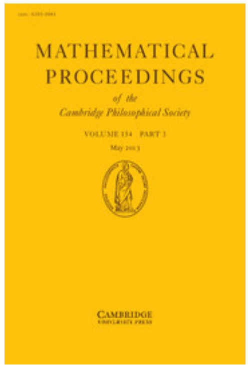

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Special values of L-functions for Saito–Kurokawa lifts

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Abstract

In this paper we obtain special value results for L-functions associated to classical and paramodular Saito–Kurokawa lifts. In particular, we consider standard L-functions associated to Saito–Kurokawa lifts as well as degree eight L-functions obtained by twisting with an automorphic form defined on GL(2). The results are obtained by combining classical and representation theoretic arguments.

1. *Introduction*

In this paper, we obtain algebraicity results for special values of L-functions associated to Saito–Kurokawa lifts with level and their twists. Starting from an elliptic cusp form $f \in S_{2k-2}(\Gamma_0(N))$ one is able to construct unique, up to scalars, Siegel cusp forms $F_f \in S_k(\Gamma_0^{(2)}(N))$ and $F_f^{\mathrm{para}} \in S_k(\Gamma^{\mathrm{para}}(N))$. These are called Saito–Kurokawa lifts of f. Here, $\Gamma_0^{(2)}(N)$ is the Siegel congruence subgroup and $\Gamma^{\mathrm{para}}(N)$ is the paramodular congruence subgroup of level N. We refer the reader to [**1**, **10**] for a classical treatment of the construction, and [**27**, **28**] for the construction using techniques of representation theory.

In [**6**], Deligne has conjectured algebraicity results for special values of motivic L-functions. If we assume the existence of motives corresponding to Siegel modular forms, then one can try to prove results in the spirit of Deligne's conjecture. If we let S be the set of primes p dividing the level N of the newform f, then the partial L-functions of the Saito–Kurokawa lifts are given

$$L^S(s, F_f, spin) = L^S(s, f)\zeta^S(s + 1/2)\zeta^S(s - 1/2),$$
$$L^S(s, F_f^{\mathrm{para}}, Std) = \zeta^S(s)L^S(s + 1/2, f)L^S(s - 1/2, f).$$

To obtain special value results we need information on all the local factors of the L-function, not just at the primes $p \nmid N$. In Section 2·4, we obtain the local L-factors at the primes $p|N$

explicitly. An example of a result obtained is that the finite part of the spin L-function is given by $L(s, f)\zeta(s + 1/2)\zeta(s - 1/2)\prod_{p\in S'} \zeta_p(s - 1/2)^{-1}$, where S' is a suitable subset of S. Now using the well-known results (see [**23**]) on the $\mathrm{Aut}(\mathbb{C})$-equivariance of local L-factors of GL_n, one sees that the algebraicity results for $L(s, F_f, spin)$ reduces to that of $L(s, f)$. Let us make two comments here. Firstly, the classically defined spinor Euler product associated to Siegel cusp forms with respect to $\Gamma_0^{(2)}(N)$ given in [**2**] does not give the correct L-function at the primes $p \mid N$, hence, we do need the representation theoretic methods of Section 2·4 to obtain the completed L-function. Secondly, one could completely avoid computing the local L-factors at the bad places and instead proceed as follows – obtain algebraicity results for partial L-functions and show the $\mathrm{Aut}(\mathbb{C})$-equivariance of local L-factors corresponding to *any* local representation of GSp_4. For the latter, one could use the results of [**9**] on local Langlands conjecture for GSp(4) and then show that the Langlands parameters are $\mathrm{Aut}(\mathbb{C})$-equivariant. This would be a separate project in itself and, hence, we do not take this approach.

If F is a Siegel cusp form of degree 2 and g is an elliptic cusp form of weight k and l, respectively, then the following is known regarding the special values of the degree 8 L-function associated to F twisted by g. In [**5**], the authors obtain special value results for both F and g full level. In the case that $l < 2k - 2$, the result is obtained under the assumption that F is a Saito–Kurokawa lift. The methods in [**5**] are classical. In [**20–22**, **26**], the authors obtain special value results for F, either full level or a newform with respect to the Borel congruence subgroup of square-free level, g of any level and nebentypus but with $k = l$. The methods used in these papers are representation theoretic. Regarding the special values of the standard L-function of a Siegel cusp form F twisted by a Dirichlet character, we refer the reader to [**4**, **12**, **18** and **33**], where results are obtained for Siegel cusp forms of genus n with respect to Siegel congruence subgroup of arbitrary levels.

In Theorems 5·1 and 5·2, we obtain special value results for the degree 8 L-function $L(s, F, g)$ where F is the Saito–Kurokawa lift F_f or F_f^{para} for $f \in S_{2k-2}^{\mathrm{new}}(\Gamma_0(N))$ and $g \in S_l^{\mathrm{new}}(\Gamma_0(N), \psi')$ with $l \neq 2k - 2$. There are two main ingredients in the proof of these theorems. The first one is an algebraicity result for the ratio of Petersson norms of f and F_f or F_f^{para}, which is obtained in Corollaries 4·2 and 4·7 using classical methods. The second one is obtaining the L-function, analyzing the special value of local L-functions and breaking the L-function into smaller pieces. This step uses representation theoretic methods. In Theorem 5·3, we obtain special value results for the degree 5 standard L-function of the Saito–Kurokawa lifts twisted by any Dirichlet character.

Let us remark that the results of this paper cannot be obtained using just the classical methods in [**5**] or the representation theoretic methods in [**20–22**, **26**]. It is the combination of the two methods as mentioned above that allows us to prove the results. Also, we would like to point out that this is the first time any special value result is obtained for Siegel modular forms with respect to the paramodular congruence subgroup.

2. *Saito–Kurokawa lifts*

In this section we set the notation and briefly review the Siegel modular forms we will be interested in, namely, Saito–Kurokawa lifts of level $\Gamma_0^{(2)}(N)$ and $\Gamma^{\mathrm{para}}(m)$. We will also give the L-functions for the automorphic representations corresponding to these lifts.

2·1. *Notation*

Let $k \geqslant 2$ be an integer and N a positive integer. Let $f \in S_{2k-2}^{\mathrm{new}}(N)$ be an elliptic new form of weight $2k-2$ and level $\Gamma_0(N)$. Given a ring R, we write $S_{2k-2}(N; R)$ to denote those cuspforms with Fourier coefficients in R. We normalize the Petersson norm of f so it is given by

$$\langle f, f\rangle := \int_{\Gamma_0(N)\backslash\mathfrak{h}_1} |f(z)|^2 y^{k-2} dx dy$$

where, $\mathfrak{h}_1 := \{z = x + iy \in \mathbb{C} : y > 0\}$ is the complex upper half plane. For more details on the classical theory one can consult [**7**] or [**17**].

Let

$$\mathrm{GSp}_4 := \{g \in \mathrm{GL}_4 : {}^t g J g = \mu(g) J, \mu(g) \in \mathrm{GL}_1\}, \text{ with } J = \begin{bmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{bmatrix}.$$

We have $\mathrm{Sp}_4 := \{g \in \mathrm{GSp}_4 : \mu(g) = 1\}$. The group $\mathrm{Sp}_4(\mathbb{R})$ acts on the Siegel upper half space $\mathfrak{h}_2 := \{Z \in \mathrm{Mat}(2, \mathbb{C}) : {}^t Z = Z, \mathrm{Im}(Z) > 0\}$ by

$$g\langle Z\rangle := (AZ + B)(CZ + D)^{-1}, \text{ where } Z \in \mathfrak{h}_2, g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}_4(\mathbb{R}).$$

For any positive integer N, define the Siegel congruence subgroup by

$$\Gamma_0^{(2)}(N) := \left\{\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}_4(\mathbb{Z}) : C \equiv 0 \pmod N\right\}.$$

For $m \geqslant 1$ we define the paramodular group to be the subgroup of $\mathrm{Sp}_4(\mathbb{Q})$ defined by

$$\Gamma^{\mathrm{para}}(m) = \left\{ g = \begin{bmatrix} * & m* & * & * \\ * & * & * & m^{-1}* \\ * & m* & * & * \\ m* & m* & m* & * \end{bmatrix} \in \mathrm{Sp}_4(\mathbb{Q}) \right\}$$

where the $*$'s denote integers.

For any positive integer k and $\Gamma = \Gamma_0^{(2)}(N)$ or $\Gamma^{\mathrm{para}}(m)$, let $S_k(\Gamma)$ denote the space of holomorphic functions $F : \mathfrak{h}_2 \to \mathbb{C}$, which satisfy

$$F(\gamma\langle Z\rangle) = \det(CZ + D)^k F(Z), \text{ for any } Z \in \mathfrak{h}_2, \gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma.$$

Let us normalize the Petersson norm as follows

$$\langle F, F\rangle := \int_{\Gamma\backslash\mathfrak{h}_2} |F(Z)|^2 \det(Y)^{k-3} dX dY$$

where $Z = X + iY$.

