}^{(j)}_{\tau\bar{\tau}}-(k-1)(\tau-\bar{\tau})\widetilde{\Theta}^{(j)}_{\bar{\tau}}-k(\tau-\bar{\tau})\widetilde{\Theta}^{(j)}_{\tau}$$

$$+k\frac{\tau-\bar\tau}{8\pi i}\widetilde{\Theta}^{(j)}_{zz}+\frac{(\tau-\bar\tau)^2}{4\pi i}\widetilde{\Theta}^{(j)}_{\bar\tau zz}=0,$$

$$2(\tau-\bar\tau)\widetilde{\Theta}^{(j)}_{\bar\tau}+\frac{1}{2}\widetilde{\Theta}^{(j)}=0,\, -2(\tau-\bar\tau)\widetilde{\Theta}^{(j)}_{\tau}+\frac{1}{2}\widetilde{\Theta}^{(j)}+\frac{\tau-\bar\tau}{4\pi i}\widetilde{\Theta}^{(j)}_{zz}=0.$$

Proof The lemma is proved by direct computation. □

For $f\in M^+_{k-1/2}(4)$ with $k\in 2\mathbb{Z}$, define $F_f:\mathcal{H}\times\mathbb{C}\mapsto\mathbb{C}$ by the formula

$$F_f(\tau,z):=f^{(0)}(\tau)\widetilde{\Theta}^{(0)}(\tau,z)+f^{(1)}(\tau)\widetilde{\Theta}^{(1)}(\tau,z). \tag{28}$$

Theorem 4.4 *Let $f\in M^+_{k-1/2}(4)$ with $k\in 2\mathbb{Z}$ and F_f be the smooth function on $\mathcal{H}\times\mathbb{C}$ defined by* (28)*. Then*

1. *F_f is a Jacobi Maaß form of weight k and index 1 with respect to Γ^J, i.e., $F_f\in J^{nh}_{k,1}$,*
2. *F_f is a Jacobi Maaß cusp form if and only if $f\in S^+_{k-1/2}(4)$,*
3. *If $\Delta_{k-1/2}f=\Lambda f$ then we have $\mathcal{C}^{k,1}F_f=2\Lambda F_f$.*

Proof Using (24) and (27) we get

$$F_f(\tau+1,z)=F_f(\tau,z),\qquad F_f\left(\frac{-1}{\tau},\frac{z}{\tau}\right)=\left(\frac{\tau}{|\tau|}\right)^k e^{2\pi i\frac{z^2}{\tau}}F_f(\tau,z),$$

for all $\tau\in\mathcal{H}$ and $z\in\mathbb{C}$. Here we have used the fact that k is even. A direct calculation gives the following for $\lambda,\mu\in\mathbb{Z}$, $j=0,1$

$$\widetilde{\Theta}^{(j)}(\tau,z+\lambda\tau+\mu)=e^{-2\pi i(\lambda^2\tau+2\lambda z)}\widetilde{\Theta}^{(j)}(\tau,z)$$
$$\Rightarrow\quad F_f(\tau,z+\lambda\tau+\mu)=e^{-2\pi i(\lambda^2\tau+2\lambda z)}F_f(\tau,z).$$

Since the group Γ^J is generated by the elements $\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right](0,0,0)$, $\left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right](0,0,0)$, and $\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right](\lambda,\mu,\kappa)$ for $\lambda,\mu,\kappa\in\mathbb{Z}$, we get

$$(F_f|_{k,1}\gamma)(\tau,z)=F_f(\tau,z)\quad\text{for all }\gamma\in\Gamma^J,\tau\in\mathcal{H},z\in\mathbb{C}.$$

This gives us the first condition from Definition 3.2. Next we have to show that F_f is an eigenfunction of the operator $\mathcal{C}^{k,1}$. Since F_f is holomorphic in the z variable (by construction) we see that all the terms in $\mathcal{C}^{k,1}$ which involve taking partial derivative with respect to $\bar z$ give 0. Using Lemma 4.3 we see that

$$\begin{aligned}\mathcal{C}^{k,1}F_f&=2\sum_{j=0,1}\left(y^2(f^{(j)}_{xx}+f^{(j)}_{yy})-\left(k-\frac{1}{2}\right)iyf^{(j)}_x\right)\widetilde{\Theta}^{(j)}\\&=2\sum_{j=0,1}(\Delta_{k-1/2}f^{(j)})\widetilde{\Theta}^{(j)}=2\Lambda F_f.\end{aligned}$$

This gives us the second condition from Definition 3.2. Since $f^{(j)}$ and $\widetilde{\Theta}^{(j)}$ are of polynomial growth in y, for $j=0,1$, we see that F_f satisfies the third condition from Definition 3.2. This gives us $F_f\in J^{nh}_{k,m}$.

Substituting the definitions of the theta series and the Fourier expansion of $f^{(j)}$ in (28) we get the following Fourier expansion of F_f

$$F_f(\tau,z)=\sum_{r,n\in\mathbb{Z}} c(4n-r^2)y^{\frac{1}{4}}W_{\operatorname{sgn}(n-\frac{r^2}{4})\frac{k-1/2}{2},\frac{il}{2}}\left(4\pi\left|n-\frac{r^2}{4}\right|y\right)e^{-2\pi\frac{r^2}{4}y}e^{2\pi inx}e^{2\pi irz}. \quad (29)$$

If $f\in S^+_{k-1/2}(4)$, then $c(0)=0$. This implies that $c(4n-r^2)=0$ for every $n,r,\in\mathbb{Z}$ such that $4n-r^2=0$. This precisely gives us the criteria (9) for cuspidality of F_f. This completes the proof of the theorem. □

Note that (29) implies that the map $f\to F_f$ is injective and hence we can conclude that, for even k, the space $J^{nh}_{k,1}$ is infinite dimensional. We will now characterize the image of this map. For any integer k and index m define

$$\hat{J}^{nh}_{k,m}:=\{F\in J^{nh}_{k,m}:Y^{k.m}_{-}F=0\}. \quad (30)$$

Elements of $\hat{J}^{nh}_{k,m}$ are precisely the Jacobi forms that are holomorphic in the z variable.

