An explicit construction of non-tempered cusp forms on $O(1, 8n + 1)^*$

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Abstract

We explicitly construct non-holomorphic cusp forms on the orthogonal group of signature $(1, 8n+1)$ for an arbitrary natural number n as liftings from Maass cusp forms of level one. In our previous works [31] and [24] the fundamental tool to show the automorphy of the lifting was the converse theorem by Maass. In this paper, we use the Fourier expansion of the theta lifts by Borcherds [4] instead.

We also study cuspidal representations generated by such cusp forms and show that they are irreducible and that all of their non-archimedean local components are non-tempered while the archimedean component is tempered, if the Maass cusp forms are Hecke eigenforms. Our non-archimedean local theory relates Sugano's local theory [39] to non-tempered automorphic forms or representations of a general orthogonal group in a transparent manner.

1 Introduction

A unique feature of automorphic forms or representations of reductive groups of higher rank (or of larger matrix size) is the existence of non-tempered cusp forms or cuspidal representations, namely cuspidal representations which have a non-tempered local component. Due to such existence the Ramanujan conjecture for $GL(2)$ can not be generalized to a general reductive group in a straightforward manner. In fact, according to the generalized Ramanujan conjecture for quasi-split reductive groups, such generalization would be possible if we impose the "genericity" on cuspidal representations, namely they are assumed to admit Whittaker models. It seems that the existence of non-tempered cuspidal representations has often been an obstruction to establish a general result of automorphic representations. Hence it is of fundamental importance to study non-tempered cusp forms or cuspidal representations in detail.

A well-known expected method for the construction of non-tempered cusp forms or representations is a lifting from a smaller group, e.g. a lifting from $GL(2)$. As related works in the case of holomorphic automorphic forms, we cite Kurokawa [20], Oda [28], Rallis-Schiffmann [35], Sugano [39], Ikeda [13], [14], Yamana [40] and Kim-Yamauchi [16] et al. We are interested in such liftings for the case of non-holomorphic real analytic automorphic forms, motivated by non-holomorphic construction of non-tempered cusp forms. We already have [31] and [24] for the groups of low rank or of small matrix size but there seems no other trial on such construction. In this paper, for a general n, we provide an explicit lifting construction of non-tempered

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cusp forms or cuspidal representations on the orthogonal group $O(1, 8n+1)$ over $\mathbb{Q}$ with non-holomorphic real analytic automorphic forms viewed as Maass cusp forms on real hyperbolic spaces of dimension $8n+1$.

Let $(\mathbb{Z}^{8n}, S)$ be an even unimodular lattice defined by a positive definite matrix S of degree $8n$. We realize the orthogonal group $O(1, 8n+1)$ by the non-degenerate symmetric matrix $Q = \begin{pmatrix} & & 1 \\ & -S & \\ 1 & & \end{pmatrix}$. Let f be a Maass cusp form of level one with Fourier coefficients $\{c(n) : n \in \mathbb{Z}\}$. We introduce a function F_f on the real Lie group $O(1, 8n+1)(\mathbb{R})$ or the real hyperbolic space of dimension $8n+1$ by a Fourier expansion whose Fourier coefficients are explicitly written in terms of those of f as follows : for $\lambda \in \mathbb{Z}^{8n}$ set

$$A(\lambda) := |\lambda|_S \sum_{d|d_\lambda} c\left(-\frac{|\lambda|_S^2}{d^2}\right) d^{4n-2}, \tag{1.1}$$

where d_λ denotes the greatest common divisor of the non-zero entries of λ and $|\lambda|_S = \sqrt{q_S(\lambda)}$ with q_S the quadratic form associated to S. Let Γ_S be an arithmetic subgroup of $O(1, 8n+1)(\mathbb{R})$ defined by the maximal lattice $(\mathbb{Z}^{8n+2}, Q)$.

Theorem 1.1 (Theorem 3.1) *The function F_f is a cusp form on $O(1, 8n+1)(\mathbb{R})$ with respect to Γ_S. If f is non-zero, so is F_f.*

In our previous works ([31] and [24]), we used the converse theorem due to Maass ([22]) to obtain automorphy of the lift. A basic limitation of the Maass converse theorem is that it provides automorphy only with respect to a discrete subgroup generated by translations and one inversion. For the case of $n > 1$, it seems difficult to determine the generators of Γ_S. Hence, the Maass converse theorem method, though applicable, does not give automorphy with respect to all of Γ_S.

To avoid this difficulty we apply a theta lift from the Maass form f, which yields an automorphic form $\Phi(\nu, f)$ (for the notation see 3.2) on $O(1, 8n+1)(\mathbb{R})$ with respect to Γ_S. We get the automorphy of F_f by explicitly computing the Fourier coefficients of $\Phi(\nu, f)$ using the calculation of Borcherds [4, Theorem 7.1], and showing that they are exactly the same as $A(\lambda)$ defined above, i.e., F_f is equal to the theta lift. We remark that the Schwartz function on $\mathbb{R}^{8n+2}$ used in this theta kernel is the product of a degree $4n$ non-harmonic polynomial and the Gaussian. In the notion of [18], this Schwartz function lies in the "polynomial Fock space".

We next show that F_f is a non-tempered cusp form, or F_f generates a cuspidal representation of $O(1, 8n+1)(\mathbb{A})$ which has a non-tempered local component. For this, it is useful to obtain an adelic reformulation of the lift F_f as a function on $O(1, 8n+1)(\mathbb{A})$. Our result is stated as follows:

Theorem 1.2 (Theorem 5.6) *Let π_{F_f} be the cuspidal representation generated by F_f and suppose that f is a Hecke eigenform with the Hecke eigenvalue λ_p for each finite prime p.*
(1) The representation π_{F_f} is irreducible and thus has the decomposition into the restricted tensor product $\otimes'_{v\le\infty}\pi_v$ of irreducible admissible representations π_v of $O(1, 8n+1)(\mathbb{Q}_v)$.

(2) For $v = p < \infty$, π_p is the spherical constituent of the unramified principal series representation of $O(1, 8n+1)(\mathbb{Q}_p)$ with the Satake parameter

$$\mathrm{diag}\left(\left(\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}\right)^2, p^{4n-1}, \cdots, p, 1, 1, p^{-1}, \cdots, p^{-(4n-1)}, \left(\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}\right)^{-2}\right).$$

(3) For every finite prime $p < \infty$, π_p is non-tempered while π_∞ is tempered.

For the first assertion of this theorem we remark that, if f is a Hecke eigenform, so is F_f (cf. Theorem 4.11). By [26, Theorem 3.1] we then see that F_f generates an irreducible cuspidal representation of $O(1, 8n+1)(\mathbb{A})$ after showing the irreducibility of the archimedean representation which F_f generates. As a consequence of this theorem we have a formula for the standard L-function of π_{F_f} or F_f in terms of the symmetric square L-function of f and the shifted Riemann zeta functions (cf. Corollary 5.7).

What is crucial to obtain the results above is Sugano's non-archimedean local theory of the "Whittaker functions" on orthogonal groups (cf. [39, Section 7]), which is now known as "special Bessel models" (cf. [27], [8]). We verify that Sugano's local theory is applicable to the Fourier coefficients of the adelic cusp form for F_f. We furthermore verify that they also satisfy the local Maass relation (cf. (4.1), (4.2)), which leads to a nice reduction of the calculation of the Hecke eigenvalues of F_f. Indeed, we see in a general setting that the space of the non-archimedean local Whittaker functions satisfying the local Maass relation admits a simple Hecke module structure (cf. Proposition 4.6). As an application of this we obtain a simple explicit formula for the Hecke eigenvalues of F_f (cf. Theorem 4.11 (2)), from which we deduce Theorem 1.2 (2) (or Theorem 5.6, 2). We now remark that, once our lifting is proved to be a theta correspondence for a symplectic-orthogonal dual pair, its non-archimedean local aspect is explained in terms of known general results, e.g. [32] and [19, Proposition 7.1.1]. However, we remark that Sugano's local theory does not use the theta correspondences. Based on such non-archimedean local theory, we can provide a general class of non-tempered automorphic forms or automorphic representations on orthogonal groups of non-compact type. The result is stated as Theorem 5.2, which can be reformulated in the modern language as follows.

Theorem 1.3 (Theorem 5.2) *Let $\pi := \otimes'_{v \le \infty} \pi_v$ be an irreducible automorphic representation of an (adelized) orthogonal group of non-compact type over a general number field generated by an automorphic form Φ. The local component π_v of π is non-tempered for a non-archimedean place v if π_v admits a "special Bessel model with the local Maass relation", by which we mean in the setting of automorphic forms that there is a Fourier coefficient of Φ belonging to the space of the local Whittaker functions satisfying the Maass relation at v.*

We can show that π_v is an irreducible unramified representation with the Satake parameter like Theorem 1.2 (2) and the result above follows from this. The general class mentioned above includes our lifts and Oda-Rallis-Schiffmann lifts [28], [35] but can be formulated without using the theta correspondence. We further remark that our simple formula for the Hecke module structure mentioned above (cf. Proposition 4.6) make it transparent how to prove the non-temperedness properties for such a class.

We discuss our results in terms of further problems. Sugano's local theory just mentioned is useful even if we replace an even umimodular lattice by a general maximal lattice. One remaining problem is to find an appropriate definition of the Fourier coefficients in such a general setting. The results could be generalized further if we discover a general definition of the Fourier coefficients which matches the general local theory by Sugano. The method by the result [4, Theorem 7.1] of Borcherds would be a useful tool to find such an appropriate definition. As we have mentioned above, we know the close connection of our lift with the theta lift. Our results would be understood totally in a representation theoretic manner, e.g. some detailed study on the Weil representation (Howe correspondence or the theta correspondence). In fact, the cuspidal representation that F_f generates admits a special Bessel model at every place and this fact is compatible with the characterization of the theta lifts by Piatetskii-Shapiro and Soudry [30] in terms of the special Bessel model. This condition at non-archimedean places is what we have remarked above and the condition at the archimedean place is verified by noting that the stabilizer group of a non-zero $\lambda \in \mathbb{Z}^{8n}$ is included in $O(8n)(\mathbb{R}) = O(S)(\mathbb{R})$, which is inside the centralizer of the torus part of the Iwasawa decomposition for $O(1, 8n + 1)(\mathbb{R})$ (cf. Section 2.1). In addition, we recall that the lifts in [24] correspond to some residual automorphic forms on $GL(4)$ via the Jacquet-Langlands correspondence. Then a natural question is "what are the automorphic forms or reprsentations (maybe non-cuspidal) on the split orthogonal group corresponding to our lifts via the Jacquet-Langlands correspondence?". Now it seems that this problem has been becoming more accessible than before due to the recent advancement of the Arthur trace formula (see [2] and [3]). On the other hand, it should be remarked that our construction of non-tempered cuspidal representations does not yet seem to be covered by Arthur's classification. In fact, the classification recorded in the well known book [2] is quite incomplete for non-tempered forms on non-quasisplit orthogonal groups (cf. [2, Theorem 9.5.3] and the remark just after it).