2·2. *Saito–Kurokawa lifts with level* $\Gamma_0^{(2)}(N)$

We will define Saito–Kurokawa lifts with level $\Gamma_0^{(2)}(N)$. For more details one can consult [**1**]. If N is odd and square-free, the space of Saito–Kurokawa lifts corresponding to f is one dimensional. However, when N is no longer required to be square-free the space of Saito–Kurokawa lifts corresponding to f is no longer necessarily a one dimensional space. In the case of general odd N, we will describe the choice of a distinguished element of the space of Saito–Kurokawa lifts corresponding to f.

Let $f \in S_{2k-2}^{\mathrm{new}}(N)$ be a newform with eigenvalues $\lambda_f(p)$ for p a prime. Let ε_p be the eigenvalue of f under the Atkin-Lehner operators W_{p^e} with $p^e \parallel N$, i.e, $p^e | N$ but $p^{e+1} \nmid N$. Let n_0 be an integer satisfying

$$\left(\frac{(-1)^{k-1} n_0}{p^e}\right) = \varepsilon_p$$

for all p with $p^e \parallel N$. Set

$$\begin{aligned} S_{k-1/2}^{+}(N; f, n_0) = \{g \in S_{k-1/2}^{+}(4N) : T(p)g = \lambda_f(p)g \text{ for all primes } p; \text{ the } n\text{th} \\ \text{Fourier coefficients of } g \text{ at infinity vanish unless } nn_0 \\ \text{is a square mod } N\}. \end{aligned}$$

We have the following theorem of Kohnen.

THEOREM 2·1. *([**14**]) Let $f \in S_{2k-2}^{\mathrm{new}}(N)$ be a newform and assume $\varepsilon_p = 1$ for those p so that e is even for $p^e \parallel N$. Let n_0 be as above. Then*

$$\dim_{\mathbb{C}} S_{k-1/2}^{+}(N; f, n_0) = 1.$$

As we are interested in algebraicity results, the following result is important for our considerations.

PROPOSITION 2·2 ([**32**, propostion 2·3·1]). *Let $f \in S_{2k-2}(N; \mathcal{O})$ be a Hecke eigenform. Then there is a nonzero complex number $\Omega \in \mathbb{C}^{\times}$ so that*

$$\theta^{\mathrm{alg}}(f) := \frac{1}{\Omega}\theta(f) \in S_{k-1/2}^{+}(4N; \mathcal{O}).$$

When constructing the Saito–Kurokawa lift of f, we will always use this algebraic Shintani lift.

One has the following result relating half-integral weight forms to Jacobi forms.

THEOREM 2·3 ([**16**, corollary 3]). *The spaces $S_{k-1/2}^{+}(4N)$ and $J_{k,1}^{\mathrm{cusp}}(\Gamma_0^J(N))$ are isomorphic as Hecke modules.*

Combining these results we see that there is a one dimensional subspace $J_{k,1}^{\mathrm{cusp}}(N; f, n_0)$ of $J_{k,1}^{\mathrm{cusp}}(\Gamma_0^J(N))$ corresponding to $S_{k-1/2}^{+}(N; f, n_0)$ so that if $\phi \in J_{k,1}^{\mathrm{cusp}}(N, f, n_0)$, then $T_J(p)\phi = \lambda_f(p)\phi$ for all primes p.

Finally, using work of Ibukiyama ([**13**]) one constructs a Siegel modular form $F_f \in S_k(\Gamma_0^{(2)}(N))$ associated to ϕ. Combining these we have the following result.

THEOREM 2·4 ([**1**, theorems 6·8, 8·4]). *Let $k \geqslant 2$ be an even integer and let $N \geqslant 1$ be an odd integer or square-free. Let $f \in S_{2k-2}^{\mathrm{new}}(N; \mathcal{O})$ be a newform. Then there exists a nonzero cuspidal Siegel eigenform $F_f \in S_k(\Gamma_0^{(2)}(N); \mathcal{O})$ satisfying for $p \nmid N$*

$$\begin{aligned} T_S(p)F_f &= (\lambda_f(p) + (p^{k-1} + p^{k-2}))F_f \\ T_S'(p)F_f &= ((p^{k-2} + p^{k-1})\lambda_f(p) + (2p^{2k-3} + p^{2k-4}))F_f \end{aligned}$$

where

$$\begin{aligned} T_S(p) &= T(\mathrm{diag}(1, 1, p, p)), \\ T_S'(p) &= pT(\mathrm{diag}(1, p, p^2, p)) + p(1 + p + p^2)T(\mathrm{diag}(p, p, p, p)) \end{aligned}$$

with the $T(g)$ the usual Hecke operators. Moreover, if N is square-free the space of $F \in S_k(\Gamma_0^{(2)}(N))$ having the same $T_S(p), T'_S(p)$-eigenvalues as F_f for all $p \nmid N$ is one-dimensional.

2·3. *Saito–Kurokawa lifts with level $\Gamma^{\mathrm{para}}(m)$*

Let $k \geqslant 2$ and $m \geqslant 1$ be integers. Let $S^{\mathrm{new},-}_{2k-2}(m)$ denote the subspace of $S^{\mathrm{new}}_{2k-2}(m)$ such that the sign of the functional equation of the associated L-function is -1. It is shown in [**31**] that this space is isomorphic to the space $J^{\mathrm{cusp,new}}_{k,m}(\mathrm{SL}_2(\mathbb{Z})^J)$. In particular, we see that given a newform $f \in S^{\mathrm{new},-}_{2k-2}(m)$, the space

$$J^{\mathrm{cusp,new}}_{k,m}(\mathrm{SL}_2(\mathbb{Z})^J;\, f) = \{\phi \in J^{\mathrm{cusp}}_{k,m}(\mathrm{SL}_2(\mathbb{Z})^J) : T_J(p)\phi = a_p\phi \ \forall p \nmid m\}$$

is one dimensional. Moreover, one has the following result implicit in [**11**, section II·4].

PROPOSITION 2·5. *Let $f \in S^{\mathrm{new},-}_{2k-2}(m; \mathcal{O})$ be a Hecke eigenform. Then there is a nonzero complex number $\Omega \in \mathbb{C}^\times$ so that*

$$\phi^{\mathrm{alg}}(f) := \frac{1}{\Omega}\phi(f) \in J^{\mathrm{cusp,new}}_{k,m}(\mathrm{SL}_2(\mathbb{Z});\, f;\, \mathcal{O}).$$

We will always use this normalization when constructing the paramodular Saito–Kurokawa lift.

Let $\phi \in J^{\mathrm{cusp,new}}_{k,m}(\mathrm{SL}_2(\mathbb{Z})^J)$. In [**10**] Gritsenko constructs a form:

$$G_\phi \in S_k(\Gamma^{\mathrm{para}}(m))$$

by setting

$$G_\phi(\tau, z, w) = \sum_{N=1}^{\infty} V_N\phi(\tau, z)e(Nmw)$$

where V_N is the index shifting operator

$$V_N\phi(\tau, z) = N^{k-1} \sum_{\substack{\gamma \in \mathrm{Mat}(2,\mathbb{Z}) \\ \det(\gamma)=N}} j(\gamma, \tau)^{-k} e\left(\frac{-mNcz^2}{c\tau + d}\right)\phi\left(\gamma\langle\tau\rangle, \frac{Nz}{c\tau + d}\right)$$

that sends a form $\phi \in J^{\mathrm{cusp}}_{k,m}(\mathrm{SL}_2(\mathbb{Z})^J)$ to a form in $J^{\mathrm{cusp}}_{k,mN}(\mathrm{SL}_2(\mathbb{Z}))$. Here, j is the usual GL_2 automorphy factor. One can check that the action of V_N on the Fourier expansion of a Jacobi form is given by

$$V_N\phi(\tau, z) = \sum_{\substack{D<0, r\in\mathbb{Z} \\ D\equiv r^2 (\mathrm{mod}\, 4mN)}} \left(\sum_{\substack{d \mid \gcd(r,N) \\ D\equiv r^2 (\mathrm{mod}\, 4mNd)}} d^{k-1} c\left(\frac{D}{d^2}, \frac{r}{d}\right)\right) e\left(\frac{r^2 - D}{4mN}\tau + rz\right)$$

where

$$\phi(\tau, z) = \sum_{\substack{D<0 \\ r\in\mathbb{Z} \\ D\equiv r^2 (\mathrm{mod}\, 4m)}} c(D, r)e\left(\frac{r^2 - D}{4m}\tau + rz\right).$$