Theorem 4.5 *If k is an even integer then we have*

$$\hat{J}^{nh}_{k,1}=\{F\in J^{nh}_{k,1}: \textit{there is a } f\in M^+_{k-1/2} \textit{such that } F=F_f\}. \quad (31)$$

Proof The Fourier expansion (29) immediately tells us that the right hand side of (31) is contained in the left hand side. Let us now show the opposite inclusion. Let $F\in\hat{J}^{nh}_{k,1}$, hence it satisfies $Y^{k,1}_{-}F=0$. We will first show that this forces F to have a Fourier expansion of the form given in (29). Applying the element $\gamma=I(0,1,0)\in\Gamma^J$ to F we get $F(\tau,z+1)=F(\tau,z)$. Set $z=u+iv$. Then we have $F(\tau,z)=\sum_{r=-\infty}^{\infty}F_r(\tau,v)e^{2\pi iru}$. Now applying $Y^{k,1}_{-}$ to each summand we get

$$\begin{aligned}
&Y^{k,1}_{-}(F_r(\tau,v)e^{2\pi iru})=0 \quad\Leftrightarrow\quad \frac{\partial}{\partial\bar{z}}(F_r(\tau,v)e^{2\pi iru})=0\\
&\Leftrightarrow\quad \frac{\partial}{\partial v}F_r(\tau,v)=-2\pi rF_r(\tau,v)\quad\Leftrightarrow\quad F_r(\tau,v)=\hat{F}_r(\tau)e^{-2\pi rv}.
\end{aligned}$$

This implies that $F(\tau,z)=\sum_{r=-\infty}^{\infty}\hat{F}_r(\tau)e^{2\pi irz}$. Next apply $\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right](0,0,0)\in\Gamma^J$ to F to get $F(\tau+1,z)=F(\tau,z)$ and hence (with $\tau=x+iy$) we have $F(\tau,z)=\sum_{n,r=-\infty}^{\infty}\hat{\hat{F}}_{n,r}(y)\cdot e^{2\pi inx}e^{2\pi irz}$. Let us now set $\hat{\hat{F}}_{n,r}(y)=y^{1/4}e^{-2\pi\frac{r^2}{4}y}g_{n,r}(y)$. For a fixed n,r, if we apply the operator $\mathcal{C}^{k,1}$ to the function $\hat{\hat{F}}_{n,r}(y)e^{2\pi inx}e^{2\pi irz}$ we get

$$-\frac{1}{2}y^{\frac{5}{4}}e^{-2\pi\frac{r^2}{4}y}e^{2\pi inx}e^{2\pi irz}\big(\pi(4n-r^2)(1-2k+(4n-r^2)\pi y)g_{n,r}(y)-4yg''_{n,r}(y)\big).$$

Using the fact that $\mathcal{C}^{k,1}F=\lambda F$ and a suitable change of variable, we see that $g_{n,r}$ satisfies the differential equation corresponding to the Whittaker function [16, p. 256]. The growth condition on F then forces $g_{n,r}(a)=c(n,r)W_{\operatorname{sgn}(n-\frac{r^2}{4})(k-1/2)/2,\frac{il}{2}}(|a|)$, which gives us

$$F(\tau,z)=\sum_{n,r=-\infty}^{\infty}c(n,r)y^{\frac{1}{4}}W_{\operatorname{sgn}(n-\frac{r^2}{4})\frac{k-1/2}{2},\frac{il}{2}}\left(4\pi\left|n-\frac{r^2}{4}\right|y\right)e^{-2\pi\frac{r^2}{4}y}e^{2\pi inx}e^{2\pi irz}.$$

Imitating the arguments given in Theorem 2.2 from [10] we can conclude that the Fourier coefficients $c(n,r)$ only depend on $4n-r^2$. If F is cuspidal then we have the extra condition that $c(n,r)=0$ if $4n-r^2=0$. Now define

$$c_f(N)=\begin{cases} c(\frac{N}{4},0), & \text{if } N\equiv 0 \pmod 4;\\ c(\frac{N+1}{4},1), & \text{if } N\equiv 3 \pmod 4;\\ 0, & \text{otherwise}\end{cases}$$

and set

$$\begin{aligned} f_F(\tau) &= \sum_{N\in\mathbb{Z}} c_f(N) W_{\mathrm{sgn}(N)\frac{k-1/2}{2},\frac{il}{2}}(4\pi|N|y)e^{2\pi iNx},\\ f_F^{(0)}(\tau) &= \sum_{N\in\mathbb{Z}} c_f(4N) W_{\mathrm{sgn}(N)\frac{k-1/2}{2},il/2}(4\pi|N|y)e^{2\pi iNx},\\ f_F^{(1)}(\tau) &= \sum_{N\in\mathbb{Z}} c_f(4N-1) W_{\mathrm{sgn}(N)\frac{k-1/2}{2},il/2}\left(4\pi\left|N-\frac{1}{4}\right|y\right)e^{2\pi i(N-\frac{1}{4})x}. \end{aligned} \tag{32}$$

Then, we have

$$f_F(\tau)=(f_F^{(0)}+f_F^{(1)})(4\tau) \quad \text{and} \quad F(\tau,z)=f_F^{(0)}(\tau)\tilde{\Theta}^{(0)}(\tau,z)+f_F^{(1)}(\tau)\tilde{\Theta}^{(1)}(\tau,z). \tag{33}$$

The automorphy condition on F and (27) imply that $f_F^{(0)}$ and $f_F^{(1)}$ satisfy the transformation property given in (24). Hence, we get

$$f_F(\tau+1)=f_F^{(0)}(4\tau+4)+f_F^{(1)}(4\tau+4)=f_F^{(0)}(4\tau)+(-i)^4f_F^{(1)}(4\tau)=f_F(\tau)$$

and

$$f_F\left(\frac{\tau}{4\tau+1}\right)=\left(\frac{4\tau+1}{|4\tau+1|}\right)^{k-1/2} f_F(\tau).$$

Since $\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1&0\\4&1\end{smallmatrix}\right]$ generate the group $\Gamma_0(4)$, we conclude that $f_F||_{k-1/2}\gamma^*=f_F$ for all $\gamma\in\Gamma_0(4)$. The Fourier expansion (32) of f_F implies that f_F has polynomial growth in y and $\Delta_{k-1/2}f_F=(\lambda/2)f_F$. Hence $f_F\in M_{k-1/2}(4)$. Once again, the Fourier expansion (32) of f_F implies that $c_F(N)=0$ if $N\equiv 1,2 \pmod 4$, which implies that $f_F\in M^+_{k-1/2}(4)$. It is clear from (33) that $F=F_{f_F}$ and hence, F belongs to the right hand side of (31). □