Finally we remark that we provide the explicit construction of F_f so that it is accessible to the people without knowledge of the representation theory, e.g. those who study automorphic forms in the classical setting. In fact, though there is a well-known approach to show the non-vanishing of theta lifts by Rallis inner product formula (cf. [33], [9]), the non-vanishing of our lift F_f is proved by the representability of every positive integer in the even unimodular lattice E_8 and some elementary argument of the L-function of f, the latter of which is also used in [24, Section 4]. We hope that such accessibility leads to a broader development of studies on automorphic forms, and that our paper serves as an interpretation of works of automorphic representations in terms of the classical setting of automorphic forms. We remark that, in general, it is never an easy problem to construct explicitly cusp forms belonging to an abstractly given cuspidal representation, especially for the case of non-holomorphic cusp forms or cuspidal represetations whose archimedean components are not holomorphic discrete series representations. We have explicitly constructed cusp forms generating (tempered) unitary spherical principal series representations at the archimedean place, whose unitarity follows from the Selberg conjecture for Maass cusp forms of level one.

Let us explain the outline of the paper. In Section 2 we introduce basic notations of algebraic groups and Lie algebras, and automorphic forms necessary for later argument. In Section 3 we introduce an automorphic form F_f by a lifting from a Maass cusp form f of level one. We first define it as an automorphic form on $O(1, 8n+1)(\mathbb{R})$. We prove that F_f (on $O(1, 8n+1)(\mathbb{R})$) is a theta lift from f, which implies the left Γ_S-invariance of F_f. We next adelize F_f. We then verify that F_f is a cusp form and show the non-vanishing of F_f. In Section 4 we develop the Hecke theory for F_f and derive the simple expression for Hecke eigenvalues of F_f, and in Section 5 we study the cuspidal representation π_{F_f} generated by F_f in detail. We determine all of its local component explicitly. That enables us to discuss its non-temperedness at finite places and have the explicit formula for the standard L-function of F_f or π_{F_f}. As we have remarked, we need Sugano's local theory [39, Section 7] to study the cuspidal representations and the standard L-functions for our lifts. In the appendix we have similar results on cuspidal representations and the standard L-functions for the lifting by Oda [28] and Rallis-Schiffmann [35], to which Sugano's local theory was originally applied. As is expected, such cuspidal representations are proved to be non-tempered at finite places.

2 Basic notations

2.1 Algebraic groups

For $N \in \mathbb{N}$, let $S \in M_N(\mathbb{Q})$ be a positive definite symmetric matrix and put $Q := \begin{pmatrix} & & 1 \\ & -S & \\ 1 & & \end{pmatrix}$.

We then define a $\mathbb{Q}$-algebraic group $\mathcal{G}$ by the group

$$\mathcal{G}(\mathbb{Q}) := \{g \in M_{N+2}(\mathbb{Q}) \mid {}^t gQg = Q\}$$

of $\mathbb{Q}$-rational points. We introduce another $\mathbb{Q}$-algebraic group $\mathcal{H}$ by the group

$$\mathcal{H}(\mathbb{Q}) := \{h \in M_N(\mathbb{Q}) \mid {}^t hSh = S\}$$

of $\mathbb{Q}$-rational points. Let q_S, resp. q_Q, denote the quadratic form on $\mathbb{Q}^N$, resp. $\mathbb{Q}^{N+2}$, associated to S, resp. Q, i.e.

$$q_S(v) = \frac{1}{2}{}^t vSv, \ q_Q(w) = \frac{1}{2}{}^t wQw$$

for $v \in \mathbb{Q}^N$ and $w \in \mathbb{Q}^{N+2}$. Then $\mathcal{H}$, resp. $\mathcal{G}$, is the orthogonal group associated to this quadratic form. For every place $v \leq \infty$ of $\mathbb{Q}$ we put $G_v := \mathcal{G}(\mathbb{Q}_v)$ and $H_v := \mathcal{H}(\mathbb{Q}_v)$.

In addition, we introduce the standard proper $\mathbb{Q}$-parabolic subgroup $\mathcal{P}$ with the Levi decomposition $\mathcal{P} = \mathcal{N}\mathcal{L}$, where the $\mathbb{Q}$-subgroups $\mathcal{N}$ and $\mathcal{L}$ are defined by

$$\mathcal{N}(\mathbb{Q}) := \left\{ n(x) = \begin{pmatrix} 1 & {}^t xS & \frac{1}{2}{}^t xSx \\ & 1_N & x \\ & & 1 \end{pmatrix} \;\middle|\; x \in \mathbb{Q}^N \right\},$$

$$\mathcal{L}(\mathbb{Q}) := \left\{ \begin{pmatrix} \alpha & & \\ & \delta & \\ & & \alpha^{-1} \end{pmatrix} \;\middle|\; \alpha \in \mathbb{Q}^\times, \ \delta \in \mathcal{H}(\mathbb{Q}) \right\}.$$

Let $J = \mathbb{Z}^2$ be the hyperbolic plane, L be a maximal lattice with respect to S, and put

$$L_0 := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{Q}^{N+2} \;\middle|\; x, z \in \mathbb{Z}, \ y \in L \right\} = L \oplus J,$$

which is a maximal lattice with respect to Q. Here see [38, Chapter II, Section 6.1] for the definition of maximal lattices. Through the bilinear form induced by the quadratic form q_S, the dual lattice $L^\sharp := \mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z})$ is identified with a sublattice of $\mathbb{Q}^N$ containing L, and is maximal with respect to S if and only if L is even unimodular. For each finite prime $p<\infty$ we introduce $L_{0,p} := L_0 \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and put

$$K_p := \{g \in G_p \mid gL_{0,p} = L_{0,p}\},$$

which forms a maximal open compact subgroup of G_p. On the other hand, let $R := \begin{pmatrix} 1 & & \\ & S & \\ & & 1 \end{pmatrix}$ and put

$$K_\infty := \{g \in G_\infty \mid {}^tgRg = R\},$$

which is a maximal compact subgroup of G_∞. Let $K_f := \prod_{p<\infty} K_p$ and $K := K_f \times K_\infty$. The groups K_f and K form maximal compact subgroups of $\mathcal{G}(\mathbb{A}_f)$ and $\mathcal{G}(\mathbb{A})$ respectively. We furthermore put $U := U_f \times H_\infty$ with $U_f := \prod_{p<\infty} U_p$, where

$$U_p := \{h \in \mathcal{H}(\mathbb{Q}_p) \mid hL_p = L_p\}$$

with $L_p := L \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We now set

$$\Gamma_S := \mathcal{G}(\mathbb{Q}) \cap K_f G_\infty = \{\gamma \in \mathcal{G}(\mathbb{Q}) \mid \gamma L_0 = L_0\}. \tag{2.1}$$

and have the following result.

Lemma 2.1 *(1) (Strong approximation theorem for $\mathcal{G}$) The class number of $\mathcal{G} = O(Q)$ with respect to $G_\infty K_f$ is one. Namely $\mathcal{G}(\mathbb{A}) = \mathcal{G}(\mathbb{Q})G_\infty K_f$*
(2) The class number of $\mathcal{H} = O(S)$ with respect to U coincides with the number of Γ_S-cusps.

Proof. For (1) see [38, Lemma 9.23 (i), Theorem 9.26], for which note that the base field is $\mathbb{Q}$ in our case. As for the second assertion we can verify that the number of the cusps coincides with the class number of the Levi subgroup $\mathcal{H}$, following the proof of [24, Lemma 2.3, 2]. More specifically, in view of the strong approximation theorem, we have the bijection

$$\mathcal{P}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{Q})/\Gamma_S \ni \mathcal{P}(\mathbb{Q})\gamma\Gamma_S \mapsto \mathcal{P}(\mathbb{Q})\gamma G_\infty K_f \in \mathcal{P}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/G_\infty K_f,$$

where $\gamma \in \mathcal{G}(\mathbb{Q})$. This yields the second assertion. In fact, by virtue of the Iwasawa decomposition $\mathcal{G}(\mathbb{A}) = \mathcal{P}(\mathbb{A})K$, we have a bijection $\mathcal{P}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/K_f G_\infty \simeq \mathcal{H}(\mathbb{Q})\backslash\mathcal{H}(\mathbb{A})/U_f U_\infty$. ∎

Remark 2.2 *The class number of $\mathcal{H}$ is also the number of equivalence classes of quadratic forms in the same genus as S (called the O-classes in [38, Chapter II, Section 9.27]). Also, there is only one element in the genus of L when S is unimodular. In that case, we see that a Γ_S-cusp corresponds to a decomposition of L_0 into the direct sum of a hyperbolic plane and a negative definite unimodular lattice.*

For the subsequent argument we remark that the real Lie group G_∞ admits an Iwasawa decomposition

$$G_\infty = N_\infty A_\infty K_\infty,$$

where

$$N_\infty := \left\{ n(x) \mid x \in \mathbb{R}^N \right\}, \qquad A_\infty := \left\{ a_y := \begin{pmatrix} y & & \\ & 1_N & \\ & & y^{-1} \end{pmatrix} \;\middle|\; y \in \mathbb{R}_+^\times \right\}. \tag{2.2}$$

From the Iwasawa decomposition we can identify the homogeneous space G_∞/K_∞ with the $(N+1)$-dimensional real hyperbolic space $H_N := \{(x,y) \mid x \in \mathbb{R}^N,\ y \in \mathbb{R}_{>0}\}$ by the natural map

$$n(x)a_y \mapsto (x,y).$$

The cusp forms we are going to study are regarded as cusp forms on the real hyperbolic space.

2.2 Lie algebras

The Lie algebra $\mathfrak{g}$ of G_∞ is defined as

$$\mathfrak{g} = \{X \in M_{N+2}(\mathbb{R}) \mid {}^tXQ + QX = 0_{N+2,N+2}\},$$

which coincides with

$$\left\{ \begin{pmatrix} a & {}^tyS & 0 \\ x & Y & y \\ 0 & {}^txS & -a \end{pmatrix} \;\middle|\; \begin{matrix} a \in \mathbb{R},\ x,\ y \in \mathbb{R}^N \\ Y \in \mathfrak{o}(S) \end{matrix} \right\},$$

where $\mathfrak{o}(S)$ denotes the Lie algebra of $\mathcal{H}(\mathbb{R})$.