If we combine the two liftings we obtain the paramodular Saito–Kurokawa lifting of f. In particular, for $f \in S^{\mathrm{new},-}_{2k-2}(\Gamma_0(m))$, we denote a paramodular Saito–Kurokawa lifting of f by F^{para}_f. Combining Proposition 2·5 and the explicit formula for the Fourier coefficients in Gritsenko's lifting, we see that if $f \in S^{\mathrm{new},-}_{2k-2}(m; \mathcal{O})$, then F^{para}_f also has Fourier coefficients in $\mathcal{O}$. One should note that in the case that $m = 1$ the Saito–Kurokawa lift and the paramodular Saito–Kurokawa lift agree and the condition that k be even is equivalent to the condition that the sign of the functional equation is -1. In the case that $m > 1$ one obtains distinct liftings. Let us also note that, using Proposition 2·2, Proposition 2·5, Theorem 2·4 and the construction of the paramodular Saito–Kurokawa lift above, we get

$$\left(F_f\right)^\sigma = F_{f^\sigma}, \qquad \left(F_f^{\mathrm{para}}\right)^\sigma = F_{f^\sigma}^{\mathrm{para}} \qquad \text{for all } \sigma \in \mathrm{Aut}(\mathbb{C}).$$

2·4. *L-functions of Saito–Kurokawa lifts and their twists*

In this section, we will describe the L-function associated to the Saito–Kurokawa lifts. All L-functions are normalized so that their completion satisfy (conjecturally) a functional equation with respect to $s \mapsto 1 - s$. The L-functions of the Siegel modular forms and their twists are defined to be those of the corresponding cuspidal automorphic representations. We refer the reader to [**3**] for details on the construction of cuspidal automorphic representations corresponding to Siegel cusp forms.

Let $f \in S_{2k-2}^{\mathrm{new}}(N)$. Let $\pi_f = \otimes \pi_p$ be the corresponding cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of $\mathbb{Q}$. We will briefly describe the representation theoretic construction of Saito–Kurokawa lifts given in [**27**]. The Saito–Kurokawa representations corresponding to π_f are automorphic representations $\Pi = \otimes \Pi_p$ of $\mathrm{GSp}_4(\mathbb{A})$ associated to the pair (π_f, S), where:

(i) S is a subset of $\{p \text{ prime } : p \mid N\} \cup \{\infty\}$;
(ii) the condition $(-1)^{\#S} = \varepsilon(1/2, \pi_f)$ is satisfied.

Such a representation is cuspidal if, in addition, we have either $L(1/2, \pi_f) = 0$ or $S \neq \emptyset$. Given such a set S, let π_S be the constituent of the global induced representation $|\cdot|^{1/2} \times |\cdot|^{-1/2}$ of $\mathrm{PGL}_2(\mathbb{A})$ which has the Steinberg representation $\mathrm{St}_{\mathrm{GL}_2}$ at the places $p \in S$ and the trivial representation $\mathbf{1}_{\mathrm{GL}_2}$ at the places $p \notin S$ as its local components. Note that, in the archimedean case, the Steinberg representation refers to the lowest discrete series representation of PGL_2 with lowest weight vector of weight 2. Then $\mathrm{SK}(\pi_f, S)$, the Saito–Kurokawa representation attached to the pair (π_f, S), is a Langlands functorial lifting of the representation $\pi_f \otimes \pi_S$ of $\mathrm{PGL}_2(\mathbb{A}) \times \mathrm{PGL}_2(\mathbb{A})$ to $\mathrm{PGSp}_4(\mathbb{A})$ under the morphism of L-groups

$$\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{Sp}_4(\mathbb{C}),$$

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \right) \longmapsto \begin{bmatrix} a & & b & \\ & a' & & b' \\ c & & d & \\ & c' & & d' \end{bmatrix}.$$

The relation between the representation theoretic construction of Saito–Kurokawa lifts above and the classical constructions from Sect. 2·2 and 2·3 are given in the following lemma.

LEMMA 2·6.
(i) *If N is odd, square-free and k is even, let F_f be the classical Saito–Kurokawa lift obtained in Section 2·2. If one chooses $S = \{\infty\} \cup \{p \mid N : \epsilon_p = -1\}$, then $\mathrm{SK}(\pi_f, S)$ is the cuspidal automorphic representation of* $\mathrm{GSp}_4(\mathbb{A})$ *corresponding to F_f.*
(ii) *If the sign in the functional equation of $L(s, f)$ is -1, let F_f^{para} be the classical Saito–Kurokawa lift obtained in Section 2·3. If one chooses $S = \{\infty\}$, then $\mathrm{SK}(\pi_f, S)$ is the cuspidal automorphic representation corresponding of* $\mathrm{GSp}_4(\mathbb{A})$ *to F_f^{para}.*

Proof. The proof follows essentially from [**28**, theorem 5·2], [**24**, theorem 6·1] and [**25**, theorem 5·5·9].

Let us remark that, for Lemma 2·6(i), we assume that N is square-free since the information on Siegel new-vectors for arbitrary local representations of GSp_4 is not known at the present time. Hence, if N is not necessarily square-free, then it is not clear how to make the choice of S so that $\mathrm{SK}(\pi_f, S)$ is the cuspidal automorphic representation corresponding to F_f. For Lemma 2.6(ii) above there is no restriction on the level N.

Let $g \in S_l^{\mathrm{new}}(N, \psi')$ and let $\tau_g = \otimes\tau_p$ be the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ corresponding to g. A calculation shows that $L(s, \mathrm{St}_{\mathrm{GL}_2} \otimes \tau_p) = L(s+1/2, \tau_p)$ and $L(s, \mathbf{1}_{\mathrm{GL}_2} \otimes \tau_p) = L(s+1/2, \tau_p)L(s-1/2, \tau_p)$. Hence, we get the following expression for the L-function of the twist of $\mathrm{SK}(\pi_f, S)$ with τ_g.

$$L(s, \mathrm{SK}(\pi_f, S) \times \tau_g) = L(s, \pi_f \times \tau_g)L(s+1/2, \tau_g)L(s-1/2, \tau_g) \times \prod_{\substack{p\in S \\ p<\infty}} \frac{1}{L(s-1/2, \tau_p)}. \tag{1}$$

Using similar arguments, we see that the degree 5 L-function of $\mathrm{SK}(\pi_f, S)$ twisted by a Dirichlet character ψ is given by

$$L^{\mathrm{St}}(s, \mathrm{SK}(\pi_f, S), \psi) = L(s, \psi)L(s+1/2, \psi\pi_f)L(s-1/2, \psi\pi_f) \times \prod_{\substack{p\in S \\ p<\infty}} \frac{1}{L(s-1/2, \psi_p\pi_p)}. \tag{2}$$

3. *Special values of* $\mathrm{GL}_2 \times \mathrm{GL}_1$ *and* $\mathrm{GL}_2 \times \mathrm{GL}_2$ *L-functions*

In this section, we will collect the relevant information on the special values of certain L-functions which will be useful for our main result. We need global special value results on the GL_1, $\mathrm{GL}_2 \times \mathrm{GL}_1$ and $\mathrm{GL}_2 \times \mathrm{GL}_2$ L-functions and local results for the primes $p \in S$.

Let $f \in S_{2k-2}(N)$ and $g \in S_l(N, \psi')$, with Fourier expansions

$$f(z) = \sum_{n>0} a_n e^{2\pi i n z}, \qquad g(z) = \sum_{n>0} b_n e^{2\pi i n z}.$$

We will assume that both f and g are primitive forms – Hecke eigenform, $a_1 = b_1 = 1$ and cannot be obtained from the forms of lower level. Let ψ be a primitive Dirichlet character modulo $N' \mid N$. Let K_ψ, K_f and K_g be the number fields generated over $\mathbb{Q}$ by the values $\psi(n)$, the Fourier coefficients a_n and the Fourier coefficients b_n, respectively. Any automorphism σ of $\mathbb{C}$ acts on modular forms by applying σ to the Fourier coefficients. Let us denote the action of σ on f by f^σ. Then f^σ is a primitive form in $S_{2k-2}(N)$ and g^σ is a primitive form in $S_l(N, \psi'^\sigma)$, where $\psi'^\sigma(t) = \sigma(\psi'(t))$. Let π_f and τ_g be the cuspidal, automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ obtained from f and g respectively. There is an action of σ on the automorphic representations as well and we have $\pi_{f^\sigma} = (\pi_f)^\sigma = \otimes\pi_p^\sigma$. First, we have the following local result.