4.3 Jacobi forms of weight $k\in\mathbb{Z}$ and index $m\geq 1$

Recall that $\hat{J}^{nh}_{k,m}:=\{F\in J^{nh}_{k,m}: Y^{k,m}_- F=0\}$ for $k\in\mathbb{Z}$ and $m\geq 1$. Let $F\in\hat{J}^{nh}_{k,m}$ with $\mathcal{C}^{k,m}F=\lambda F$. Imitating the arguments in the proof of Theorem 4.5 we see that F has the Fourier expansion

$$\begin{aligned} F(\tau,z) &= \sum_{n,r\in\mathbb{Z}} \hat{c}(n,r,y)e^{2\pi in\tau}e^{2\pi irz},\\ \hat{c}(n,r,y) &= c(n,r)y^{\frac{1}{4}}W_{\mathrm{sgn}(nm-\frac{r^2}{4})\frac{k-1/2}{2},\frac{il}{2}}\left(4\pi\left|nm-\frac{r^2}{4}\right|y\right)e^{2\pi(nm-\frac{r^2}{4})y}. \end{aligned} \tag{34}$$

Here, the coefficients $c(n,r)$ only depend on $4nm-r^2$ and $r \pmod{2m}$. For $\mu \in \mathbb{Z}/2m\mathbb{Z}$ and all integers $N \equiv -\mu^2 \pmod{4m}$, set

$$c_\mu(N) := c\left(\frac{N+r^2}{4m}, r\right) \quad \text{for any } r \in \mathbb{Z}, r \equiv \mu \pmod{2m} \tag{35}$$

and extend to all N by setting $c_\mu(N) = 0$ if $N \not\equiv -\mu^2 \pmod{4m}$. Define

$$f^{(\mu)}(\tau) := \sum_{N\in\mathbb{Z}} c_\mu(N) W_{\operatorname{sgn}(N)\frac{k-1/2}{2}, \frac{il}{2}}\left(4\pi\left|\frac{N}{4m}\right|y\right) e^{2\pi i \frac{N}{4m}x}. \tag{36}$$

Then $\Delta_{k-1/2} f^{(\mu)} = \lambda/2 f^{(\mu)}$ for all μ. Define the theta functions

$$\widetilde{\Theta}_m^{(\mu)}(\tau,z) := y^{1/4} \sum_{\substack{r\in\mathbb{Z} \\ r\equiv\mu \pmod{2m}}} e^{2\pi i\tau\frac{r^2}{4m}} e^{2\pi i z r}. \tag{37}$$

From (34), (36) and (37) it is clear that

$$F(\tau,z) = \sum_{\mu \pmod{2m}} f^{(\mu)}(\tau)\widetilde{\Theta}_m^{(\mu)}(\tau,z). \tag{38}$$

The theta functions $\widetilde{\Theta}_m^{(\mu)}$ satisfy transformation properties (see [10, pp. 58–59]) and differential equations analogous to (27) and Lemma 4.3. Using the theta transformation property and the automorphy of F, we conclude that the functions $f^{(\mu)}$, $\mu \pmod{2m}$ satisfy

$$f^{(\mu)}(\tau+1) = e^{-2\pi i \frac{\mu^2}{4m}} f^{(\mu)}(\tau), \tag{39}$$

$$f^{(\mu)}\left(\frac{-1}{\tau}\right) = \frac{1+i}{2\sqrt{m}}\left(\frac{\tau}{|\tau|}\right)^{k-1/2} \sum_{\nu \pmod{2m}} e^{2\pi i \frac{\mu\nu}{2m}} f^{(\nu)}(\tau). \tag{40}$$

In the reverse direction, if one starts with functions $(f^{(\mu)}, \mu \pmod{2m})$ with Fourier expansion (36), satisfying properties (39), (40) and bounded as $\operatorname{Im}(\tau) \to \infty$, then the function F defined by (38) clearly satisfies the first condition from Definition 3.2 and has the Fourier expansion (34). One can check that F is an eigenfunction of $\mathcal{C}^{k,m}$ directly from the Fourier expansion (34) or by using the differential equations satisfied by the $\widetilde{\Theta}_m^{(\mu)}$ and $f^{(\mu)}$. The third condition from Definition 3.2 follows from the boundedness of $f^{(\mu)}$ as $\operatorname{Im}(\tau) \to \infty$. Finally, the Fourier expansion (34) tells us that $Y_-^{k.m} F = 0$. Let us summarize in the following theorem.

Theorem 4.6 *For any integer k and $m \geq 1$, the equation (38) gives an isomorphism between $\hat{J}_{k,m}^{nh}$ and the space of vector valued modular forms $(f^{(\mu)}, \mu \pmod{2m})$ with Fourier expansion (36), satisfying properties (39), (40) and bounded as $\operatorname{Im}(\tau) \to \infty$. In addition, for k even and $m = 1$ the space $\hat{J}_{k,1}^{nh}$ is isomorphic to $M_{k-1/2}^+(4)$.*

Remark The above theorem states that the subspace $\hat{J}_{k,m}^{nh}$ is obtained from scalar or vector valued half integer weight Maaß forms. If $F \in J_{k,m}^{nh}$, then one can actually write $F = \sum_{l\leq k} \alpha_l Y_+^{k-1,m} \cdots Y_+^{l,m} F_l$ for some choice of functions $F_l \in \hat{J}_{l,m}^{nh}$. (This is obtained by looking at how the differential operators act on the vectors in the archimedean representations for $G^J(\mathbb{R})$ [5, pp. 35–36].) In this sense, one can say that all of $J_{k,m}^{nh}$ is "essentially" obtained from half integer weight Maaß forms.

5 Known examples

In this section we will consider the three known examples of Jacobi forms: the holomorphic Jacobi forms considered by Eichler-Zagier, the skew-holomorphic Jacobi forms due to Skoruppa and the real-analytic Jacobi Eisenstein series due to Arakawa. We will show that if F is any one of the above Jacobi forms then a suitable modification $\hat{F}$ is actually a Jacobi Maaß form in the sense of Definition 3.2. We will first give the definition of the three types of Jacobi forms mentioned above.

Holomorphic Jacobi forms [10, p. 9] A holomorphic function $F^h : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ is called a holomorphic Jacobi form of weight $k > 0$ and index m if F^h satisfies

$$F^h\left(\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda\tau+\mu}{c\tau+d}\right) = e^{-2\pi i m(-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d}+\lambda^2\tau+2\lambda z)}(c\tau+d)^k F^h(\tau,z) \tag{41}$$

where $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right](\lambda,\mu,\kappa) \in \Gamma^J$ and has a Fourier expansion of the form

$$F^h(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z} \\ 4nm-r^2\geq 0}} c(n,r) e^{2\pi i(n\tau+rz)}.$$

Skew-holomorphic Jacobi forms [20, p. 179] A smooth function $F^{sh} : \mathcal{H}\times\mathbb{C} \to \mathbb{C}$ is called a skew-holomorphic Jacobi form of weight $k > 0$ and index m if it satisfies the following conditions