Let θ be the Cartan involution of $\mathfrak{g}$ defined by

$$\mathfrak{g} \ni X \to -R^tXR^{-1} \in \mathfrak{g}.$$

We put

$$\mathfrak{k} := \{X \in \mathfrak{g} \mid \theta(X) = X\}, \quad \mathfrak{p} := \{X \in \mathfrak{g} \mid \theta(X) = -X\}.$$

Then a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is obtained. Let $\mathfrak{a}$ be a maximal abelian subalgebra given by

$$\mathfrak{a} := \left\{ \begin{pmatrix} t & & \\ & 0_{N,N} & \\ & & -t \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

The algebra $\mathfrak{g}$ has an Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k},$$

where

$$\mathfrak{n} := \left\{ \begin{pmatrix} 0 & {}^txS & 0 \\ 0_N & 0_{N,N} & x \\ 0 & {}^t0_N & 0 \end{pmatrix} \;\middle|\; x \in \mathbb{R}^N \right\}.$$

We next consider the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Let $H := \begin{pmatrix} 1 & & \\ & 0_{N,N} & \\ & & -1 \end{pmatrix}$ and α be the linear form of $\mathfrak{a}$ such that $\alpha(H) = 1$. Then $\{\pm\alpha\}$ is the set of roots for $(\mathfrak{g},\mathfrak{a})$. Let

$\{e_i \mid 1 \le i \le N\}$ be an orthonormal basis of the Euclidean space $\mathbb{R}^N$ with respect to S. For e_i with $1 \le i \le N$ we put

$$E_\alpha^{(i)} := \begin{pmatrix} 0 & {}^t e_i S & 0 \\ 0_N & 0_{N,N} & e_i \\ 0 & {}^t 0_N & 0 \end{pmatrix}, \quad E_{-\alpha}^{(i)} := \begin{pmatrix} 0 & {}^t 0_N & 0 \\ e_i & 0_{N,N} & 0_N \\ 0 & {}^t e_i S & 0 \end{pmatrix}.$$

The set $\{E_\alpha^{(i)} \mid 1 \le i \le N\}$ (respectively $\{E_{-\alpha}^{(i)} \mid 1 \le i \le N\}$) forms a basis of $\mathfrak{n}$ (respectively a basis of $\bar{\mathfrak{n}} := \left\{ \begin{pmatrix} 0 & {}^t 0_N & 0 \\ Sx & 0_{N,N} & 0_N \\ 0 & {}^t x & 0 \end{pmatrix} \middle| \; x \in \mathbb{R}^N \right\}$). Let $\mathfrak{z}_{\mathfrak{a}}(\mathfrak{k}) := \{X \in \mathfrak{k} \mid [X, A] = 0 \; \forall A \in \mathfrak{a}\}$, which coincides with

$$\left\{ \begin{pmatrix} 0 & {}^t 0_N & 0 \\ 0_N & Y & 0_N \\ 0 & {}^t 0_N & 0 \end{pmatrix} \middle| \; Y \in \mathfrak{so}(S) \right\}.$$

Then $\mathfrak{z}_{\mathfrak{a}}(\mathfrak{k}) \oplus \mathfrak{a}$ is the eigen-space with the eigenvalue zero. We then see from the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$ that $\mathfrak{g}$ decomposes into

$$\mathfrak{g} = (\mathfrak{z}_{\mathfrak{a}}(\mathfrak{k}) \oplus \mathfrak{a}) \oplus \mathfrak{n} \oplus \bar{\mathfrak{n}}.$$

We now introduce the differential operator Ω defined by the infinitesimal action of

$$\Omega := \frac{1}{2N} H^2 - \frac{1}{2} H + \frac{1}{N} \sum_{i=1}^{N} {E_\alpha^{(i)}}^2. \tag{2.3}$$

This differential operator Ω coincides with the infinitesimal action of the Casimir element of $\mathfrak{g}$ (see [17, p.293]) on the space of right K-invariant smooth functions of G_∞. To check this we note $[E_\alpha^{(i)}, E_{-\alpha}^{(i)}] = H$ and $E_\alpha^{(i)} - E_{-\alpha}^{(i)} \in \mathfrak{k}$ for $1 \le i \le N$. In what follows, we call Ω the Casimir operator.

2.3 Automorphic forms

For $\lambda \in \mathbb{C}$ and a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ we denote by $S(\Gamma, \lambda)$ the space of Maass cusp forms of weight 0 on the complex upper half plane $\mathfrak{h} := \{u + \sqrt{-1}v \in \mathbb{C} \mid v > 0\}$ whose eigenvalue with respect to the hyperbolic Laplacian is $-\lambda$.

We continue with the same notations from the previous two sections. For $r \in \mathbb{C}$ we denote by $\mathcal{M}(\Gamma_S, r)$ the space of smooth functions F on G_∞ satisfying the following conditions:

1. $\Omega \cdot F = \dfrac{1}{2N}\left(r^2 - \dfrac{N^2}{4}\right) F$, where Ω is the Casimir operator defined at (2.3),
2. for any $(\gamma, g, k) \in \Gamma_S \times G_\infty \times K_\infty$, we have $F(\gamma g k) = F(g)$,
3. F is of moderate growth.

As usual we can say that $F \in \mathcal{M}(\Gamma_S, r)$ is a cusp form if it vanishes at all the cusps of Γ_S. We will explain this in Section 3.3 with the help of the adelic language. Though we assume there that L is even unimodular, the explanation works for any maximal lattice L.

Let K_α, with $\alpha \in \mathbb{C}$, denote the modified Bessel function (see [1, Section 4.12]), which satisfies the differential equation

$$y^2 \frac{d^2 K_\alpha}{dy^2} + y \frac{dK_\alpha}{dy} - (y^2 + \alpha^2) K_\alpha = 0.$$

With K_α we can describe the Fourier expansion of F as follows:

Proposition 2.3 *Let $L^\sharp$ be the dual lattice of L. An automorphic form $F \in \mathcal{M}(\Gamma_S, r)$ admits a Fourier expansion*

$$F(n(x)a_y) = \sum_{\lambda \in L^\sharp} W_\lambda(a_y) \exp(2\pi\sqrt{-1}({}^t\lambda S x)).$$

Here

$$W_\lambda(a_y) = \begin{cases} C_\lambda y^{N/2} K_r\left(4\pi y\sqrt{q_S(\lambda)}\right) & (\lambda \neq 0_N), \\ c_1 y^{N/2-r} + c_2 y^{N/2+r} & (\lambda = 0_N,\ r \neq 0), \\ c_1 y^{N/2} + c_2 y^{N/2} \log(y) & (\lambda = 0_N,\ r = 0), \end{cases}$$

where C_λ, c_1 and c_2 are constants.

Proof. The condition $\Omega \cdot F = \frac{1}{2N}(r^2 - \frac{N^2}{4})F$ implies that W_λ satisfies the same condition. We note that W_λ is determined by its restriction to A. For simplicity of the notation we put $W_\lambda(y) := W_\lambda(a_y)$ and $\hat{W}_\lambda(y) := y^{-\frac{N-1}{2}} W_\lambda(y)$.

We then verify that $\hat{W}_\lambda(y)$ satisfies the differential equation

$$\frac{\partial^2}{\partial y^2}\hat{W}_\lambda(y) - \left((8\pi^2)2q_S(\lambda) + \frac{r^2 - \frac{1}{4}}{y^2}\right)\hat{W}_\lambda(y) = 0.$$

When $\lambda = 0_N$ it is easy to show that we have

$$W_\lambda(y) = \begin{cases} c_1 y^{N/2+r} + c_2 y^{N/2-r} & (\lambda = 0_N,\ r \neq 0), \\ c_1 y^{N/2} + c_2 y^{N/2} \log(y) & (\lambda = 0_N,\ r = 0). \end{cases}$$

Now assume $\lambda \neq 0_N$. Putting $Y := 8\pi y\sqrt{q_S(\lambda)}$, the differential equation above is rewritten as

$$(\frac{\partial^2}{\partial Y^2} + (-\frac{1}{4} + \frac{\frac{1}{4} - r^2}{Y^2}))\hat{W}_\lambda\left(\frac{Y}{8\pi\sqrt{q_S(\lambda)}}\right) = 0.$$

This is precisely the differential equation for the Whittaker function (see [1, Section 4.3]). With the Whittaker function $W_{0,r}$ parametrized by $(0, r)$ we thereby have the moderate growth solution

$$\hat{W}_\lambda(y) = C_\lambda W_{0,r}\left(8\pi y\sqrt{q_S(\lambda)}\right)$$

with a constant C_λ depending only on λ. We now note the relation (see [29, Section 13, 13.18 (iii)])

$$W_{0,r}(2y) = \sqrt{\frac{2y}{\pi}} K_r(y).$$

This means that F has the Fourier expansion as in the statement of the proposition. ∎

3 An explicit lifting construction

Let us now assume that L is an even unimodular lattice. We can identify L with the quadratic module $(\mathbb{Z}^N, q_S)$, where S is a positive definite symmetric matrix satisfying $S^{-1}\mathbb{Z}^N = \mathbb{Z}^N$ and $N = 8n$ for some $n \in \mathbb{N}$. Then the dual lattice $L^\sharp = S^{-1}\mathbb{Z}^N$ is the same as L.

Let $\mathfrak{h}$ denote the complex upper half plane and let

$$f(\tau) = \sum_{n \neq 0} c(n) W_{0, \frac{\sqrt{-1}r}{2}}(4\pi|n|v) \exp(2\pi\sqrt{-1}nu) \in S\left(SL_2(\mathbb{Z}); -\frac{r^2+1}{4}\right)$$

be a Maass cusp form on $\mathfrak{h}$, where note that the Selberg conjecture means that r is a real number. For $\lambda \in \mathbb{R}^n$ we put $|\lambda|_S := \sqrt{q_S(\lambda)}$. To consider the lifting for an even unimodular lattice L, we define the Fourier coefficient $A(\lambda)$ for a non-zero $\lambda \in L^\sharp = L$ as in (1.1) by

$$A(\lambda) := |\lambda|_S \sum_{d | d_\lambda} c\left(-\frac{|\lambda|_S^2}{d^2}\right) d^{4n-2},$$

where d_λ denotes the greatest common divisor of the non-zero entries of λ. The lifting from f to F_f on G_∞ is defined by

$$F_f(n(x)a_y) = \sum_{\lambda \in L^\sharp \setminus \{0\}} A(\lambda) y^{4n} K_{\sqrt{-1}r}(4\pi|\lambda|_S y) \exp(2\pi\sqrt{-1}{}^t\lambda S x).$$

The aim of this section is to prove the following result:

Theorem 3.1 *The automorphic form F_f is a cusp form in $M(\Gamma_S, \sqrt{-1}r)$. Furthermore, if f is non-zero, so is F_f.*

As mentioned in the introduction, we obtain the Γ_S-invariance of F_f by recognizing it as a theta lift from modular forms on $\mathfrak{h}$. For this, we need to calculate the Fourier expansion of the theta lift explicitly using [4, Theorem 7.1] and check that it agrees with the Fourier expansion of F_f. Recognizing F_f as a theta lift does not give us cuspidality or non-vanishing directly. The former follows from reinterpreting F_f in the adelic setting, whereas the latter is a consequence of representability of integers by the E_8 lattice and properties of the L-function of f.