LEMMA 3·1. *Let π_p be an irreducible, admissible representation of* $\mathrm{GL}_2(\mathbb{Q}_p)$. *Let $m \in \mathbb{Z} \cup (1/2+\mathbb{Z})$. If $m \in \mathbb{Z}$ let $K = \mathbb{Q}(\sqrt{p})$, and if $m \in 1/2+\mathbb{Z}$ let $K = \mathbb{Q}$. Then, for any automorphism $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$, we have*

$$\sigma\left(L(m, \tau_p)\right) = L(m, \tau_p^\sigma).$$

Proof. The proof is exactly as the proof of [**23**, proposition 3·17].

Let us define the following Dirichlet series.

$$D(s, f, \psi) := \sum_{n=1}^{\infty} \frac{a_n \psi(n)}{n^s}, \qquad D(s, g) := \sum_{n=1}^{\infty} \frac{b_n}{n^s},$$

$$D(s, f, g) := L(2s + 4 - 2k - l, \psi') \sum_{n=1}^{\infty} \frac{a_n b_n}{n^s}, \text{ with } L(s, \psi') = \sum_{n=1}^{\infty} \frac{\psi'(n)}{n^s}.$$

Then we obtain

$$D(s, f, \psi) = \prod_{p \nmid N'} L_p\left(s - \frac{2k-3}{2}, \psi_p \pi_p\right), \quad D(s, g) = L\left(s - \frac{l-1}{2}, \tau_g\right),$$

$$D(s, f, g) = L\left(s - k - \frac{l}{2} + 2, \pi_f \times \tau_g\right).$$

The last equality follows from [**29**, lemma 1].

LEMMA 3·2. *Let the notation be as above. Let $g(\psi)$, $g(\psi')$ be the Gauss sums associated to the characters ψ, ψ'.*

(i) *Set*

$$A(m, f, \psi) := \frac{L(m, \psi\pi_f)L(m+1, \psi\pi_f)}{(2\pi i)^{2m+2k-2} g(\psi)^2 i^{3-2k} \langle f, f \rangle}$$

Then, for $m \in (1/2 + \mathbb{Z}) \bigcap [(-2k+5)/2, (2k-5)/2]$, we have $\sigma(A(m, f, \psi)) = A(m, f^\sigma, \psi^\sigma)$ for any $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. In particular, $A(m, f, \psi) \in K_f K_\psi$.

(ii) *Set*

$$B(m, g) := \frac{L(m, \tau_g)L(m+1, \tau_g)}{(2\pi i)^{2m+l} i^{1-l} g(\psi') \langle g, g \rangle}$$

Then, for $m \in [(-l+3)/2, (l-3)/2] \bigcap ((l+1)/2)\mathbb{Z}$, we have

$$\sigma(B(m, g)) = B(m, g^\sigma) \text{ for any } \sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}). \text{ In particular, } B(m, g) \in K_g.$$

(iii) *Set*

$$C(m, f, g) := \begin{cases} \dfrac{L(m, \pi_f \times \tau_g)}{(2\pi i)^{2m+2k-3} g(\psi') i^{3-2k} \langle f, f \rangle} & \text{if } l < 2k-2, \\ \dfrac{L(m, \pi_f \times \tau_g)}{(2\pi i)^{2m+l-1} g(\psi') i^{1-l} \langle g, g \rangle} & \text{if } l > 2k-2. \end{cases}$$

If $l < 2k-2$, then let $m \in [l/2 - k + 2, k - l/2 - 1] \bigcap (l/2)\mathbb{Z}$, and, if $l > 2k-2$, then let $m \in [k - l/2, l/2 - k + 1] \bigcap (l/2)\mathbb{Z}$. Then, we have $\sigma(C(m, f, g)) = C(m, f^\sigma, g^\sigma)$ for any $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. In particular, $C(m, f, g) \in K_f K_g$.

(iv) *For positive integers m satisfying $\psi(-1) = (-1)^m$, we have that*

$$\sigma\left(\frac{L(m, \psi)}{(2\pi i)^m}\right) = \frac{L(m, \psi^\sigma)}{(2\pi i)^m} \text{ for any } \sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}). \text{ In particular, } \frac{L(m, \psi)}{(2\pi i)^m} \in K_\psi.$$

Proof. [**30**, theorem 1] states the special value result for the Dirichlet series $D(s, f, \psi)$ and $D(s, g)$. This gives us parts (i) and (ii) above. For part (i), we also have to use Lemma 3·1 to obtain information at the primes $p \mid N'$. [**30**, theorem 4] states the special value result for the Dirichlet series $D(s, f, g)$, which gives part (iii) of the lemma. Finally, for part (iv), the result is given in [**19**, VII.2].

4. *Ratios of Petersson norms*

In this section, we will obtain algebraicity results for the ratio of the Petersson norms of f and its Saito–Kurokawa lifts.

4·1. *Ratios of Petersson norms: the level $\Gamma_0^{(2)}(N)$ case*

In [**1**, corollary 6·15], the following relation between the Petersson norms of F_f and f has been obtained.

THEOREM 4·1. *Let $f \in S^{\text{new}}_{2k-2}(N)$, with k even and N odd. Assume that $\varepsilon_p = 1$ for all p so that e is even for $p^e \parallel N$. Let $F_f \in S_k(\Gamma_0^{(2)}(N))$ be the Saito–Kurokawa lift of f. Let D be a fundamental discriminant with $D < 0$, $\gcd(N, D) = 1$ and $a_{\theta^{\text{alg}}(f)}(|D|) \neq 0$. Then,*

$$\frac{\langle F_f, F_f\rangle}{\langle f, f\rangle} = C_{k,N}\frac{|a_{\theta^{\text{alg}}(f)}(|D|)|^2 D(k, f)}{\pi |D|^{k-\frac{3}{2}} D(k-1, f, \chi_D)}, \tag{3}$$

where $C_{k,N}$ is an explicitly determined rational number and χ_D is the quadratic character corresponding to the quadratic field $\mathbb{Q}(\sqrt{D})$.

Note that [**1**, corollary 4·12] is stated for the particular case of N odd and square-free, but the entire argument used in the proof is given for general odd N and is only specialized to make the formula cleaner.

We now get the following corollary from (3), [**30**, theorem 1] and the fact that the action of $\text{Aut}(\mathbb{C})$ commutes with the correspondence that associates a half integral weight modular form to f as stated in Proposition 2·2.

COROLLARY 4·2. *Let f, F_f be as in Theorem 4·1. Then,*

$$\sigma\left(\frac{\langle F_f, F_f\rangle}{\langle f, f\rangle}\right) = \frac{\langle F_{f^\sigma}, F_{f^\sigma}\rangle}{\langle f^\sigma, f^\sigma\rangle} \text{ for any } \sigma \in \text{Aut}(\mathbb{C}). \text{ In particular, } \frac{\langle F_f, F_f\rangle}{\langle f, f\rangle} \in K_f. \tag{4}$$

4·2. *Ratio of Petersson norms: the $\Gamma^{\text{para}}(m)$ case*

Let $k \geqslant 2$ and $m \geqslant 1$ be integers and let $f \in S^{\text{new}}_{2k-2}(m)$ be a newform. Let $\phi^{\text{alg}} \in J^{\text{cusp,new}}_{k,m}(\text{SL}_2(\mathbb{Z})^J)$ be a lifting of f as in [**31**]. We drop the "alg" to ease notation in this section. We apply [**11**, corollary 1] to conclude the following result.

LEMMA 4·3. *Let f and ϕ be as above. Let D be a negative fundamental discriminant so that $\gcd(D, m) = 1$ and D is a square modulo $4m$. Then we have*

$$\frac{|a_\phi(D)|^2}{\langle \phi, \phi\rangle} = \frac{(k-2)!|D|^{k-3/2}}{2^{2k-3}m^{k-2}\pi^{k-1}}\frac{D(k-1, f, \chi_D)}{\langle f, f\rangle}$$

where $a_\phi(D)$ is the Dth Fourier coefficient of ϕ.