1. $\partial_{\bar{z}} F = (8\pi i m \partial_\tau - \partial_z^2) F = 0$
2. For every $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right](\lambda,\mu,\kappa) \in \Gamma^J$ we have

$$\begin{aligned} &F\left(\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda\tau+\mu}{c\tau+d}\right) \\ &\quad = e^{-2\pi i m(-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d}+\lambda^2\tau+2\lambda z)}\left(\frac{c\tau+d}{|c\tau+d|}\right)^{1-k} |c\tau+d|^k F(\tau,z) \end{aligned} \tag{42}$$

3. F^{sh} has a Fourier expansion of the form

$$F(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z} \\ 4nm-r^2\leq 0}} c(n,r) e^{2\pi i(n\tau+iy(r^2-4nm)/(2m)+rz)}.$$

Real-analytic Jacobi Eisenstein series [1, p. 132] Let $\Gamma^J_{\infty,+}$ be the subgroup of Γ^J consisting of all elements of the form $\left[\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right](0,\mu,\kappa)$ with $n,\mu,\kappa \in \mathbb{Z}$. Fix $k \in \mathbb{Z}$ and $m \in \mathbb{N}$. For each integer r such that $r^2 \equiv 0 \pmod{4m}$ and $s \in \mathbb{C}$ define a function $\phi_{r,s} : \mathcal{H}\times\mathbb{C} \to \mathbb{C}$ by

$$\phi_{r,s}(\tau,z) = e^{2\pi i m(\frac{r^2\tau}{4m^2}+\frac{rz}{m})} y^{s-\frac{k-1/2}{2}}$$

where $\tau = x+iy$. Then the real analytic Eisenstein series $E_{k,m,r}((\tau,z),s)$ (defined in [1, p. 132]) of weight k and index m is given by

$$
\begin{aligned}
&E_{k,m,r}((\tau,z),s)\\
&\quad=\sum_{\gamma\in\Gamma^J_{\infty,+}\backslash\Gamma^J} e^{2\pi i m(-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d}+\lambda^2\tau+2\lambda z)}(c\tau+d)^{-k}\phi_{r,s}\left(M\langle\tau\rangle,\frac{z+\lambda\tau+\mu}{c\tau+d}\right) \qquad (43)
\end{aligned}
$$

where $\gamma=M(\lambda,\mu,\kappa)\in\Gamma^J$ with $M=\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]$. The Eisenstein series is absolutely convergent for $Re(s)>5/4$ and, if $k>3$, and s is evaluated at $(k-1/2)/2$ then $E_{k,m,r}((\tau,z),(k-1/2)/2)$ coincides with the holomorphic Eisenstein series of [10]. The Eisenstein series satisfies the following transformation law:

$$
\begin{aligned}
&E_{k,m,r}\left(\left(\frac{a\tau+b}{c\tau+d},\frac{z+\lambda\tau+\mu}{c\tau+d}\right),s\right)\\
&\quad=e^{-2\pi i m(-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d}+\lambda^2\tau+2\lambda z)}(c\tau+d)^kE_{k,m,r}((\tau,z),s) \qquad (44)
\end{aligned}
$$

where $\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right](\lambda,\mu,\kappa)\in\Gamma^J$. It is known that $E_{k,m,r}$ depends only on $r \pmod{2m}$.

Theorem 5.1 *Let $F:\mathcal{H}\times\mathbb{C}\to\mathbb{C}$ be one of the functions F^h or F^{sh} or $E_{k,m,r}$ defined above. Set $\hat F(\tau,z):=y^{k/2}F(\tau,z)$. If $F=F^h$ or $E_{k,m,r}$, then $\hat F\in\hat J^{nh}_{k,m}$ and, if $F=F^{sh}$, then $\hat F\in\hat J^{nh}_{1-k,m}$.*

Proof For $M=\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]$, we have $\mathrm{Im}(M\langle\tau\rangle)=|c\tau+d|^{-2}\mathrm{Im}(\tau)$. Using this fact and the transformation properties (41), (42) and (44) we see that the function $\hat F$ satisfies the first condition of Definition 3.2 in all three cases. In [5, p. 83], it has been shown that

$$
\begin{aligned}
\mathcal{C}(\Phi_{k,m}\widehat{F^h})&=\big((k-1/2)(k-5/2)/2\big)(\Phi_{k,m}\widehat{F^h}),\\
\mathcal{C}(\Phi_{1-k,m}\widehat{F^{sh}})&=\big((k-1/2)(k-5/2)/2\big)(\Phi_{1-k,m}\widehat{F^{sh}}),\\
\mathcal{C}(\Phi_{k,m}\hat F_s)&=\big(((2s-1)^2-1)/2\big)(\Phi_{k,m}\hat F_s).
\end{aligned}
$$

Using (7), we can now conclude that in all the three cases, $\hat F$ is an eigenfunction of the differential operator $\mathcal{C}^{*,m}$ ($*$ is the weight of $\hat F$), which gives us the second condition from Definition 3.2. The third condition from Definition 3.2 follows from the definition of the Jacobi forms F. The Fourier expansion of F^h, F^{sh} and the definition of $E_{k,m,r}$ implies that in each case $\hat F$ is annihilated by the appropriate $Y_-^{*,m}$ operator. Hence we get the theorem. □

6 Hecke equivariance

We want to relate the representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ obtained from a half integer weight Maaß form f to the representation of $G^J(\mathbb{A})$ obtained from the corresponding Jacobi Maaß form F_f. To relate the non-archimedean representations, the first step is to show the Hecke equivariance of the map $f\mapsto F_f$. Let us assume that $f\in M^+_{k-1/2}(4)$ (k even) is a Hecke eigenform for all odd primes p. Then we will show that F_f is also an eigenfunction of the Hecke operators for every odd prime p. The proof closely follows the proof of the holomorphic analogue in [10, Theorem 4.5].

Let us first recall the definition of the Jacobi Hecke operator T_p, for an odd prime p. From [5, p. 168] or [10, p. 41], we have the following operator acting on Jacobi Maaß forms $F \in J_{k,1}^{nh}$

$$T_p F := p^{k-4} \sum_{\substack{M \in \mathrm{SL}_2(\mathbb{Z})/M_2(\mathbb{Z}) \\ \det(M)=p^2 \\ \gcd(M)=1}} \sum_{(\lambda,\mu)\in(\mathbb{Z}/p\mathbb{Z})^2} F|_{k,1}(\det(M)^{-\frac{1}{2}} M(\lambda,\mu,0)). \tag{45}$$

Theorem 6.1 *Let $f \in M_{k-1/2}^{+}(4)$ (k even) be a Hecke eigenform with eigenvalue λ_p for every odd prime p. Then for every odd prime p, the corresponding Jacobi Maaß form F_f defined in (28) is an eigenfunction of the operator T_p defined above. Moreover, if $T_p F_f = \mu_p F_f$ then we have $\mu_p = p^{k-3/2}\lambda_p$.*