3.1 Real hyperbolic space as a Grassmanian manifold

For this section, we let $(\mathbb{R}^N, q_S)$ be any positive definite, real quadratic space, where q_S denotes a quadratic form defined by an arbitrary positive symmetric matrix S. To use the result of Borcherds mentioned above we need an identification of the real hyperbolic space H_N with one of two connected components of the Grassmanian $\mathcal{D}$ of positive oriented lines in the quadratic space $V_N := (\mathbb{R}^{N+2}, q_Q)$, where $q_Q(a, x, b) := ab - q_S(x)$. Let B_Q denote the bilinear form associated with q_Q. For every $(x, y) \in H_N$, we have a vector

$$\nu(x, y) := \frac{1}{\sqrt{2}}{}^t(y + y^{-1}q_S(x), -y^{-1}x, y^{-1}) \in V_N \tag{3.1}$$

satisfying $B_Q(\nu(x,y), \nu(x,y)) = 1$. It generates the positive, oriented line $\mathbb{R}\cdot\nu(x,y)$, which is an element in $\mathcal{D}$. In fact, we see that $\mathcal{D}^+ := \{\mathbb{R}\cdot\nu(x,y) \mid (x,y) \in H_N\}$ is one of the two connected components of $\mathcal{D}$.

We now note that the quadratic space V_N is isometric to $\mathbb{R}^{1,N+1}$, where $\mathbb{R}^{p,q}$ denotes the real vector space $\mathbb{R}^{p+q}$ with the quadratic form

$$Q_{p,q}(x_1, x_2, \cdots, x_{p+q}) := \frac{1}{2}\left(\sum_{i=1}^{p} x_i^2 - \sum_{j=1}^{q} x_{p+j}^2\right).$$

We slightly abuse the notation by using ν to represent the line generated by $\nu(x,y)$. Every line $\nu \in \mathcal{D}^+$ induces an isometry

$$\begin{aligned} \iota_\nu : V_N &\to \mathbb{R}\cdot\nu \oplus (\nu^\perp, q_S|_{\nu^\perp}) \simeq \mathbb{R}^{1,N+1} \\ \lambda &\mapsto (\iota_\nu^+(\lambda), \iota_\nu^-(\lambda)), \end{aligned}$$

where

$$\iota_\nu^+(\lambda) := B_Q(\lambda, \nu)\nu, \;\; \iota_\nu^-(\lambda) := \lambda - \iota_\nu^+(\lambda) \in \nu^\perp$$

are the components of λ. Note that $\iota_{\gamma\cdot\nu}^+(\gamma\cdot\lambda) = \gamma\cdot\iota_\nu^+(\lambda)$ for any $\gamma \in \mathcal{G}(\mathbb{R})$ and $\lambda \in V_N$.

3.2 Theta lift from f and its coincidence with F_f

We now resume the assumption that L is even unimodular, thus $N = 8n$ for some $n \in \mathbb{N}$. We denote it also by (L, q_S) and its direct sum with a hyperbolic plane by (L_0, q_Q) as in section 2.1. Since L is even unimodular, so is L_0. Let $\mathcal{D}^+$ be the a connected component of the Grassmannian associated to $V_N = L_0 \otimes_{\mathbb{Z}} \mathbb{R}$.

To introduce the theta lift of a Maass cusp form f of level one, let $P_n(x) := 2^{-2n-3}x^{4n}$ be a polynomial on $\mathbb{R}$. The operator $\exp(-\partial_x^2/(8\pi v))$ acts on P_n and produces a polynomial $\mathcal{P}_{n,v}$ on $\mathbb{R}$, which is closely related to the physicists' Hermite polynomial by a change of variable. Furthermore, we can consider $\mathcal{P}_{n,v}$ as a polynomial on $\mathbb{R}^{1,8n+1}$ after precomposing it with the surjection $\mathbb{R}^{1,8n+1} \to \mathbb{R}$ that sends $(x_1, ..., x_{8n+2})$ to x_1. Now, we define the theta series Θ_{L_0} on $\mathfrak{h} \times \mathcal{D}^+$ by the following absolutely convergent sum

$$\Theta_{L_0}(\tau, \nu) := \sum_{\lambda \in L_0} \mathcal{P}_{n,v}(B_Q(\iota_\nu(\lambda), \nu)) \exp(2\pi\sqrt{-1}(q_Q(\iota_\nu^+(\lambda))\tau + q_Q(\iota_\nu^-(\lambda)\bar{\tau})).$$

Comparing to [4, Section 4], it is easy to see that this is the theta kernel used by Borcherds with a homogeneous polynomial of degrees $(4n, 0)$. By its definition and the Poisson summation formula, we know that $v^{4n+1/2}\Theta_{L_0}(\tau, \nu)$ is a modular function in $\tau \in \mathfrak{h}$ with respect to $SL_2(\mathbb{Z})$ for any $\nu \in \mathcal{D}^+$ (see e.g. [4, Theorem 4.1]). Then the theta lift $\Phi_{L_0}(\nu, f)$ of the Maass cusp form f is defined as

$$\Phi_{L_0}(\nu, f) := \int_{SL_2(\mathbb{Z})\backslash\mathfrak{h}} f(\tau)\overline{\Theta_{L_0}(\tau, \nu)}v^{4n+\frac{1}{2}}d\mu(\tau),$$

where $d\mu(\tau) := v^{-2}dudv$. Since f is a cusp form, the integral above converges absolutely. On the other hand, since

$$B_Q(\iota_{\gamma\cdot\nu}(\lambda), \gamma\cdot\nu) = B_Q(\iota_{\gamma\cdot\nu}^+(\lambda), \gamma\cdot\nu) = B_Q(\iota_\nu^+(\gamma^{-1}\lambda), \nu) = B_Q(\iota_\nu(\gamma^{-1}\lambda), \nu)$$

for any $\lambda \in V_{8n}$ and $\gamma \in \mathcal{G}(\mathbb{R})$, we have $\Theta_{L_0}(\tau, \gamma \cdot \nu) = \Theta_{L_0}(\tau, \nu)$ for any $\gamma \in \Gamma_S$ and $\tau \in \mathfrak{h}$. The integral $\Phi_{L_0}(\nu, f)$ is also left Γ_S-invariant.

If we choose another unimodular lattice $(L', q_{S'})$ with the same rank as (L, q_S), then $L'_0 := L' \oplus J$ is isomorphic to L_0 by the classification of indefinite unimodular lattices [37, Chapitre V, Section 2.2, Theorem 6]. Different choices of such decomposition correspond to different cusps of the hyperbolic space H_N (cf. Remark 2.2), each of which gives the coordinates to express the Fourier expansion. To do this, we follow [4, Theorem 7.1] and choose the isotropic vectors

$$l := {}^t(1, 0, 0),\ \check{l} := {}^t(0, 0, 1)$$

in L_0. We then have

$$l_\nu := \iota_\nu^+(l) = \frac{1}{\sqrt{2}y}\nu,\ l_{\nu^\perp} := l - l_\nu,\ B_Q(l_\nu, l_\nu) = -B_Q(l_{\nu^\perp}, l_{\nu^\perp}) = \frac{1}{2y^2},$$
$$\mu = -\check{l} + \frac{1}{2B_Q(l_\nu, l_\nu)}l_\nu + \frac{1}{2B_Q(l_{\nu^\perp}, l_{\nu^\perp})}l_{\nu^\perp} = -\check{l} + y^2(2l_\nu - l),$$

and

$$P_n(B_Q(\iota_\nu(\lambda), \nu)) = 2^{-2n-3}B_Q(\lambda, \nu)^{4n} = 2^{-3}y^{4n}B_Q(\lambda, l_\nu)^{4n}.$$

We furthermore note that the orthogonal complement of $l_{\nu^\perp}$ in $\nu^\perp$ is $L \otimes \mathbb{R}$, which means

$$B_Q(\lambda, \mu) = 2y^2 B_Q(\lambda, l_\nu) = \sqrt{2}yB_Q(\lambda, \nu) = {}^t\lambda Sx$$

for any $\lambda \in L \otimes \mathbb{R} \subset V_{8n}$. With the input datum above, we can apply [4, Theorem 7.1] to write out the Fourier expansion of $\Phi(\nu, f)$. In the notation loc. cit., we have $M = L_0 = L_0^\sharp = M'$, $K' = L^\sharp = L = K$, $\lambda_{w^+} = 0$, $\lambda_{w^-} = \lambda$,

$$z_{v^+} = l_\nu,\ |z_{v^+}| = \sqrt{z_{v^+}^2} = \sqrt{2q_Q(l_\nu)},\ p_{w,h^+,h^-}(x) = \begin{cases} x^{4n}/8, & (h^+, h^-) = (4n, 0), \\ 0, & \text{otherwise.} \end{cases}$$