We now turn our attention to relating $\langle \phi, \phi\rangle$ and $\langle F_f^{\text{para}}, F_f^{\text{para}}\rangle$. Set $\Gamma_{\mathbb{Z}}^{\text{para}}(m) = \text{Sp}_4(\mathbb{Z}) \cap \Gamma^{\text{para}}(m)$ and Γ_∞ to be the parabolic subgroup of $\text{Sp}_4(\mathbb{Z})$ given by

$$\Gamma_\infty = \left\{\begin{bmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}\right\}.$$

Given a matrix $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ in block form, write g_1 for a. Define an Eisenstein series $E(Z, s)$ by

$$E(Z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_{\mathbb{Z}}^{\mathrm{para}}(m)} \left(\frac{\det \mathrm{Im}(\gamma\langle Z\rangle)}{\mathrm{Im}(\gamma\langle Z\rangle_1)} \right)^s .$$

We can relate this Eisenstein series to Epstein zeta functions as follows. For $Z = X + iY \in \mathfrak{h}_2$, define a positive definite quadratic form P_Z by

$$P_Z = \begin{pmatrix} Y & 0 \\ 0 & Y^{-1} \end{pmatrix} \left[\begin{pmatrix} 1_2 & 0_2 \\ X & 1_2 \end{pmatrix} \right]$$

where we define $A[B] = {}^tBAB$. One can then show, see [**10**], that

$$\begin{aligned} \pi^{-s}\Gamma(s)L(2s, \chi_m)E(Z, s) &= \pi^{-s}\Gamma(s) \sum_{\substack{A = {}^t(a_1, a_2, a_3, a_4) \in \mathbb{Z}^4 \\ a_1, a_2, a_3 \equiv 0 (\mathrm{mod}\, m) \\ \gcd(a_4, m) = 1}} P_Z[A]^{-s} \\ &= m^{-2s} \sum_{\substack{B = (0,0,0,b_4) \\ b_4 \in \frac{1}{m}\mathbb{Z}/\mathbb{Z} \\ \gcd(mb_4, m) = 1}} \zeta^*(s, B, 0, P_Z) \end{aligned}$$

where

$$\zeta(s, g, h, P_Z) = \sum_{\substack{A \in \mathbb{Z}^4 \\ A + g \neq 0}} \exp(2\pi i\, {}^tAh) P_Z[A + g]^{-s}$$

and $\zeta^*(s, g, h, P_Z) = \pi^{-s}\Gamma(s)\zeta(s, g, h, P_Z)$. It is known that $\zeta^*(s, g, h, P_Z)$ has meromorphic continuation to $\mathbb{C}$ with only a simple pole at $s = 2$ with residue 1 if h is integral and a simple pole at $s = 0$ with residue -1 if g is integral. It also has a functional equation, but we will not need this. In particular, since the Eisenstein series is a sum over these functions, we do not get a nice functional equation for the Eisenstein series. However, if we set $E^*(Z, s) = \pi^{-s}\Gamma(s)L(2s, \chi_m)E(Z, s)$, we do get that the Eisenstein series $E^*(Z, s)$ has a meromorphic continuation to $\mathbb{C}$. In our case, g cannot be integral so the only pole is at $s = 2$ and has residue $\varphi(m)m^{-4}$ where φ denotes Euler's phi function.

Following [**15**] we consider the Rankin–Selberg convolution of two cuspidal Siegel eigenforms F and G of weight k and paramodular level $\Gamma_{\mathbb{Z}}^{\mathrm{para}}(m)$. In particular, consider

$$\int_{\Gamma_{\mathbb{Z}}^{\mathrm{para}}(m)\backslash \mathfrak{h}_2} F(Z)\overline{G(Z)}E(Z, s)\det(Y)^{k-2} dX dY.$$

One now unfolds the integral and follows the same method as in [**15**] to obtain

$$\int_{\Gamma_{\mathbb{Z}}^{\mathrm{para}}(m)\backslash \mathfrak{h}_2} F(Z)\overline{G(Z)}E(Z, s)\det(Y)^{k-2} dX dY = \frac{\Gamma(s + k - 2)}{(4\pi m)^{s+k-2}} \sum_{N \geqslant 1} \langle \phi_N, \psi_N \rangle N^{-(s+k-2)}$$

where the ϕ_N are the Fourier Jacobi coefficients of F and the ψ_N likewise for G. One should note that an analogous formula is given in [**10**, lemma 2] in a more restrictive setting, but the same argument works in this case. As in [**15**], define

$$D_{F,G}(s) = L(2s - 2k + 4, \chi_m) \sum_{N \geqslant 1} \langle \phi_N, \psi_N \rangle N^{-s}$$

and

$$D^*_{F,G}(s) = (2\sqrt{m}\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)D_{F,G}(s).$$

Our equation above can be written as

$$\int_{\Gamma^{\mathrm{para}}_{\mathbb{Z}}(m)\backslash\mathfrak{h}_2} F(Z)\overline{G(Z)}E^*(Z,s)\det(Y)^{k-2}dXdY = \pi^{k-2}D^*_{F,G}(s+k-2). \tag{5}$$

Using the meromorphic continuation of $E^*(Z,s)$ we obtain a meromorphic continuation of $D^*_{F,G}(s)$ as well. However, unlike the case studied in [**15**] we do not obtain a functional equation for $D^*_{F,G}(s)$ via this formula as we do not have a functional equation for $E^*(Z,s)$. Taking the residue at $s=2$ of each side of equation (5) gives

$$\frac{\varphi(m)\langle F,G\rangle}{m^4} = \left(\frac{(k-1)!}{(4m)^k\pi^{k+2}}\right)\operatorname*{res}_{s=k} D_{F,G}(s).$$

We now specialize to the case that $F=G=F^{\mathrm{para}}_f$. Let $\phi\in J^{\mathrm{cusp,new}}_{k,m}(\mathrm{SL}_2(\mathbb{Z})^J)$ be the lifting of f as in [**31**]. Then we have

$$D_{F^{\mathrm{para}}_f,F^{\mathrm{para}}_f}(s) = L(2s-2k+4,\chi_m)\sum_{N\geqslant 1}\langle V_N\phi, V_N\phi\rangle N^{-s},$$

and so we see that we need to study $\langle V_N\phi, V_N\phi\rangle = \langle V_N^*V_N\phi,\phi\rangle$ where V_N^* is the adjoint of V_N with respect to the Petersson product. A thorough analysis of this has been done in [**15**] in the case that $m=1$. We now follow these arguments for general m.

Let $\psi\in J^{\mathrm{cusp}}_{k,mN}(\mathrm{SL}_2(\mathbb{Z})^J)$. For $c\in\mathbb{C}$, set $\psi_c(\tau,z)=\psi(\tau,cz)$. It is shown in [**15**] that

$$V_N^*\psi = \sum_{X\,(\mathrm{mod}\,N\mathbb{Z}^2)}\ \sum_{A\in\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{Mat}(2,\mathbb{Z})_N}\psi_{\sqrt{N}^{-1}}|_{k,mN}A|_{k,mN}X$$

in the case of index 1. However, the exact same argument given there translates to the case of general index and gives the formula listed. Using the matrices $\left(\begin{smallmatrix} a & b\\ 0 & d\end{smallmatrix}\right)$ with $ad=N$ and b running modulo d for representatives of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{Mat}(2,\mathbb{Z})_N$ one can calculate

$$\begin{aligned}
V_N^*\psi(\tau,z) &= N^{k/2-3}\sum_{\lambda,\mu\,(\mathrm{mod}\,N)}\ \sum_{\substack{ad=N\\ b\,(\mathrm{mod}\,d)}}\left(\frac{d}{\sqrt{N}}\right)^{-k}\psi\left(\frac{a\tau+b}{d},\frac{z+\lambda\tau+\mu}{d}\right)e(m\lambda^2\tau+2\lambda mz)\\
&= N^{k-3}\sum_{\lambda,\mu\,(\mathrm{mod}\,N)}\ \sum_{\substack{ad=N\\ b\,(\mathrm{mod}\,d)}} d^{-k}\sum_{\substack{D<0,r\in\mathbb{Z}\\ D\equiv r^2\,(\mathrm{mod}\,4Nm)}} c(D,r)e\left(\left(\frac{r^2-D}{4Nm}\frac{a}{d}+\frac{r\lambda}{d}+m\lambda^2\right)\tau\right)\\
&\quad\cdot e\left(\left(\frac{r}{d}+2\lambda m\right)z+\left(\frac{r^2-D}{4Nm}\frac{b}{d}+\frac{r\mu}{d}\right)\right).
\end{aligned}$$

We have that the sum

$$\sum_{\substack{\mu\,(\mathrm{mod}\,N)\\ b\,(\mathrm{mod}\,d)}} e\left(\frac{r^2-D}{4Nm}\frac{b}{d}+\frac{r\mu}{d}\right)$$

is Nd if $d \mid r^2 - D/4Nm$ and $d \mid r$ and 0 otherwise. Using this, we make a change of variables to obtain

$$\begin{aligned} V_N^*\psi(\tau,z) &= N^{k-2} \sum_{\lambda \pmod N} \sum_{ad=N} d^{1-k} \sum_{\substack{D<0, r\in\mathbb{Z} \\ D\equiv r^2 \pmod{4Nm/d}}} c(d^2D, dr) \\ &\quad \times e\left(\left(\frac{r^2 - 4mr\lambda + 4m^2\lambda^2}{4m} - \frac{D}{4m}\right)\tau + (r+2\lambda m)z\right) \\ &= N^{k-2} \sum_{\lambda \pmod N} \sum_{ad=N} d^{1-k} \sum_{\substack{D<0, r\in\mathbb{Z} \\ D\equiv r^2 \pmod{4Nm/d}}} c(d^2D, dr) \\ &\quad \times e\left(\left(\frac{(r+2\lambda m)^2 - D}{4m}\right)\tau + (r+2\lambda m)z\right). \end{aligned}$$