Proof We have the following coset decomposition (see [5, p. 170])

$$\begin{aligned}
&\{M \in M_2(\mathbb{Z}) : \det(M) = p^2, \gcd(M) = 1\} \\
&\quad = \mathrm{SL}_2(\mathbb{Z}) \begin{bmatrix} p^2 & 0 \\ 0 & 1 \end{bmatrix} \sqcup \Bigg(\bigsqcup_{h \in (\mathbb{Z}/p\mathbb{Z})^\times} \mathrm{SL}_2(\mathbb{Z}) \begin{bmatrix} p & h \\ 0 & p \end{bmatrix} \Bigg) \\
&\quad\quad \sqcup \Bigg(\bigsqcup_{b \in \mathbb{Z}/p^2\mathbb{Z}} \mathrm{SL}_2(\mathbb{Z}) \begin{bmatrix} 1 & b \\ 0 & p^2 \end{bmatrix} \Bigg).
\end{aligned} \tag{46}$$

First consider the function F_1 given by

$$\begin{aligned}
F_1(\tau, z) &= \sum_{\substack{M \in \mathrm{SL}_2(\mathbb{Z})/M_2(\mathbb{Z}) \\ \det(M)=p^2 \\ \gcd(M)=1}} (F_f|_{k,1}(\det(M)^{-\frac{1}{2}} M))(\tau, z) \\
&= F_f(p^2\tau, pz) + \sum_{h=1}^{p-1} F_f\left(\tau + \frac{h}{p}, z\right) + \sum_{b=0}^{p^2-1} F_f\left(\frac{\tau}{p^2} + \frac{b}{p^2}, \frac{z}{p}\right).
\end{aligned} \tag{47}$$

Recall, from (29) that F_f has the Fourier expansion

$$F_f(\tau, z) = \sum_{n,r\in\mathbb{Z}} \hat{c}(n, r, y) e^{2\pi i n\tau} e^{2\pi i r z},$$

$$\hat{c}(n, r, y) = c(4n - r^2) y^{\frac{1}{4}} W_{\mathrm{sgn}(n-\frac{r^2}{4})\frac{k-1/2}{2}, \frac{il}{2}}\left(4\pi|n - \frac{r^2}{4}|y\right) e^{2\pi(n-\frac{r^2}{4})y}.$$

Set $\delta_{t,m} = 1$ if $t = m$ and $\delta_{t,m} = 0$ if $t \neq m$. Hence, for two integers a, b, we have $a|b \Leftrightarrow \delta_{(a,b),a} = 1$. From (47) we get

$$F_1(\tau, z) = \sum_{n\in\mathbb{Z}, r\in\frac{1}{p}\mathbb{Z}} \hat{c}_1(n, r, y) e^{2\pi i n\tau} e^{2\pi i r z} \tag{48}$$

where

$$\hat{c}_1(n,r,y)=\begin{cases} p^2\hat{c}(np^2,rp,\frac{y}{p^2}), \quad \text{if } r\notin\mathbb{Z} \\ \delta_{(p^2,n),p^2}\delta_{(p,r),p}\hat{c}(\frac{n}{p^2},\frac{r}{p},p^2y)+(p\delta_{(p,n),p}-1)\hat{c}(n,r,y)+p^2\hat{c}(np^2,rp,\frac{y}{p^2}), \\ \quad \text{if } r\in\mathbb{Z}. \end{cases}$$

Next let us consider the function

$$F_2(\tau,z)=\sum_{\mu=0}^{p-1}(F_1|_{k,1}(0,\mu,0))(\tau,z)=\sum_{n,r\in\mathbb{Z}}p\hat{c}_1(n,r,y)e^{2\pi in\tau}e^{2\pi irz}. \tag{49}$$

Corresponding to the three terms in $\hat{c}_1(n,r,y)$ let us define for $j=1,2,3$

$$\phi_j(\tau,z)=\sum_{n,r\in\mathbb{Z}}c_j^*(n,r,y)e^{2\pi in\tau}e^{2\pi irz}$$

where

$$\begin{aligned} c_1^*(n,r,y)&=p\delta_{(p^2,n),p^2}\delta_{(p,r),p}\hat{c}\left(\frac{n}{p^2},\frac{r}{p},p^2y\right),c_2^*(n,r,y) \\ &=p(p\delta_{(p,n),p}-1)\hat{c}(n,r,y),c_3^*(n,r,y)=p^3\hat{c}\left(np^2,rp,\frac{y}{p^2}\right). \end{aligned}$$

Now

$$\begin{aligned} (T_pF_f)(\tau,z)&=p^{k-4}\sum_{\lambda=0}^{p-1}(F_2|_{k,1}(\lambda,0,0))(\tau,z)=p^{k-4}\sum_{j=1}^{3}\sum_{\lambda=0}^{p-1}e^{2\pi i(\lambda^2\tau+2\lambda z)}\phi_j(\tau,z+\lambda\tau) \\ &=p^{k-4}\sum_{j=1}^{3}\sum_{N,R\in\mathbb{Z}}\sum_{\lambda=0}^{p-1}c_j^*(N-R\lambda+\lambda^2,R-2\lambda,y)e^{2\pi iN\tau}e^{2\pi iRz}. \end{aligned} \tag{50}$$

For $j=1,2,3$ we need to evaluate the term $\mu_j:=\sum_{\lambda=0}^{p-1}c_j^*(N-R\lambda+\lambda^2,R-2\lambda,y)$. We have

$$\mu_1=p^{\frac{3}{2}}c\left(\frac{4N-R^2}{p^2}\right)y^{\frac{1}{4}}W_{\mathrm{sgn}(N-\frac{R^2}{4})\frac{k-1/2}{2},\frac{il}{2}}\left(4\pi\left|N-\frac{R^2}{4}\right|y\right)e^{2\pi(N-\frac{R^2}{4})y}, \tag{51}$$

$$\mu_2=p^2\left(\frac{R^2-4N}{p}\right)c(4N-R^2)y^{\frac{1}{4}}W_{\mathrm{sgn}(N-\frac{R^2}{4})\frac{k-1/2}{2},\frac{il}{2}}\left(4\pi\left|N-\frac{R^2}{4}\right|y\right)e^{2\pi(N-\frac{R^2}{4})y}, \tag{52}$$

$$\mu_3=p^{\frac{7}{2}}c((4N-R^2)p^2)y^{\frac{1}{4}}W_{\mathrm{sgn}(N-\frac{R^2}{4})\frac{k-1/2}{2},\frac{il}{2}}\left(4\pi\left|N-\frac{R^2}{4}\right|y\right)e^{2\pi(N-\frac{R^2}{4})y}. \tag{53}$$