Here $p_{w,4n,0}$ is nothing but $4^n P_n$ and can be viewed as a polynomial on $\mathbb{R}^{1,8n+1}$. Therefore, the term involving $\Phi_K(w, p_{w,h,h}, F_K)$ vanishes identically and $(-\Delta)^j \overline{p}_{w,h^+,h^-}$ is identically zero for $j \geq 1$. Since L is unimodular, the term $\sum_{\delta \in M'/M, \delta|L=\lambda} \mathbf{e}(n(\delta, z'))$ in the third line of Borcherds' Theorem 7.1 becomes the factor 1. Furthermore, the coefficient $c_{\delta,\lambda^2/2}(y)$ is $c(-q_S(\lambda))W_{0,\sqrt{-1}r/2}(4\pi q_S(\lambda)v)v^{4n+1/2}$, which is the Fourier coefficient of $v^{4n+1/2}f(\tau)$. The extra term of $v^{4n+1/2}$ comes from the way Borcherds normalized his input (which has an extra factor of $v^{m^+ + b^+/2}$). With these in hand and that $c(0) = 0$, Theorem 7.1 of [4] simplifies to

$$\begin{aligned}\Phi_{L_0}(\nu(x, y), f) = \frac{y^{4n+1}}{4} &\sum_{\lambda \in L^\sharp \setminus \{0\}} c(-|\lambda|_S^2) \sum_{m \geq 1} m^{4n} \exp(2\pi\sqrt{-1}m\,{}^t\lambda Sx) \\ &\int_0^\infty W_{0,\sqrt{-1}r/2}(4\pi|\lambda|_S^2 v) e^{-\frac{\pi m^2 y^2}{v} - 2\pi v|\lambda|_S^2} v^{-2} dv \\ = y^{4n} &\sum_{\lambda \in L^\sharp \setminus \{0\}} c(-|\lambda|_S^2) \sum_{m \geq 1} m^{4n-1} \exp(2\pi\sqrt{-1}m\,{}^t\lambda Sx)|\lambda|_S K_{\sqrt{-1}r}(4\pi m|\lambda|_S y)\end{aligned}$$

$$= y^{4n} \sum_{\lambda \in L^\sharp \setminus \{0\}} A(\lambda) K_{\sqrt{-1}r}(4\pi|\lambda|_S y) \exp(2\pi\sqrt{-1}{}^t\lambda S x),$$

where $A(\lambda)$ is defined in (1.1). Here, for the second equation, we have made the change of variable $v \mapsto 1/v$ and used the following integral identity

$$\int_0^\infty \exp(-pt - a/(2t)) W_{0,\sqrt{-1}r/2}(a/t) dt = 2\sqrt{a/p} K_{\sqrt{-1}r}(2\sqrt{ap})$$

(cf. [7, 4.22 (22)]) with $a = 4\pi|\lambda|_S^2$ and $p = \pi m^2 y^2$. We now immediately see that F_f coincides with $\Phi_{L_0}(\nu, f)$, which is left Γ_S-invariant.

3.3 Adelic formulation of the lifting and the proof of Theorem 3.1

We reformulate the lifting F_f in the adelic setting to complete the proof of Theorem 3.1.

For this purpose we introduce the special orthogonal group $SO(S) := \mathcal{H} \cap SL_N$ over $\mathbb{Q}$, where SL_N denotes the $\mathbb{Q}$-algebraic group defined by the special linear group of degree N. It is easy to verify that the cosets $\mathcal{H}(\mathbb{Q})\backslash\mathcal{H}(\mathbb{A})/U_f U_\infty$ have representatives in $SO(S)(\mathbb{A})$, as is explained soon below.

We shall recall that the cosets $SO(S)(\mathbb{Q})\backslash SO(S)(\mathbb{A})/(SO(S)(\mathbb{A}) \cap U_f U_\infty)$ are in bijection with the equivalence classes of the quadratic forms of the same genus with S (cf. [38, Chapter II, Section 9.25]). To describe this bijection we recall that each $h \in SO(S)(\mathbb{A})$ has a decomposition $h = au^{-1}$ with $(a, u) \in SL_N(\mathbb{Q}) \times (\prod_{p<\infty} SL_N(\mathbb{Z}_p) \times SL_N(\mathbb{R}))$ by the strong approximation theorem of SL_N. Then $S_h := {}^t aSa = {}^t uSu$ is in the same genus with S. The bijection above is induced by the mapping

$$h \mapsto S_h.$$

Furthermore we remark that, if $h \in \mathcal{H}(\mathbb{A}) \setminus SO(S)(\mathbb{A})$, there exists $\delta_0 \in \mathcal{H}(\mathbb{Q}) \setminus SO(S)(\mathbb{Q})$ such that $\delta_0 h \in SO(S)(\mathbb{A})$. We can thus say that $h \in \mathcal{H}(\mathbb{A})$ has a decomposition $h = au^{-1}$ with $(a, u) \in GL_N(\mathbb{Q}) \times (\prod_{p<\infty} SL_N(\mathbb{Z}_p) \times SL_N(\mathbb{R}))$ in general. We put $L_h := (\prod_{p<\infty} h_p \mathbb{Z}_p^N \times \mathbb{R}^N) \cap \mathbb{Q}^N$ for $h = (h_v)_{v \le \infty} \in \mathcal{H}(\mathbb{A})$. Then we have $L_h = a\mathbb{Z}^N$.

We see that $\mathcal{H}(\mathbb{Q})\backslash\mathcal{H}(A)/U_f U_\infty$ can be viewed as a subset of the double coset space $SO(S)(\mathbb{Q})$ $\backslash SO(S)(\mathbb{A})/(SO(S)(\mathbb{A}) \cap U_f U_\infty)$. We let $C(S) := \{S_1, \ S_2, \ldots, S_c\}$ be the classes of the same genus with S corresponding bijectively to a complete set $\{h_i \in SO(S)(\mathbb{A}_f) \mid 1 \le i \le c\}$ of representatives for $\mathcal{H}(\mathbb{Q})\backslash\mathcal{H}(\mathbb{A})/U_f U_\infty$ (called the O-classes in [38, Chapter II, Section 9.27]).

Let $f \in S(SL_2(\mathbb{Z}); -(\frac{1}{4} + \frac{r^2}{4}))$ be a Maass cusp form with the Fourier expansion $f(z) = \sum_{n \ne 0} c(n) W_{0,\frac{\sqrt{-1}r}{2}}(4\pi|n|y) \exp(2\pi\sqrt{-1}nx)$. Let Λ be the standard additive character of $\mathbb{A}/\mathbb{Q}$. We introduce the Fourier series

$$F_f(n(x)a_y kg) := \sum_{\lambda \in \mathbb{Q}^N \setminus \{0\}} F_{f,\lambda}(n(x)a_y kg) \quad \forall (x, y, k, g) \in \mathbb{A}^N \times \mathbb{R}_+^\times \times K_\infty \times \mathcal{G}(\mathbb{A}_f) \tag{3.2}$$

with

$$F_{f,\lambda}(n(x)a_y kg) := A_\lambda(g) y^{4n} K_{\sqrt{-1}r}(4\pi|\lambda|_S y) \Lambda({}^t\lambda S x),$$

where $A_\lambda(g)$ is defined by the following three conditions:

$$A_\lambda\left(\begin{pmatrix} 1 & & \\ & h & \\ & & 1 \end{pmatrix}\right) := \begin{cases} |\lambda|_S \sum_{d|d_\lambda} c(-\frac{|\lambda|_S^2}{d^2}) d^{4n-2} & (\lambda \in L_h) \\ 0 & (\lambda \in \mathbb{Q}^N \setminus L_h) \end{cases}, \tag{3.3}$$

$$A_\lambda\left(\begin{pmatrix}\beta & & \\ & h & \\ & & \beta^{-1}\end{pmatrix}\right) := ||\beta||_{\mathbb{A}}^{4n} A_{||\beta||_{\mathbb{A}}^{-1}\lambda}\left(\begin{pmatrix}1 & & \\ & h & \\ & & 1\end{pmatrix}\right), \tag{3.4}$$

$$A_\lambda(n(x)gk) := \Lambda({}^t\lambda Sx)A_\lambda(g)\ \forall(x,g,k)\in\mathbb{A}_f^N\times\mathcal{G}(\mathbb{A}_f)\times K_f. \tag{3.5}$$

Here

1. $(\beta,h)\in\mathbb{A}_f^\times\times\mathcal{H}(\mathbb{A}_f)$ and $||\beta||_{\mathbb{A}}$ denotes the idele norm of β,

2. d_λ denotes the greatest common divisor of the non-zero entries in $a^{-1}\lambda\in\mathbb{Z}^N$ $(=a^{-1}L_h)$.

Let us remark that we are using the same notation F_f for both the adelic lift above and the non-adelic lift of the previous section. This should not lead to any confusion since it will be clear from the context which lift we are discussing.

We note that the definition of d_λ does not depend on the decomposition $h=au^{-1}$, as we are going to see in the proof of the following lemma.

Lemma 3.2 *The adelic Fourier series defining the adelic F_f is well-defined and is a left $\mathcal{P}(\mathbb{Q})$-invariant and right $K(=K_fK_\infty)$-invariant smooth function on $\mathcal{G}(\mathbb{A})$.*

Proof. We should first note that A_λ defines a well-defined function in $h\in\mathcal{H}(\mathbb{A}_f)$ for each fixed $\lambda\in\mathbb{Q}^N$. To see this we have to check that d_λ does not depend on the decomposition $h=au^{-1}$. If we take another decomposition $h=a'u'^{-1}$ with $(a',u')\in GL_N(\mathbb{Q})\times(\prod_{p<\infty}SL_N(\mathbb{Z}_p)\times SL_N(\mathbb{R}))$, we have that $a^{-1}a'=u^{-1}u'\in SL_N(\mathbb{Z})$ and see that the definition of d_λ remains the same even if we replace a by a'.