Another change of variables gives

$$\begin{aligned} V_N^*\psi(\tau,z) = N^{k-2} \sum_{\lambda \pmod N} \sum_{ad=n} d^{1-k} & \sum_{\substack{D<0, r\in\mathbb{Z} \\ D\equiv (r-2\lambda m)^2 \pmod{4Nm/d}}} \\ & \times c(d^2D, d(r-2\lambda m)) e\left(\left(\frac{r^2-D}{4m}\right)\tau + rz\right). \end{aligned}$$

Set $\lambda = s + Ns' \pmod N)/d$ with s running over $\mathbb{Z}/\frac{N}{d}\mathbb{Z}$ and s' running over $\mathbb{Z}/d\mathbb{Z}$. We have that $2\lambda m \equiv 2ms + 2Ns'm \pmod{2Nm}/d$ and so $d(r-2\lambda m) \equiv d(r-2ms) \pmod{2Nm}$. Similarly, we have $D \equiv (r-2ms)^2 \pmod{4Nm/d}$. We know that the coefficients $c(D,r)$ depend only on the pair (D,r) with $r \pmod{2Nm}$ and $D \equiv r^2 \pmod{4Nm}$. Thus, we have

$$\begin{aligned} V_N^*\psi(\tau,z) = N^{k-2} \sum_{d\mid N} d^{2-k} \sum_{s \pmod{N/d}} & \sum_{\substack{D<0, r\in\mathbb{Z} \\ D\equiv (r-2ms)^2 \pmod{4Nm/d}}} \\ & \times c(d^2D, d(r-2ms)) e\left(\left(\frac{r^2-D}{4m}\right)\tau + rz\right). \end{aligned}$$

Finally, making a change of variables replacing d by N/d and $r-2ms$ by s, we obtain

$$V_N^*\psi(\tau,z) = \sum_{\substack{D<0, r\in\mathbb{Z} \\ D\equiv r^2 \pmod{4m}}} \left(\sum_{d\mid N} d^{k-2} \sum_{s\in S(r,m,d,D)} c\left(\frac{N^2}{d^2}D, \frac{N}{d}s\right)\right) e\left(\left(\frac{r^2-D}{4m}\right)\tau + rz\right),$$

where $S(r,m,d,D) := \{s \pmod{2md} : s \equiv r \pmod{2m}, s^2 \equiv D \pmod{4md}\}$. This reduces to the calculation given in [**15**] if one sets $m = 1$. In particular, we have shown the first part of the following proposition.

PROPOSITION 4·4. *Let $V_N^* : J_{k,mN}^{\mathrm{cusp}} \to J_{k,m}^{\mathrm{cusp}}$ be the adjoint of V_N with respect to the Petersson product as above. Then we have*:

(i) *Let $\psi \in J_{k,mN}^{\mathrm{cusp}}$ with*

$$\psi(\tau,z) = \sum_{\substack{D<0, r\in\mathbb{Z} \\ D^2\equiv r^2 \pmod{4mN}}} c_\psi(D,r) e\left(\left(\frac{r^2-D}{4mN}\right)\tau + rz\right).$$

The action of V_N^* on the Fourier coefficients is given by

$$V_N^*\psi(\tau,z) = \sum_{\substack{D<0, r\in\mathbb{Z}\\ D\equiv r^2 (mod\, 4m)}} \left(\sum_{d|N} d^{k-2} \sum_{s\in S(r,m,d,D)} c_\psi\left(\frac{N^2}{d^2}D, \frac{N}{d}s\right)\right) \times e\left(\left(\frac{r^2-D}{4m}\right)\tau + rz\right).$$

(ii) *The map $V_N^*V_N : J_{k,m}^{\mathrm{cusp}} \to J_{k,m}^{\mathrm{cusp}}$ is given by*

$$V_N^*V_N = \sum_{d|N} \varsigma(d) d^{k-2} T\left(\frac{N}{d}\right)$$

with $T(n)$ the nth Hecke operator on $J_{k,m}^{\mathrm{cusp}}$ and ς the arithmetical function given by

$$\varsigma(d) = d\prod_{p|d}\left(1+\frac{1}{p}\right).$$

Proof. The first part has already been shown. In the case of $m = 1$, the second part is given in [**15**] and left as an exercise to the reader. They point out that it is enough to check it for Fourier coefficients indexed by fundamental discriminants. We include the calculation for $N = p$ an odd prime for the convenience of the reader.

Let $\phi \in J_{k,m}^{\mathrm{cusp}}$ with

$$\phi(\tau,z) = \sum_{\substack{D<0, r\in\mathbb{Z}\\ D\equiv r^2 (\mathrm{mod}\, 4m)}} c_\phi(D,r) e\left(\left(\frac{r^2-D}{4m}\right)\tau + rz\right).$$

Write $\psi = V_p\phi \in J_{k,mp}^{\mathrm{cusp}}$ with

$$\psi(\tau,z) = \sum_{\substack{D<0, r\in\mathbb{Z}\\ D\equiv r^2 (\mathrm{mod}\, 4mp)}} c_\psi(D,r) e\left(\left(\frac{r^2-D}{4mp}\right)\tau + rz\right).$$

Finally, write $\varphi = V_p^*V_p\phi \in J_{k,m}^{\mathrm{cusp}}$ with

$$\varphi(\tau,z) = \sum_{\substack{D<0, r\in\mathbb{Z}\\ D\equiv r^2 (\mathrm{mod}\, 4m)}} c_\varphi(D,r) e\left(\left(\frac{r^2-D}{4m}\right)\tau + rz\right).$$

We have

$$\sum_{t|p} \varsigma(t) t^{k-2} T\left(\frac{p}{t}\right) = T(p) + p^{k-2}(p+1).$$

Thus, on Fourier coefficients we have that

$$\sum_{t|p} \varsigma(t) t^{k-2} T\left(\frac{p}{t}\right)\phi(\tau,z) = \sum_{\substack{D<0, r\in\mathbb{Z}\\ D\equiv r^2 (\mathrm{mod}\, 4m)}} (c(p^2D, pr) + p^{k-2}(p+1+\chi_D(p))c(D,r)) \times e\left(\left(\frac{r^2-D}{4m}\right)\tau + rz\right).$$

We have

$$c_\psi(D,r) = \sum_{\substack{d \mid \gcd(r,p) \\ D \equiv r^2 (\mathrm{mod}\, 4mpd)}} d^{k-1} c_\phi\left(\frac{D}{d^2}, \frac{r}{d}\right).$$

We break into cases. If $p \nmid r$, then $\gcd(r, p) = 1$ and so we obtain

$$\begin{aligned} c_\psi(D,r) &= \sum_{D \equiv r^2 (\mathrm{mod}\, 4mp)} c_\phi(D,r) \\ &= c_\phi(D,r) \end{aligned}$$

since the condition $D \equiv r^2 (\mathrm{mod}\, 4mp)$ is already contained in the sum over r and D. If $p \mid r$, then $\gcd(r, p) = p$ and so we obtain

$$\begin{aligned} c_\psi(D,r) &= \sum_{D \equiv r^2 (\mathrm{mod}\, 4mp)} c_\phi(D,r) + \sum_{D \equiv r^2 (\mathrm{mod}\, 4mp^2)} p^{k-1} c_\phi\left(\frac{D}{p^2}, \frac{r}{p}\right) \\ &= c_\phi(D,r) + p^{k-1} c_\phi\left(\frac{D}{p^2}, \frac{r}{p}\right) \end{aligned}$$

where we again use that r and D must already satisfy $D \equiv r^2 (\mathrm{mod}\, 4mp)$ and that $c_\phi\left(D/p^2, r/p\right) = 0$ unless $p^2 \mid D$ and $p \mid r$, so the condition $D \equiv r^2 (\mathrm{mod}\, 4p^2)$ is already accounted for in the notation. Thus, we obtain

$$c_\psi(D,r) = \begin{cases} 0 & D \not\equiv r^2 (\mathrm{mod}\, 4mp) \\ c_\phi(D,r) + p^{k-1} c_\phi\left(\frac{D}{p^2}, \frac{r}{p}\right) & \text{otherwise,} \end{cases}$$

i.e., we have

$$\psi(\tau, z) = \sum_{\substack{D<0, r \in \mathbb{Z} \\ D \equiv r^2 (\mathrm{mod}\, 4mp)}} \left(c_\phi(D,r) + p^{k-1} c_\phi\left(\frac{D}{p^2}, \frac{r}{p}\right)\right) e\left(\left(\frac{r^2 - D}{4mp}\right)\tau + rz\right).$$