Substituting (51), (52) and (53) in (50) we get

$$(T_pF_f)(\tau,z)=\sum_{N,R\in\mathbb{Z}}p^{k-\frac{3}{2}}\Bigg(pc((4N-R^2)p^2)+p^{-\frac{1}{2}}\left(\frac{-(4N-R^2)}{p}\right)c(4N-R^2)$$

$$+ p^{-1}c\left(\frac{4N-R^2}{p^2}\right)\Big)$$

$$\times y^{\frac{1}{4}} W_{\mathrm{sgn}(N-\frac{R^2}{4})\frac{k-1/2}{2},\frac{il}{2}}\left(4\pi\left|N-\frac{R^2}{4}\right|y\right)e^{2\pi(N-\frac{R^2}{4})y}e^{2\pi i N\tau}e^{2\pi i Rz}$$

$$= p^{k-\frac{3}{2}}\lambda_p F_f. \tag{54}$$

We have used (15) to get the last step. This completes the proof of the theorem. □

7 Automorphic representation

The main purpose of this section is to show that the classical definition of Jacobi Maaß forms made in Definition 3.2 is compatible with the representation theory of the Jacobi group. For this, we first recall the correspondence between the automorphic representations $\tilde{\pi}$ of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ and π of $G^J(\mathbb{A})$. Let π^m_{SW} be the global Schrödinger-Weil representation of $G^J(\mathbb{A})$ as defined in [5, Sect. 7.2]. Then from [5, Sect. 7.3] the map

$$\tilde{\pi} \mapsto \pi := \tilde{\pi} \otimes \pi^m_{SW} \tag{55}$$

gives a 1–1 correspondence between (genuine) automorphic representations of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ and automorphic representations of $G^J(\mathbb{A})$ with central character ψ^m. Here ψ is a fixed character on $\mathbb{Q}\backslash\mathbb{A}$ and $\psi^m(x) = \psi(mx)$ for any $x \in \mathbb{A}$ and $m \in \mathbb{Q}$.

There is also a local correspondence similar to the one above. Namely, for any prime p (including ∞), if $\tilde{\pi}_p$ is a representation of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ then

$$\tilde{\pi}_p \mapsto \pi_p := \tilde{\pi} \otimes \pi^m_{SW,p} \tag{56}$$

gives a 1–1 correspondence between genuine representations of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ and representations of $G^J(\mathbb{Q}_p)$. These local and global correspondences are compatible.

We will now show that if $f \in S^+_{k-1/2}(4)$ and $F_f \in \hat{J}^{nh,cusp}_{k,1}$ is the corresponding Jacobi Maaß cusp form then the representations obtained from these forms obey (55). To do this let us first give some details on the local correspondence (56).

Proposition 7.1 Archimedean Case: *We have the following complete list of genuine unitarizable representations of the Lie algebra of* $\widetilde{\mathrm{SL}}_2(\mathbb{R})$.

(A) Principal series representation: *For $s \in \mathbb{C}\backslash\{\mathbb{Z}+1/2\}$, $\nu = \pm 1/2$ we have the representation $\tilde{\pi} = \tilde{\pi}_{s,\nu}$ with weights $2\mathbb{Z}+\nu+1/2$. The Laplace Beltrami operator $\Delta = -\frac{1}{4}(Z^2 + 2X_+X_- + 2X_-X_+)$ acts by the scalar $\frac{1}{4}(s^2-1)$.*
(B) Discrete series representation: *For an integer $k_0 \geq 1$, we have the two representations $\tilde{\pi} = \tilde{\pi}^{\pm}_{k_0-\frac{1}{2}}$ with weights $\pm(k_0 - \frac{1}{2} + l)$ for $l \in 2\mathbb{N}_0$. The Laplace Beltrami operator Δ acts by the scalar $\frac{1}{4}((k_0-\frac{3}{2})^2-1)$.*

Non-Archimedean Case: *The non-archimedean principal series representations of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ are obtained as follows: Let $m \in \mathbb{Q}_p^\times$ and χ be a character of $\mathbb{Q}_p^\times$ such that $\chi^2 \neq |\,|^{\pm 1}$. Then we have the representation $\tilde{\pi} = \tilde{\pi}_{\chi,-m}$ of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ induced from the character $\left(\left[\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right], \epsilon\right) \mapsto \epsilon\delta_{-m}(a)\chi(a)$ of the torus, where δ_{-m} is the Weil character defined using ψ^{-m} by the formula given in* [5, p. 26].

For details on this proposition refer to [5, Sects. 3.1, 3.2, 5.3]. For the proof of Theorem 7.5 below, we need the strong multiplicity one theorem [22, Theorem 3] for representations of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$. Following the notations of [22], let $\tilde{A}_{00}$ be the space of genuine cuspidal automorphic forms on $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ which are orthogonal to the space spanned by theta series corresponding to quadratic forms in one variable. Then we have

Theorem 7.2 *Let $\tilde{\pi}_1$ and $\tilde{\pi}_2$ be two genuine cuspidal automorphic representations of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ lying in $\tilde{A}_{00}$. Then $\tilde{\pi}_1 = \tilde{\pi}_2$ if and only if the following two conditions are satisfied*

1. *$\tilde{\pi}_{1,p} \simeq \tilde{\pi}_{2,p}$ for almost all primes p,*
2. *$\tilde{\pi}_1$ and $\tilde{\pi}_2$ have the same central character.*

For a nice description of the space $\tilde{A}_{00}$ and the space generated by theta series we refer the reader to [11]. In particular, one has the fact that the archimedean component of a representation corresponding to theta series can be only one of $\{\tilde{\pi}^{\pm}_{\frac{1}{2}}, \tilde{\pi}^{\pm}_{\frac{3}{2}}\}$.

Next we state the representations of the Jacobi group corresponding to the representations of $\widetilde{\mathrm{SL}}_2$ given in Proposition 7.1 under the correspondence (56).