By the definition of $A_\lambda\left(\begin{pmatrix}\beta & & \\ & h & \\ & & \beta^{-1}\end{pmatrix}\right)$ in (3.3) and (3.4) we can verify that this is right $(\prod_{p<\infty}\mathbb{Z}_p^\times)\times U_f(=\mathcal{L}(\mathbb{A}_f)\cap K_f)$-invariant as a function of $(\beta,h)\in\mathbb{A}_f^\times\times\mathcal{H}(\mathbb{A}_f)(=\mathcal{L}(\mathbb{A}_f))$ and that $\alpha^{\frac{N}{2}}A_\lambda\left(\begin{pmatrix}\alpha_f & & \\ & \delta_f h & \\ & & \alpha_f^{-1}\end{pmatrix}\right)=A_{\alpha\delta^{-1}\lambda}\left(\begin{pmatrix}1 & & \\ & h & \\ & & 1\end{pmatrix}\right)$ for $(\alpha,\delta)\in\mathbb{Q}_+^\times\times\mathcal{H}(\mathbb{Q})$, where (α_f,δ_f) denotes the finite adele part of $(\alpha,\delta)\in\mathbb{Q}_+^\times\times\mathcal{H}(\mathbb{Q})$. From the latter we deduce that F_f is left $\mathcal{P}(\mathbb{Q})$-invariant and that, as a result of (3.5), A_λ is well-defined on $\mathcal{G}(\mathbb{A}_f)$ and right K_f-invariant. To finish the proof we note that the archimedean part $y^{4n}K_{\sqrt{-1}r}(4\pi|\lambda|_S y)\exp(2\pi\sqrt{-1}({}^t\lambda Sx))$ with $(y,x)\in\mathbb{R}_+^\times\times\mathbb{R}^N$ is a smooth right K_∞-invariant function on G_∞ and is a rapidly decreasing with respect to y. From this and the K_f-invariance above we deduce the convergence and smoothness of the Fourier series as a function on $\mathcal{G}(\mathbb{A})$. As a result we have seen that F_f satisfies the desired property in the assertion, and we are thus done. ∎

For $r\in\mathbb{C}$ we now introduce the space $\mathcal{M}(\mathcal{G},r)$ of smooth functions F on $\mathcal{G}(\mathbb{A})$ satisfying the following conditions:

1. $\Omega\cdot F=\frac{1}{2N}\left(r^2-\frac{N^2}{4}\right)F$, where Ω is the Casimir operator defined at (2.3),

2. for any $(\gamma,g,k)\in\mathcal{G}(\mathbb{Q})\times\mathcal{G}(\mathbb{A})\times K$, we have $F(\gamma gk)=F(g)$,

3. F is of moderate growth.

We further remark that $F \in \mathcal{M}(\mathcal{G}, r)$ has the Fourier expansion

$$F(g) = \sum_{\lambda \in \mathbb{Q}^N} F_\lambda(g), \quad F_\lambda(g) := \int_{\mathbb{A}^N/\mathbb{Q}^N} F(n(x)g)\Lambda(-{}^t\lambda S x)dx,$$

where dx is the invariant measure normalized so that the volume of $\mathbb{A}^N/\mathbb{Q}^N$ is one. We call F a cusp form if $F_0 \equiv 0$ in the Fourier expansion.

Now let us explain that this cuspidality condition of $F \in \mathcal{M}(\mathcal{G}, r)$ implies the vanishing of the constant term of the non-adelic Fourier expansion of F at every Γ_S-cusp. Let $c \in \mathcal{G}(\mathbb{Q})$ be a representative of a Γ_S-cusp and decompose c into $c_f c_\infty$ with $c_f \in \mathcal{G}(\mathbb{A}_f)$ and $c_\infty \in G_\infty$. Furthermore let c correspond to a representative h of $\mathcal{H}(\mathbb{Q})\backslash\mathcal{H}(\mathbb{A})/U_f U_\infty$ via the bijection in Lemma 2.1. From Proposition 2.3 we then see that the non-adelic Fourier expansion of $F(n(x_\infty)a_{y_\infty}\begin{pmatrix}1 & & \\ & h & \\ & & 1\end{pmatrix})$ for $(x_\infty, y_\infty) \in \mathbb{R}^N \times \mathbb{R}_+^\times$ is expressed as

$$\sum_{\lambda \in L_h \setminus \{0\}} C_\lambda(h) W_\lambda(a_{y_\infty}) \exp(2\pi\sqrt{-1}({}^t\lambda S x_\infty))$$

with Fourier coefficients $C_\lambda(h)$. Here note that the above F translated by $\begin{pmatrix}1 & & \\ & h & \\ & & 1\end{pmatrix}$ is right $\begin{pmatrix}1 & & \\ & h & \\ & & 1\end{pmatrix} K_f \begin{pmatrix}1 & & \\ & h & \\ & & 1\end{pmatrix}^{-1}$-invariant, which implies that the summation of the expansion runs over $L_h \setminus \{0\}$ with the even unimodular lattice L_h. On the other hand, we have that $F(g_\infty \begin{pmatrix}1 & & \\ & h & \\ & & 1\end{pmatrix}) = F(c_f g'_\infty) = F(c_\infty^{-1} g'_\infty)$ with a suitable change of variables $g_\infty \to g'_\infty$ in G_∞. The Fourier expansion of $F(c^{-1}g'_\infty)$ is nothing but the non-adelic expansion at a Γ_S-cusp c^{-1}. We therefore see the vanishing of F at every Γ_S-cusp in the sense of the non-adelic Fourier expansion (for this see Remark 2.2).

Theorem 3.3 *Given a Maass cusp form $f \in S(SL_2(\mathbb{Z}); -(\frac{1}{4} + \frac{r^2}{4}))$ we define F_f as in (3.2). Then F_f is a cusp form in $\mathcal{M}(\mathcal{G}, \sqrt{-1}r)$.*

Proof. By the strong approximation theorem for $\mathcal{G}$ (cf. Lemma 2.1 (1)) we have an isomorphism $\mathcal{M}(\mathcal{G}(\mathbb{Q}), \sqrt{-1}r) \simeq \mathcal{M}(\Gamma_S, \sqrt{-1}r)$ given by $F \mapsto F|_{\mathcal{G}(\mathbb{R})}$. The right K-invariance for the adelized F_f follows immediately from Lemma 3.2. We have proved the left Γ_S-invariance of $F_f|_{\mathcal{G}(\mathbb{R})}$ since it coincides with the non-adelic lift. By the standard argument in terms of the strong approximation theorem we can deduce that F_f is a left $\mathcal{G}(\mathbb{Q})$-invariant function. To see that F_f is of moderate growth, note that this is determined by its restriction to $\mathcal{G}(\mathbb{R})$. In fact, $F_f|_{\mathcal{G}(\mathbb{R})}$ is given by the Fourier series with rapidly decreasing terms and its Fourier coefficients $A(\lambda)$ satisfy $|A(\lambda)| = O(|\lambda|_S^\kappa)$ with some $\kappa > 0$, which follows from the growth property of the Fourier coefficients $c(n)$ of f. From this we verify that $F_f|_{\mathcal{G}(\mathbb{R})}$ is at most of polynomial order by estimating the Fourier series of F_f. The action of the Casimir Ω follows from the Fourier expansion of F_f. Hence, we get $F_f \in \mathcal{M}(\mathcal{G}, \sqrt{-1}r)$. From the Fourier series of F_f it is straightforward to see that F_f is cuspidal. ∎

We are left with discussing the non-vanishing of F_f. To this end we need the following lemma, which is similar to [24, Lemma 4.5].

Lemma 3.4 *Let $f \in S(SL_2(\mathbb{Z}); -(\frac{1}{4}+\frac{r^2}{4}))\setminus\{0\}$ with Fourier coefficients $c(m)$. Then, there exist $M>0, M\in\mathbb{Z}$, such that $c(-M)\neq 0$.*

Proof. Assume that $c(m)=0$ for all $m<0$. Set $f_1(z)=(f(z)+f(-\bar{z}))/2$ and $f_2(z)=(f(z)-f(-\bar{z}))/2$. Then, f_1, f_2 are elements of $S(SL_2(\mathbb{Z}); -(\frac{1}{4}+\frac{r^2}{4}))$. In addition, f_1 is an even Maass form and f_2 is an odd Maass form, with the property that they have the exact same Fourier coefficients corresponding to positive indices. This implies that the L-functions for f_1 and f_2 satisfy $L(s,f_1)=L(s,f_2)$. On the other hand, $L(s,f_1)$ and $L(s,f_2)$ satisfy functional equations with the gamma factors shifted by 1. If $L(s,f_1)\neq 0$, we obtain an identity of gamma factors, which can be checked to be impossible. This gives us that f has to be zero, a contradiction. ∎

By Lemma 3.4, there is the smallest positive integer M_0 such that $c(-M_0)\neq 0$. For $n\in\mathbb{N}$, let E_8^n be the direct sum of n copies of the E_8 lattice. Then there exists $\lambda_0\in E_8^n$ with norm M_0. This follows from the case $n=1$, which holds since the theta function associated to E_8 is the Eisenstein series of weight 4 on $SL_2(\mathbb{Z})$. From the Fourier expansion near the cusp determined by $L_0\cong J\oplus E_8^n$ (cf. Remark 2.2), we see that $A_{\lambda_0}\neq 0$. Thus $F_f\not\equiv 0$ for a non-zero f, which finishes the proof of Theorem 3.1.

In addition, let us note that Weyl's law for $SL_2(\mathbb{Z})$ by Selberg (cf. [15, Section 11.1]) implies the existence of non-zero Maass cusp forms for $SL_2(\mathbb{Z})$. As a result the argument so far implies the following:

Proposition 3.5 *There exists non-zero F_f.*

4 Hecke theory for the lifting

We are going to discuss the Hecke theory of our lifting. In fact, we will show that if f is a Hecke eigenform then so is the lift F_f, and we can compute the Hecke eigenvalues of F_f explicitly in terms of those of f. The method is to use the non-archimedean local theory by Sugano [39, Section 7] for the Jacobi form formulation of the Oda-Rallis-Schiffmann lifting [28], [35].

4.1 Sugano's local theory

In this section we work over a non-archimedean local field F of characteristic not equal to two. Let ϖ be a prime element of F and let $\mathfrak{o}$ be the ring of integers in F. We put $\mathfrak{p}:=\varpi\mathfrak{o}$, which is nothing but the prime ideal of $\mathfrak{o}$, and put $q:=\sharp\mathfrak{o}/\mathfrak{p}$, namely the number of the residue field. Let $n_0\leq 4$ and let $S_0\in M_{n_0}(F)$ be an anisotropic even symmetric matrix of degree n_0. We introduce the $m\times m$ matrix $J_m:=\begin{pmatrix} & & 1\\ & \iddots & \\ 1 & & \end{pmatrix}$. We denote by G_m the group of F-valued points of the orthogonal group of degree $2m+n_0$ defined by the symmetric matrix $Q_m:=\begin{pmatrix} & & J_m\\ & S_0 & \\ J_m & & \end{pmatrix}$. In what follows, we suppose that $L_m:=\mathfrak{o}^{2m+n_0}$ is a maximal lattice with respect to Q_m. Let K_m be the maximal open compact subgroup of G_m defined by the maximal lattice (L_m,Q_m), namely

$$K_m:=\{g\in G_m\mid gL_m=L_m\}.$$

We regard G_i with $i \le m$ as a subgroup of G_m by embedding G_i into the middle $(2i+n_0) \times (2i+n_0)$ block of G_m. We also regard K_i with $i \le m$ as a subgroup of K_m similarly.

Hereafter we normalize the invariant measure of G_m so that the volume of K_i is one for each $i \le m$, which is justified in view of the existence of the quotient measure for G_{i+1}/G_i.