We have

$$\begin{aligned} \varphi(\tau, z) &= \sum_{\substack{D<0, r \in \mathbb{Z} \\ D \equiv r^2 (\mathrm{mod}\, 4m)}} \left(\sum_{d \mid p} d^{k-2} \sum_{s \in S(r,m,d,D)} c_\psi\left(\frac{p^2}{d^2} D, \frac{p}{d} s\right)\right) e\left(\left(\frac{r^2 - D}{4m}\right)\tau + rz\right) \\ &= \sum_{\substack{D<0, r \in \mathbb{Z} \\ D \equiv r^2 (\mathrm{mod}\, 4m)}} \left(\sum_{s \in S(r,m,1,D)} c_\psi(p^2 D, ps) + p^{k-2} \sum_{s \in S(r,m,p,D)} c_\psi(D,s)\right) \\ &\quad \times e\left(\left(\frac{r^2 - D}{4m}\right)\tau + rz\right). \end{aligned}$$

Observe that

$$\begin{aligned} \sum_{s \in S(r,m,1,D)} c_\psi(p^2 D, ps) = c_\psi(p^2 D, pr) &= c_\phi(p^2 D, pr) + p^{k-1} c_\phi(D,r) \\ &= c_\phi(p^2 D, pr) + p^{k-2} p c_\phi(D,r). \end{aligned}$$

Similarly, we have

$$\begin{aligned} p^{k-2}\sum_{s\in S(r,m,p,D)} c_{\psi}(D,s) &= p^{k-2}\sum_{s\in S(r,m,p,D)}\left(c_{\phi}(D,s)+p^{k-1}c_{\phi}\left(\frac{D}{p^2},\frac{s}{p}\right)\right)\\ &= p^{k-2}c_{\phi}(D,r)\sum_{s\in S(r,m,p,D)}1+\sum_{s\in S(r,m,p,D)}p^{-1}c_{\phi}\left(\frac{D}{p^2},\frac{s}{p}\right)\\ &= p^{k-2}(1+\chi_D(p))c_{\phi}(D,r)+\sum_{s\in S(r,m,p,D)}p^{-1}c_{\phi}\left(\frac{D}{p^2},\frac{s}{p}\right)\end{aligned}$$

where we have used that D is a square modulo $4m$ by assumption, so the first sum is just counting if D is a square modulo p or not. Now we have that $c_{\phi}(D/p^2, s/p) = 0$ unless $p^2 \mid D$ and $p \mid s$. However, since we have assumed D is a fundamental discriminant, we cannot have the square of an odd prime dividing D and so this term vanishes. Thus, we have

$$\begin{aligned} V_p^*V_p\phi(\tau,z) = &\sum_{\substack{D<0,r\in\mathbb{Z}\\ D\equiv r^2 (\mathrm{mod}\, 4m)}}(c(p^2D,pr)+p^{k-2}(p+1+\chi_D(p))c(D,r))\\ &\times e\left(\left(\frac{r^2-D}{4m}\right)\tau+rz\right),\end{aligned}$$

as claimed.

We now return to $D_{F_f^{\text{para}},F_f^{\text{para}}}(s)$. We have

$$\begin{aligned} D_{F_f^{\text{para}},F_f^{\text{para}}}(s) &= L(2s-2k+4,\chi_m)\sum_{N\geqslant 1}\langle V_N\phi, V_N\phi\rangle N^{-s}\\ &= L(2s-2k+4,\chi_m)\sum_{N\geqslant 1}\langle V_N^*V_N\phi,\phi\rangle N^{-s}\\ &= \langle\phi,\phi\rangle L(2s-2k+4,\chi_m)\sum_{N\geqslant 1}\left(\sum_{d\mid N}\varsigma(d)d^{k-2}\lambda_f\left(\frac{N}{d}\right)\right)N^{-s}\\ &= \langle\phi,\phi\rangle L(2s-2k+4,\chi_m)\left(\frac{\zeta(s-k+1)\zeta(s-k+2)}{\zeta(2s-2k+4)}\right)D(s,f)\end{aligned}$$

where we have used that

$$\sum_{N\geqslant 1}\varsigma(N)N^{-s} = \frac{\zeta(s-1)\zeta(s)}{\zeta(2s)}.$$

Thus, we have

$$\operatorname*{res}_{s=k} D_{F_f^{\text{para}},F_f^{\text{para}}}(s) = \langle\phi,\phi\rangle\frac{\zeta(2)L(4,\chi_m)}{\zeta(4)}D(k,f).$$

Combining this with our previous calculations gives the following result.

THEOREM 4·5. *Let $\phi \in J_{k,m}^{\text{cusp,new}}(\mathrm{SL}_2(\mathbb{Z})^J)$ be the lifting of f as in* [**31**] *with F_f^{para} the paramodular Saito–Kurokawa lift of f. Then we have*

$$\frac{\langle F_f^{\text{para}}, F_f^{\text{para}}\rangle}{\langle\phi,\phi\rangle} = \frac{3\cdot 5\cdot (k-1)!L(4,\chi_m)D(k,f)}{2^{2k}\cdot m^{k-4}(m-1)\pi^{k+4}}.$$

We can combine this result with Lemma 4·3 to obtain the following result.

COROLLARY 4·6. *Let* $f \in S^{\mathrm{new}}_{2k-2}(\Gamma_0(m))$ *be a newform. Let* $\phi \in J^{\mathrm{cusp,new}}_{k,m}(\mathrm{SL}_2(\mathbb{Z})^J)$ *be a lifting of* f *as in* [**31**]. *Let* D *be a negative fundamental discriminant so that* $m \nmid D$ *and* D *is a square modulo* $4m$. *Let* $F^{\mathrm{para}}_f \in S_k(\Gamma^{\mathrm{para}}(m))$ *be the paramodular Saito–Kurokawa lift of* f. *Then we have*

$$\frac{\langle F^{\mathrm{para}}_f, F^{\mathrm{para}}_f\rangle}{\langle f, f\rangle} = \mathcal{C}_{k,m}\frac{|a_\phi(D)|^2 L(4, \chi_m) D(k, f)}{\pi^5 |D|^{k-3/2} D(k-1, f, \chi_D)}$$

where

$$\mathcal{C}_{k,m} = \frac{3\cdot 5\cdot (k-1)m^2(m+1)}{2^3\cdot (m-1)}$$

and $a_\phi(D)$ *is the* D*th Fourier coefficient of* ϕ.

From this along with Proposition 2·5 we obtain the following algebraicity result.

COROLLARY 4·7. *Let notations be as above. Then we have*

$$\sigma\left(\frac{\langle F^{\mathrm{para}}_f, F^{\mathrm{para}}_f\rangle}{\langle f, f\rangle}\right) = \frac{\langle F^{\mathrm{para}}_{f^\sigma}, F^{\mathrm{para}}_{f^\sigma}\rangle}{\langle f^\sigma, f^\sigma\rangle} \text{ for all } \sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}). \text{ In particular, } \frac{\langle F^{\mathrm{para}}_f, F^{\mathrm{para}}_f\rangle}{\langle f, f\rangle} \in K_f.$$

5. *Special values of* L*-functions*

In this section, we will obtain special value results for L-functions of Saito–Kurokawa lifts and their twists in the spirit of Deligne's conjecture. In the $\Gamma_0^{(2)}(N)$ case, we will assume that N is odd and square-free and, in the $\Gamma^{\mathrm{para}}(m)$ case, m will be arbitrary.