Proposition 7.3 *We have the following correspondence of local representations.*

1. Archimedean principal series: *For any $m > 0$, $s \in \mathbb{C}\backslash\{\mathbb{Z}+1/2\}$, $\nu = \pm 1/2$, the principal series representation of the complexified Lie algebra of the Jacobi group is given by $\pi_{m,s,\nu} = \tilde{\pi}_{s,\nu} \otimes \pi^m_{SW,\infty}$. The Casimir operator $\mathcal{C}$ defined in (6) acts by the scalar $\frac{1}{2}(s^2-1)$.*
2. Archimedean discrete series representation: *For integers $k_0, m > 0$, the two discrete series representations of the complexified Lie algebra of the Jacobi group are given by $\pi^{\pm}_{m,k_0} = \tilde{\pi}^{\pm}_{k_0-\frac{1}{2}} \otimes \pi^m_{SW,\infty}$. The Casimir operator $\mathcal{C}$ acts by the scalar $\frac{1}{2}(k_0-1/2)(k_0-5/2) = \frac{1}{2}((k_0-\frac{3}{2})^2-1)$.*
3. Non-archimedean principal series: *Let $m \in \mathbb{Q}_p^\times$ and χ be a character of $\mathbb{Q}_p^\times$ such that $\chi^2 \neq ||^{\pm 1}$. The principal series representation of $G^J(\mathbb{Q}_p)$ given by $\pi_{\chi,m} = \tilde{\pi}_{\chi,-m} \otimes \pi^m_{SW,p}$ is the representation induced from the character $\left[\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right](0,0,\kappa) \mapsto \chi(a)\psi^m(\kappa)$. If $p \nmid 2m$, the generator $\hat{T}_p$ of the p-adic Hecke algebra (defined in* [5, Sect. 6.1]*) of the Jacobi group acts on the spherical vector of $\pi_{\chi,m}$ by the constant $p^{3/2}(\chi(p)+\chi(p)^{-1})$.*

For details on this proposition and precise formulas for the action of the Lie algebra on the representation space refer to [5, Sects. 3.1, 3.2, 5.4, 5.8, 6.4].

Now let us fix a Hecke eigenform $f \in S^+_{k-1/2}(4)$ (k even) such that for every odd prime p we have $T_{p^2} f = \lambda_p f$ and $\Delta_{k-1/2} f = \Lambda f$ with $\Lambda = \frac{1}{4}(s^2-1)$. Let $\tilde{\pi}_f = \otimes \tilde{\pi}_p$ be the irreducible cuspidal (genuine) automorphic representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ corresponding to f (see [21, p. 386]). The local representations $\tilde{\pi}_p$ ($p \neq 2$) are described in the next lemma.

Lemma 7.4 *Let $f \in S^+_{k-1/2}(4)$ be as given above. Then we have*

$$\tilde{\pi}_\infty = \begin{cases} \tilde{\pi}_{s,-\frac{1}{2}}, & \text{if } s \in \mathbb{C}\backslash\{\mathbb{Z}+1/2\}; \\ \tilde{\pi}^+_{k_0-\frac{1}{2}}, & \text{if } s = (k_0-\frac{3}{2}) \text{ with an integer } k_0 \geq 1 \text{ and } k > 0; \\ \tilde{\pi}^-_{k_0-\frac{1}{2}}, & \text{if } s = (k_0-\frac{3}{2}) \text{ with an integer } k_0 \geq 1 \text{ and } k < 0. \end{cases} \tag{57}$$

For an odd prime p we have

$$\tilde{\pi}_p = \tilde{\pi}_{\chi,-1} \text{ with } \chi(p) + \chi(p)^{-1} = \lambda_p. \tag{58}$$

Proof The case of the archimedean representation is clear. To deduce the non-archimedean representation one needs to use the non-holomorphic analogue of the calculation in Lemmas 3, 4 and Proposition 10 of [21]. □

Let $F_f \in \hat{J}_{k,1}^{nh,cusp}$ be the Jacobi Maaß cusp form corresponding to f constructed in (28). Then, from Theorem 4.4, we have $\mathcal{C}^{k,1}F_f = 2\Lambda = \frac{1}{2}(s^2-1)$ and, from Theorem 6.1, we have $T_pF_f = \mu_pF_f$, where $\mu_p = p^{k-3/2}\lambda_p$, for every odd prime p. Let π_F be the irreducible cuspidal automorphic representation of $G^J(\mathbb{A})$ corresponding to F_f. To construct π_F, one first lifts F_f to a function ϕ_F on $G^J(\mathbb{A})$ using the decomposition $G^J(\mathbb{Z})G^J(\mathbb{R})\prod_{p<\infty}G^J(\mathbb{Z}_p)$ as follows. If $g = \gamma g_\infty k_0 \in G^J(\mathbb{A})$, with $\gamma \in G^J(\mathbb{Q})$, $g_\infty \in G^J(\mathbb{R})$ and $k_0 \in \prod_{p<\infty}G^J(\mathbb{Z}_p)$, then set

$$\Phi_F(g) := (F_f|_{k,m}g_\infty)(i,0) = j_{k,m}^{nh}(g_\infty,(i,0))F_f(g_\infty(i,0)).$$

Then π_F is obtained as the space of all right translates of ϕ_F and the group $G^J(\mathbb{A})$ acts by right translation. See [5, Chapter 7] for details.

Theorem 7.5 *With notations as above, we have*

$$\pi_F = \tilde{\pi}_f \otimes \pi_{SW}^1.$$

Proof Let $\tilde{\pi}'$ be the irreducible cuspidal automorphic representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ such that $\pi_F = \tilde{\pi}' \otimes \pi_{SW}^1$. Such a representation exists by (55). We will use Theorem 7.2 to show that $\tilde{\pi}_f = \tilde{\pi}'$. Let $\pi_F = \otimes\pi_{F,p}$ and $\tilde{\pi}' = \otimes\tilde{\pi}'_p$.

The archimedean representation $\pi_{F,\infty}$ is given by

$$\pi_{F,\infty} = \begin{cases} \pi_{1,s,-\frac{1}{2}}, & \text{if } s \in \mathbb{C}\backslash\{\mathbb{Z}+1/2\}; \\ \pi_{1,k_0}^{+}, & \text{if } s = (k_0 - \frac{3}{2}) \text{ with an integer } k_0 \geq 1 \text{ and } k > 0; \\ \pi_{1,k_0}^{-}, & \text{if } s = (k_0 - \frac{3}{2}) \text{ with an integer } k_0 \geq 1 \text{ and } k < 0. \end{cases} \tag{59}$$

For an odd prime p, we know that the local representation is unramified and hence $\pi_{F,p} = \pi_{\hat{\chi},1}$ for an unramified character $\hat{\chi}$ of $\mathbb{Q}_p^\times$. In Theorem 6.4.6 of [5], it is shown that the p-adic Hecke operator $\hat{T}_p$ acts on a spherical vector in $\pi_{\hat{\chi},1}$ by the scalar $p^{3/2}(\hat{\chi}(p)+\hat{\chi}^{-1}(p))$. Using the relation $T_p = p^{k-3}\hat{T}_p$ from Proposition 6.6.5 of [5] and the fact that the classical Hecke eigenvalue of F_f is $\mu_p = p^{k-3/2}\lambda_p$ we get $\hat{\chi}(p)+\hat{\chi}^{-1}(p) = \lambda_p \Rightarrow \hat{\chi}(p) = \chi(p)^{\pm 1}$, where χ is the unramified character obtained in Lemma 7.4. Hence for an odd prime p we get

$$\pi_{F,p} = \pi_{\chi,1} \quad \text{with } \chi(p) + \chi(p)^{-1} = \lambda_p. \tag{60}$$