By $\mathcal{H}_m$ we denote the Hecke algebra for (G_m, K_m). Let $C_m^{(r)} \in \mathcal{H}_m$ be defined by the double coset $K_m c_m^{(r)} K_m$, where

$$c_m^{(r)} := \mathrm{diag}(\varpi, \cdots, \varpi, 1, \ldots, 1, \varpi^{-1}, \cdots, \varpi^{-1}) \in G_m,$$

which is the diagonal matrix whose first r entries and last r entries are ϖ and ϖ^{-1} respectively. As is remarked in [39, Section 7], $\{C_m^{(r)} \mid 1 \le r \le m\}$ forms generators of the Hecke algebra $\mathcal{H}_m$. Note that the Satake isomorphism also holds for these orthogonal groups although they are not connected (cf. [36, Theorem 5, Remark 1 after Theorem 3]).

Assume that $m \ge 2$ or $n_0 > 0$ and,with $1 \le i \le m$, let $n_i(x) := \begin{pmatrix} 1 & -{}^t x Q_{i-1} & -\frac{1}{2}{}^t x Q_{i-1} x \\ & 1_{2i-2+n_0} & x \\ & & 1 \end{pmatrix} \in G_i$ for $x \in F^{2i-2+n_0}$. By $L_{m-1}^\sharp$ we denote the dual lattice of L_{m-1} with respect to Q_{m-1}. We need the notion that $\lambda \in L_{m-1}^\sharp \setminus \{0\}$ is primitive or reduced as follows (cf. [39, p44]):

Definition 4.1 *(1) A vector $\lambda \in L_{m-1}^\sharp$ is defined to be primitive if its $(2m-2+n_0)$-th entry is equal to 1.*
(2) A primitive $\lambda \in L_{m-1}$ is called reduced (with respect to Q_{m-1}) if

$$\begin{pmatrix} \varpi^{-1} & & \\ & 1_{2m-4+n_0} & \\ & & \varpi \end{pmatrix} n_{m-1}(x)\lambda \notin \varpi L_{m-1}^\sharp$$

for any $x \in F^{2m-4+n_0}$.

Lemma 4.2 *Suppose that $L_{m-1}^\sharp = L_{m-1}$, namely L_{m-1} is self-dual. For a primitive $\lambda \in L_{m-1}^\sharp$, λ is reduced if and only if the ϖ-adic order of $\frac{1}{2}{}^t\lambda Q_{m-1}\lambda$ is not greater than 1.*

Proof. We write $\lambda = \begin{pmatrix} a \\ \alpha \\ 1 \end{pmatrix}$ with $a \in \mathfrak{o}$, $\alpha \in L_{m-2}^\sharp$. We note that $L_{m-1}^\sharp = L_{m-1}$ implies $L_{m-2}^\sharp = L_{m-2}$. Suppose first that λ is reduced. Assume that the ϖ-adic order of $\frac{1}{2}{}^t\lambda Q_{m-1}\lambda$ is not less than two, and take $\beta \in F^{2m-4+n_0}$ so that $\alpha + \beta \in \varpi L_{m-2}^\sharp$. We then verify that

$$\begin{pmatrix} \varpi^{-1} & & \\ & 1_{2m-4+n_0} & \\ & & \varpi \end{pmatrix} n_{m-1}(\beta)\lambda = \begin{pmatrix} \varpi^{-1}(\frac{1}{2}{}^t\lambda Q_{m-2}\lambda - \frac{1}{2}{}^t(\alpha+\beta)Q_{m-2}(\alpha+\beta)) \\ \alpha + \beta \\ \varpi \end{pmatrix} \in \varpi L_{m-1}^\sharp.$$

In fact, noting $\varpi L_{m-2}^\sharp = \varpi L_{m-2}$, we see that the ϖ-adic order of the first entry for the vector above is not less than one. This contradicts the assumption that λ is reduced.

Suppose next that the ϖ-adic order of $\frac{1}{2}{}^t\lambda Q_{m-1}\lambda$ is not greater than one. Then, with $\beta \in F^{2m-4+n_0}$ such that $\alpha + \beta \in \varpi L_{m-2}^\sharp$, the first entry of $\begin{pmatrix} \varpi^{-1} & & \\ & 1_{2m-4+n_0} & \\ & & \varpi \end{pmatrix} n_{m-1}(\beta)\lambda$

has the ϖ-adic order less than or equal to 0. This suffices to prove that λ is reduced since the proof is straightforward for β such that $\alpha+\beta \notin \varpi L_{m-2} = \varpi L_{m-2}^\sharp$. ■

Let us now introduce $M_k := \begin{pmatrix} \varpi^{-k} & & \\ & 1_{2m-4+n_0} & \\ & & \varpi^k \end{pmatrix} \in G_{m-1}$ for a non-negative integer k. This is useful to describe elements in $L_{m-1}^\sharp$ in terms of reduced ones under the assumption that L_{m-1} is self-dual.

Lemma 4.3 *Let $L_{m-1} = L_{m-1}^\sharp$ (which implies $L_{m-2} = L_{m-2}^\sharp$). Any $\lambda \in L_{m-1}^\sharp \setminus \{0\}$ can be written as $u\lambda = \varpi^{k+l} M_k^{-1}\eta$ with some non-negative integers k and l, $u \in K_{m-1}$ and a reduced $\eta \in L_{m-1}^\sharp$.*

Proof. Without loss of generality we may assume that λ is primitive since we can take $l \geq 0$ and $u \in K_{m-1}$ so that $u\lambda = \varpi^l \lambda_0$ with a primitive λ_0. Note that the case of $l = 0$ means that λ is primitive, up to the K_{m-1}-action on the left.

Let us take $x \in L_{m-2}^\sharp$ so that $n_{m-1}(x)\lambda = \begin{pmatrix} y \\ 0_{2m-4+n_0} \\ 1 \end{pmatrix}$ with $y \in \mathfrak{o}$. Note that $n_{m-1}(x) \in K_{m-1}$ for $x \in L_{m-2}^\sharp = L_{m-2}$. Thus there is $u_0 \in K_{m-1}$, $t \in \mathfrak{o}^\times$ and a non-negative integer f such that $u_0\lambda = \begin{pmatrix} \varpi^f t \\ 0_{2m-4+n_0} \\ 1 \end{pmatrix}$. We may now assume $\lambda = \begin{pmatrix} \varpi^f t \\ 0_{2m-4+n_0} \\ 1 \end{pmatrix}$. Put $k := \left[\frac{f}{2}\right]$, then we have $\lambda = \varpi^k M_k^{-1} \begin{pmatrix} a_0 \\ 0_{2m-4+n_0} \\ 1 \end{pmatrix}$ with $a_0 \in \mathfrak{o}$ whose ϖ-adic order is equal to 0 or 1. We are therefore done since $\begin{pmatrix} a_0 \\ 0_{2m-4+n_0} \\ 1 \end{pmatrix}$ is reduced in view of Lemma 4.2. ■

We are now ready to introduce the notion of "local Whittaker functions" on G_m in the sense of [39, Section 7, p47]. Though this does not come from the "Whittaker model" in the usual sense, it can be understood in terms of "special Bessel models" for unramified principal series representations of G_m. We furthermore review the notion of the "local Maass relation" as in [39, Section 7, p52], which leads to a nice reduction for the calculation of the Hecke eigenvalues. Sugano's local theory deals with the case of general maximal lattices. Though what we need for the Hecke theory of F_f is the local theory under the assumption "$\partial = n_0 = 0$" (see the notation below and [39, p6]), our review on this is going to be given in such a general setting. We will show that the Fourier coefficients of F_f belong to the space of the local Whittaker functions satisfying the local Maass relation.

Let $\lambda \in L_{m-1}^\sharp$ be reduced and put H_λ to be the stabilizer of λ in G_{m-1}. We then introduce the space of the local Whittaker functions as follows:

$$\mathcal{W}_\lambda^{\mathcal{F}} := \left\{ W : G_m \to \mathbb{C} \;\middle|\; \begin{array}{c} W\left(n_m(x) \begin{pmatrix} 1 & & \\ & h & \\ & & 1 \end{pmatrix} gk \right) = \Lambda_F({}^t\lambda(-Q_{m-1})x) W(g) \\ \forall (x,h,g,k) \in F^{2m-2+n_0} \times H_\lambda \times G_m \times K_m \end{array} \right\},$$

where Λ_F denotes the additive character of F trivial on $\mathfrak{o}$ but non-trivial on $\mathfrak{p}^{-1}$.

For $W \in \mathcal{W}_\lambda^{\mathcal{F}}$, $l \in \mathbb{Z}$ and a non-negative integer k we put

$$W_{k,l} := W\left(\begin{pmatrix} \varpi^{k+l} & & \\ & M_k & \\ & & \varpi^{-(k+l)} \end{pmatrix}\right),$$

for which note that $W_{k,l}$ is checked to be zero for a negative l. We see that W is determined by the $W_{k,l}$'s in view of the Iwasawa decomposition of G_m and the following coset decomposition of G_{m-1} (cf. [39, Lemma 7.2]):

Lemma 4.4 *We have*

$$G_{m-1} = H_\lambda K_{m-1} \sqcup \bigsqcup_{k\geq 1} H_\lambda M_k K_{m-1}^*,$$

where

$$K_{m-1}^* := \{h \in K_{m-1} \mid (h-1)L_{m-1}^\sharp \subset L_{m-1}\}.$$

We say that $W \in \mathcal{W}_\lambda^{\mathcal{F}}$ satisfies the local Maass relation if

$$W_{k,l} - W_{k+1,l-1} = q^{-l} W_{k,0} \quad \forall k \geq 0,\ \forall l \geq 0, \tag{4.1}$$

which is equivalent to

$$W_{k,l} = \sum_{i=0}^{l} q^{-i} W_{k+l-i,0}. \tag{4.2}$$

By $\mathcal{W}_\lambda^{\mathcal{M}}$ we denote the subspace of $\mathcal{W}_\lambda^{\mathcal{F}}$ consisting of those satisfying the local Maass relation.