5·1. *Special values: the level* $\Gamma_0^{(2)}(N)$ *case*

Let $f \in S^{\mathrm{new}}_{2k-2}(\Gamma_0(N))$ be a primitve form, with k even, N odd and square-free. Let $F_f \in S^{\mathrm{new}}_k(\Gamma_0^{(2)}(N))$ be the Saito–Kurokawa lift of f obtained in Section 2·2. Let S be the subset of $\{p \text{ prime} : p \mid N\} \cup \{\infty\}$ such that $\mathrm{SK}(\pi_f, S)$ is the cuspidal, automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ corresponding to F_f. Let $g \in S^{\mathrm{new}}_l(\Gamma_0(N), \psi')$ be a primitive form. The critical points for the L-function $L(s, F_f, g)$ are given by:

$$B_{k,l} := \begin{cases} [k-\frac{l}{2}, -k+\frac{l}{2}+1] \cap \frac{l}{2}\mathbb{Z} & \text{if } l > 2k-2; \\ [-k+\frac{l}{2}+2, k-\frac{l}{2}-1] \cap \frac{l}{2}\mathbb{Z} & \text{if } k \leqslant l < 2k-2; \\ [-\frac{l}{2}+2, \frac{l}{2}-1] \cap \frac{l}{2}\mathbb{Z} & \text{if } l \leqslant k. \end{cases} \tag{6}$$

If $l = 2k-2$, then there are no critical points. Set

$$K_l := \begin{cases} \mathbb{Q} & \text{if } l \text{ is even;} \\ \mathbb{Q}\left(\prod_{\substack{p\in S\\ p<\infty}} p^{\frac{1}{2}}\right) & \text{if } l \text{ is odd.} \end{cases}$$

THEOREM 5·1. *Let the notations be as above. Set*:

$$\Theta(m, F_f, g) := \begin{cases} \dfrac{L(m, F_f, g)}{(2\pi i)^{4m+2k+l-4} g(\psi')^2 i^{4-l-2k}\langle F_f, F_f\rangle\langle g, g\rangle} & \text{if } l < 2k-2; \\ \dfrac{L(m, F_f, g)}{(2\pi i)^{4m+2l-2} g(\psi')^2 i^{2-2l}\langle g, g\rangle^2} & \text{if } l > 2k-2. \end{cases}$$

Then, for $m \in B_{k,l}$ and $\sigma \in \mathrm{Aut}(\mathbb{C}/K_l)$, we have

$$\sigma(\Theta(m, F_f, g)) = \Theta(m, F_{f^\sigma}, g^\sigma).$$

In particular,

$$\Theta(m, F_f, g) \in K_f K_g K_l.$$

Proof. We have $L(s, F_f, g) = L(s, \mathrm{SK}(\pi_f, S) \times \tau_g)$. Hence, from (1), we see that $\Theta(m, F_f, g)$ is equal to

$$\frac{L(m, \pi_f \times \tau_g)}{(2\pi i)^{2m+2k-3} g(\psi')i^{3-2k}\langle f, f\rangle} \frac{L(m-\frac{1}{2}, \tau_g)L(m+\frac{1}{2}, \tau_g)}{(2\pi i)^{2m+l-1} i^{1-l} g(\psi')\langle g, g\rangle} \frac{\langle f, f\rangle}{\langle F_f, F_f\rangle}$$
$$\times \prod_{\substack{p\in S\\ p<\infty}} \frac{1}{L(s-1/2, \tau_p)} \quad \text{if } l < 2k-2;$$

$$\frac{L(m, \pi_f \times \tau_g)}{(2\pi i)^{2m+l-1} g(\psi')i^{1-l}\langle g, g\rangle} \frac{L(m-\frac{1}{2}, \tau_g)L(m+\frac{1}{2}, \tau_g)}{(2\pi i)^{2m+l-1} i^{1-l} g(\psi')\langle g, g\rangle} \prod_{\substack{p\in S\\ p<\infty}} \frac{1}{L(s-1/2, \tau_p)} \quad \text{if } l > 2k-2.$$

Hence,

$$\Theta(m, F_f, g) = \begin{cases} C(m, f, g) B(m-\frac{1}{2}, g) \dfrac{\langle f, f\rangle}{\langle F_f, F_f\rangle} \displaystyle\prod_{\substack{p\in S\\ p<\infty}} \frac{1}{L(s-1/2, \tau_p)} & \text{if } l < 2k-2; \\ C(m, f, g) B(m-\frac{1}{2}, g) \displaystyle\prod_{\substack{p\in S\\ p<\infty}} \frac{1}{L(s-1/2, \tau_p)} & \text{if } l > 2k-2. \end{cases}$$

For $m \in B_{k,l}$, we see that $m - 1/2 \in [(-l+3)/2, (l-3)/2] \cap ((l+1)/2)\mathbb{Z}$. The theorem now follows from Lemma 3·1, Lemma 3·2 (ii), (iii) and Corollary 4·2.

5·2. *Special values: the $\Gamma^{\mathrm{para}}(m)$ case*

Let $f \in S_{2k-2}^{\mathrm{new},-}(\Gamma_0(m))$ be a primitive form, with m any positive integer. The minus in the superscript means that the sign in the functional equation of $L(s, \pi_f)$ is -1. Let $F_f^{\mathrm{para}} \in S_k(\Gamma^{\mathrm{para}}(m))$ be the Saito–Kurokawa lift of f obtained in Section 2·3. If we take $S = \{\infty\}$ then $\mathrm{SK}(\pi_f, S)$ is the cuspidal, automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ corresponding to F_f^{para}. Let $g \in S_l^{\mathrm{new}}(\Gamma_0(m), \psi')$ be a primitive form. Let ψ be a primitive Dirichlet character modulo $N' \mid m$.

The critical points for the L-function $L(s, F_f^{\mathrm{para}}, g)$ are given by $B_{k,l}$ defined in (6).

THEOREM 5·2. *Let the notations be as above. Set*:

$$\Phi(m', F_f^{\mathrm{para}}, g) := \begin{cases} \dfrac{L(m', F_f^{\mathrm{para}}, g)}{(2\pi i)^{4m'+2k+l-4} g(\psi')^2 i^{4-l-2k}\langle F_f^{\mathrm{para}}, F_f^{\mathrm{para}}\rangle\langle g, g\rangle} & \textit{if } l < 2k-2; \\ \dfrac{L(m', F_f^{\mathrm{para}}, g)}{(2\pi i)^{4m'+2l-2} g(\psi')^2 i^{2-2l}\langle g, g\rangle^2} & \textit{if } l > 2k-2. \end{cases}$$

Then, for $m' \in B_{k,l}$ and $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, we have

$$\sigma\Big(\Phi(m', F_f^{\mathrm{para}}, g)\Big) = \Phi(m', F_{f^\sigma}^{\mathrm{para}}, g^\sigma).$$

In particular,

$$\Phi(m', F_f^{\mathrm{para}}, g) \in K_f K_g.$$

Proof. Since $S = \{\infty\}$, we have

$$L\left(s, F_f^{\mathrm{para}}, g\right) = L(s, \mathrm{SK}(\pi_f, S) \times \tau_g) = L(s, \pi_f \times \tau_g) L\left(s - \frac{1}{2}, \tau_g\right) L\left(s + \frac{1}{2}, \tau_g\right).$$

Proceeding as in the proof of Theorem 5·1 we obtain the result.

The critical points for the L-function $L^{\mathrm{St}}(s, F_f^{\mathrm{para}}, \psi)$ are given by

$$B_k := \{m' \in \mathbb{Z} : 1 \leqslant m' \leqslant k-2, \psi(-1) = (-1)^{m'}\}.$$

THEOREM 5·3. *Let the notations be as above. Set*

$$\Psi(m', F_f^{\mathrm{para}}, \psi) = \frac{L^{\mathrm{St}}(m', F_f^{\mathrm{para}}, \psi)}{(2\pi i)^{3m'+2k-3} g(\psi)^2 i^{3-2k} \langle F_f^{\mathrm{para}}, F_f^{\mathrm{para}} \rangle}$$

Then, for any $m' \in B_k$ and $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, we have

$$\sigma\left(\Psi(m', F_f^{\mathrm{para}}, \psi)\right) = \Psi(m', F_{f^\sigma}^{\mathrm{para}}, \psi^\sigma).$$

In particular,

$$\Psi(m', F_f^{\mathrm{para}}, \psi) \in K_f K_\psi.$$

Proof. We have $L^{\mathrm{St}}(s, F_f^{\mathrm{para}}, \psi) = L^{\mathrm{St}}(s, \mathrm{SK}(\pi_f, S), \psi)$. Hence, by (2) and using the fact that $S = \{\infty\}$ in this case, we get

$$\begin{aligned}\Psi(m', F_f^{\mathrm{para}}, \psi) &= \frac{L(m', \psi)}{(2\pi i)^{m'}} \frac{L(m' - \frac{1}{2}, \psi\pi_f) L(m' + \frac{1}{2}, \psi\pi_f)}{(2\pi i)^{2m'+2k-3} g(\psi)^2 i^{3-2k} \langle f, f \rangle} \frac{\langle f, f \rangle}{\langle F_f^{\mathrm{para}}, F_f^{\mathrm{para}} \rangle} \\ &= \frac{L(m', \psi)}{(2\pi i)^{m'}} A\left(m' - \frac{1}{2}, f, \psi\right) \frac{\langle f, f \rangle}{\langle F_f^{\mathrm{para}}, F_f^{\mathrm{para}} \rangle}.\end{aligned}$$

Now, we get the theorem using Lemma 3·2(i), (iv) and Corollary 4·7.

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