Using Proposition 7.3 and Lemma 7.4, we see that for every prime $p \neq 2$ (including ∞) we have

$$\tilde{\pi}'_p = \tilde{\pi}_p.$$

If $\omega_{\tilde{\pi}_p}$ and $\omega_{\tilde{\pi}'_p}$ are central characters of $\tilde{\pi}_p$ and $\tilde{\pi}'_p$ respectively, then we have $\omega_{\tilde{\pi}_p} = \omega_{\tilde{\pi}'_p}$ for every prime $p \neq 2$ (including ∞). We claim that this forces the central characters for

$p=2$ to be equal as well. For any prime p, the center of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ is given by $\{(\pm\mathbf{1}_p, \pm 1)\}$, where $\mathbf{1}_p$ is the identity element in $\mathrm{SL}_2(\mathbb{Q}_p)$. Since the representations we are considering here are genuine representations, the central character is completely determined by its value on $(-\mathbf{1}_p, 1)$. Now consider the element $(-\mathbf{1}, 1) = \otimes(-\mathbf{1}_p, 1) \in \widetilde{\mathrm{SL}}_2(\mathbb{Q})$. Since the representations $\tilde{\pi}$ and $\tilde{\pi}'$ are automorphic, we get

$$1 = \omega_{\tilde{\pi}_f}((-\mathbf{1}, 1)) = \prod_{p\le\infty} \omega_{\tilde{\pi}_p}((-\mathbf{1}_p, 1)) \quad \Rightarrow \quad \omega_{\tilde{\pi}_2}((-\mathbf{1}_2, 1)) = \prod_{\substack{p\le\infty \\ p\neq 2}} \omega_{\tilde{\pi}_p}((-\mathbf{1}_p, 1))^{-1}$$

$$\text{and} \quad 1 = \omega_{\tilde{\pi}'}((-\mathbf{1}, 1)) = \prod_{p\le\infty} \omega_{\tilde{\pi}'_p}((-\mathbf{1}_p, 1))$$

$$\Rightarrow \quad \omega_{\tilde{\pi}'_2}((-\mathbf{1}_2, 1)) = \prod_{\substack{p\le\infty \\ p\neq 2}} \omega_{\tilde{\pi}'_p}((-\mathbf{1}_p, 1))^{-1}$$

which gives us the claim. From [15] and Lemma 7.4, we see that both $\tilde{\pi}_\infty$ and $\tilde{\pi}'_\infty$ are not one of $\{\tilde{\pi}^{\pm}_{\frac{1}{2}}, \tilde{\pi}^{\pm}_{\frac{3}{2}}\}$. Hence $\tilde{\pi}$ and $\tilde{\pi}'$ belong to $\tilde{A}_{00}$. Now the theorem follows from the strong multiplicity one Theorem 7.2 for representations of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$. □

Note that for the case when f corresponds to a holomorphic half integer weight form, the above theorem is essentially done in [5]. For the case of holomorphic and skew-holomorphic Jacobi forms also see [3], [4] for the archimedean representation. Theorem 7.5 indeed confirms that the classical definition of Jacobi Maaß form given in Definition 3.2 is compatible with the representation theory of the Jacobi group. In that sense, it is the correct notion of Jacobi Maaß forms.

8 Concluding remarks

1. To the best of our knowledge, the first attempt to define Jacobi Maaß forms is in [5, Chapter 4]. If one considers Definition 4.1.8 in [5] of automorphic forms on the group $G^J(\mathbb{R})$ and pulls it back to functions on $\mathcal{H}\times\mathbb{C}$ via the non-holomorphic automorphy factor $j^{nh}_{k,m}$ defined in (2) we get exactly the Definition 3.2. However, in Definition 4.1.7 of [5], the authors define Jacobi Maaß forms via a pull-back with the holomorphic automorphy factor and consider eigenfunctions of the degree 2 differential operator $\Delta^{k,m}$ (a slight modification of the operator $\Delta_1^{k,m}$ defined in Sect. 3). There has been some work on the Jacobi forms as per Definition 4.1.7 of [5], for example, see [23] (no examples are constructed here). Recently, in [7], Jacobi Maaß forms over complex quadratic fields are defined using higher dimensional analogue of $\Delta^{k,m}$ and examples in the form of Eisenstein series are constructed. It would be interesting to see what one would obtain if the higher dimensional analogue of the operator $\mathcal{C}^{k,m}$ is used instead.
2. The representation theory of the Jacobi group further tells us why Definition 3.2 of Jacobi Maaß forms is the appropriate one. Suppose, one defines a Jacobi form $\tilde{F}$ using some other automorphy factor and differential operator and assume that $\tilde{F}$ is a Hecke eigenform. Then looking at the automorphic representation of $G^J(\mathbb{A})$ obtained from $\tilde{F}$, one can extract a vector corresponding to a Jacobi Maaß Hecke eigen-form F (according to Definition 3.2) which has the same eigenvalue as $\tilde{F}$ for almost all primes p.

3. In [2], Arakawa has defined holomorphic Jacobi forms for odd weight and index 1 using holomorphic half integer weight forms of weight $k-1/2$ (k odd) in the Kohnen plus space. Arakawa obtains Jacobi forms not with respect to the full discrete group Γ^J but the subgroup $\Gamma_0(4)\ltimes H(\mathbb{Z})$. One should be able to obtain Jacobi Maaß forms of odd weight and index 1 with respect to $\Gamma_0(4)\ltimes H(\mathbb{Z})$ in a similar fashion from functions in $M^+_{k-1/2}(4)$ with k odd.
4. One can speculate that it is possible to construct a certain class of non-holomorphic Siegel modular forms using the Jacobi Maaß forms of index 1 and suitable non-holomorphic index raising operators V_l^{nh}. We expect that these would give us representations of $\mathrm{GSp}_4(\mathbb{A})$ that are not cuspidal but residual.
5. Following the methods in [5], one should be able to study the L-functions associated to Jacobi Maaß forms. It would be nice to see that these L-functions have analytic continuation and functional equation. (For the holomorphic case see [9]) These L-functions can be used to prove an appropriate converse theorem in the non-holomorphic case. (For the holomorphic case see [17])
6. It would be interesting to obtain a multiplicity one theorem for Jacobi Maaß forms similar to Corollary 7.5.6 in [5]. We have not worked out the details of this problem.

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