We now review an explicit structure of $\mathcal{W}_\lambda^{\mathcal{F}}$ and $\mathcal{W}_\lambda^{\mathcal{M}}$ as $\mathcal{H}_m$-modules. For that purpose let $L'_m := \{x \in L_m^\sharp \mid \frac{1}{2}{}^t x Q_m x \in \mathfrak{p}^{-1}\}$ and denote by ∂ the dimension of L'_m/L_m as a vector space over the residual field $\mathfrak{o}/\mathfrak{p}$ of F (cf. [39, p6]). For a non-negative integer m we put

$$f_{m,j} := \frac{q^{j-1}(q^{m-j+1}-1)(q^{m-j+n_0}+q^\partial)}{q^j - 1} \quad (\forall j \in \mathbb{Z}\setminus\{0\}). \tag{4.3}$$

We note that this is a modification of what has been introduced at [39, (7.11)] (for this see also Remark 4.10 (2)). For a positive integers m, r, set $R_m^{(r)} := K_m/(K_m \cap c_m^{(r)} K_m (c_m^{(r)})^{-1})$, and let $|R_m^{(r)}|$ denote the cardinality of $R_m^{(r)}$. We have

$$|R_m^{(r)}| := \begin{cases} \prod_{j=1}^{r} f_{m,j} & (1 \leq r \leq m) \\ 1 & (r = 0). \end{cases} \tag{4.4}$$

Without difficulty it is verified that $\mathcal{W}_\lambda^{\mathcal{F}}$ is stable under the action of $\mathcal{H}_m$. In [39, Corollary 7.5, Corollary 7.8] the aforementioned explicit $\mathcal{H}_m$-module structure of $\mathcal{W}_\lambda^{\mathcal{F}}$ and $\mathcal{W}_\lambda^{\mathcal{M}}$ is given (for this see Remark 4.10 (2)). We state it with the notation $f_{m,j}$ as follows:

Proposition 4.5 *1. Suppose that $m \geq 3$. On $\mathcal{W}_\lambda^{\mathcal{F}}$ the Hecke operators $C_m^{(r)}$ for $r \geq 3$ act as*

$$C_m^{(r)} = |R_{m-2}^{(r-2)}| \left(C_m^{(2)} - \frac{q^{r-2}-1}{q^{r-1}-1} \cdot f_{m-1,1} \cdot C_m^{(1)} + \frac{q^{r-2}-1}{q(q^r-1)} f_{m-1,1} f_{m+1,2} \right).$$

2. *The subspace $\mathcal{W}_\lambda^{\mathcal{M}}$ of $\mathcal{W}_\lambda^{\mathcal{F}}$ is stable under the action by $\mathcal{H}_m$. In addition to the above formula, for $m \geq 2$, $C_m^{(2)}$ coincides with*

$$\begin{cases} f_{m-1,1}C_m^{(1)} + q^4 f_{m-2,1}f_{m-2,2} - q^3 f_{m-2,1}^2 - q^2(q^{2m-4+n_0} - (q-2)q^{\partial})f_{m-2,1} \\ +(q-1)q^{\partial} f_{m-1,1} - q(q^{2m-4+n_0} + q^{\partial})^2 & (m \geq 3), \\ f_{1,1}(C_2^{(1)} - \dfrac{q-1}{q^2-1} f_{2,1}) & (m = 2). \end{cases}$$

as Hecke operators acting on $\mathcal{W}_\lambda^{\mathcal{M}}$.

In particular, the Hecke eigenvalues of an eigenvector in $\mathcal{W}_\lambda^{\mathcal{M}}$ with respect to the $\mathcal{H}_m$-action is determined by the eigenvalue of $C_m^{(1)}$.

This is checked by a direct calculation. Regarding the formula in the second assertion for the case of $m = 2$ we should note that $f_{0,1}$ is defined to be 0, from which we deduce that the formula for $m = 2$ is compatible with that for $m \geq 3$.

We are now able to point out that the description of the $\mathcal{H}_m$-module structure of $\mathcal{W}_\lambda^{\mathcal{M}}$ above can be simplified as follows:

Proposition 4.6 *Suppose that $m \geq 2$. As Hecke operators on $\mathcal{W}_\lambda^{\mathcal{M}}$, we have the coincidence*

$$C_m^{(r)} = |R_{m-1}^{(r-1)}| \left(C_m^{(1)} - \frac{q^{r-1}-1}{q^r - 1} f_{m,1} \right)$$

for $2 \leq r \leq m$.

Proof. The case of $m = 2$ is already shown in Proposition 4.5, 2, for which note that $|R_1^{(1)}| = f_{1,1}$. The proof for the case of $m \geq 3$ starts with the following lemma:

Lemma 4.7 *If we assume that $C_m^{(2)} = |R_{m-1}^{(1)}| \left(C_m^{(1)} - \dfrac{q-1}{q^2-1} f_{m,1} \right)$ holds, we then have the formulas for $C_m^{(r)}$ with $r \geq 3$.*

Proof. Insert the assumed formula for $C_m^{(2)}$ into those in Proposition 4.5, 1. Furthermore note that

$$|R_{m-1}^{(r-1)}| = f_{m-1,1} \times p^{r-2} \cdot \frac{q-1}{q^{r-1}-1} \times |R_{m-2}^{(r-2)}|$$

for $r \geq 3$. Then the lemma is settled by a direct calculation. ∎

What is remaining now is to deduce the formula for $C_m^{(2)}$ from that in Proposition 4.5, 2. This needs the following technical lemma:

Lemma 4.8 *We have*

$$\begin{aligned} & q^2 f_{m-1,1} f_{m-1,2} - q f_{m-1,1}^2 \\ & \qquad = q^4 f_{m-2,1} f_{m-2,2} - q^3 f_{m-2,1}^2 - q^2(q^{2m-4+n_0} - (q-2)q^{\partial}) f_{m-2,1} - q(q^{2m-4+n_0} + q^{\partial})^2. \end{aligned}$$

Proof. To show this we need

$$f_{m-1,1} = qf_{m-2,1} + (q^{2m-4+n_0} + q^{\partial}),$$
$$f_{m-1,2} = qf_{m-2,2} + \frac{q}{q+1}(q^{2m-6+n_0} + q^{\partial}) = \frac{1}{q+1}(f_{m-1,1} - (p^{2m-4+n_0} + q^{\partial})).$$

This verifies the coincidence of both sides. ∎

We therefore see that the formula in Proposition 4.5, 2 implies

$$C_m^{(2)} = |R_{m-1}^{(1)}|(C_m^{(1)} + (q-1) + q^2 f_{m-1,2} - qf_{m-1,1}),$$

for which note that $f_{m-1,1} = |R_{m-1}^{(1)}|$. This is verified to coincide with the desired formula for $C_m^{(2)}$ by a direct computation. As a result we have completed the proof of Proposition 4.6. ∎

For the application to the action of the Hecke operators on F_f we assume that $m = 4n+1$ for $n \geq 1$, $F = \mathbb{Q}_p$, $q = p$ and $\partial = n_0 = 0$. We describe the actions of the Hecke operators $\{C_{4n+1}^{(r)} \mid 1 \leq r \leq 4n+1\}$ on $W \in \mathcal{W}_\lambda^{\mathcal{F}}$ in terms of a recurrence relation of the $W_{k,l}$'s (see [39, Theorem 7.4]).

Proposition 4.9 *Let $C_{4n+1}^{(r)} * W \in \mathcal{W}_\lambda^{\mathcal{F}}$ denote the action of $C_{4n+1}^{(r)}$ on $W \in \mathcal{W}_\lambda^{\mathcal{F}}$. For two non-negative integer k, l we have the following formula:*

$$\begin{aligned}(C_{4n+1}^{(r)} * W)_{k,l} =& |R_{4n-1}^{(r-2)}|\{p^{8n}(W_{k-1,l+2} + u_{r-1}W_{k,l+1} + p^{8n-2}W_{k+1,l}) \\ &+ (p(p^{r-2}-1)u_{r-2} + p^r f_{4n-1,r-1}u_r)W_{k,l} \\ &+ pu_{r-1}W_{k-1,l+1} + p^{8n-1}u_{r-1}W_{k+1,l-1} \\ &+ W_{k-1,l} + u_{r-1}W_{k,l-1} + p^{8n-2}W_{k+1,l-2} - \delta(l=0)p^{8n-1}W_{k,0} \\ &+ \delta(k=0)\{p^{4n-1}\beta_\lambda(p^{8n}(W_{0,l+1} - W_{1,l}) + pu_{r-1}(W_{0,l} - W_{1,l-1}) \\ &+ (W_{0,l-1} - W_{1,l-2})) + p^{8n}W_{0,l+1} + pu_{r-1}W_{0,l} + W_{0,l-1}\} \\ &+ \delta(k=l=0)p^{4n}\beta_\lambda W_{0,0}\},\end{aligned}$$

where we put $u_r := p^r f_{4n-1,r} + p^{r-1} - 1$ and

$$\beta_\lambda := \begin{cases} 0 & (p\text{-adic order of } \frac{1}{2}{}^t\lambda J_{4n}\lambda = 1) \\ -1 & (p\text{-adic order of } \frac{1}{2}{}^t\lambda J_{4n}\lambda = 0) \end{cases}.$$

Here the formula for $r = 1$ needs the following interpretation

$$|R_{4n-1}^{(-1)}| = 0, \ |R_{4n-1}^{(-1)}|f_{4n-1,0} = |R_{4n-1}^{(-1)}|u_0 = 1.$$

We furthermore have $W_{k',l'} = 0$ for negative k', l'.

Remark 4.10 *For this proposition we have two remarks on Sugano's formula [39, Theorem 7.4] in the general case.*
(1) The formula need the notation "ρ_λ" as well as "β_λ" (for their definitions see [39, Proposition

7.3, (2.17)]). For the proposition we see $\rho_\lambda = 0$. We further remark that the proof of Proposition 4.5 is based on Sugano's formula in the general case.
(2) The formula for the case of "$r = \nu + 2$" need the interpretation $f_{\nu,\nu+1} = 0$, which is not referred to in [39]. This is a reason for our modified definition of $f_{m,j}$ (see also the remark just after Proposition 4.5). In addition, we remark that "$3 \le r \le \nu + 1$" should be replaced by "$3 \le r \le \nu + 2$" in [39, Corollary 7.5].

4.2 Hecke theory for our lift F_f

We are now in a position to state our result on the Hecke theory for F_f.

Theorem 4.11 *Suppose that f is a Hecke eigenform and let λ_p be the Hecke eigenvalue of f at $p < \infty$.*
(1) F_f is a Hecke eigenform.
(2) Let μ_i be the Hecke eigenvalue with respect to the Hecke operator $C^{(i)}_{4n+1}$ for $1 \le i \le 4n+1$. We have

$$\mu_1 = p^{4n}(\lambda_p^2 - 2) + pf_{4n,1} = p^{4n}(\lambda_p^2 + p^{4n-1} + \cdots + p + p^{-1} + \cdots + p^{-(4n-1)}),$$
$$\mu_i = |R^{(i-1)}_{4n}| \left(\mu_1 - \frac{p^{i-1}-1}{p^i - 1} f_{4n+1,1}\right), \ (2 \le i \le 4n+1).$$

Proof. To apply Sugano's local theory as in Section 4.1 we need the following:

Lemma 4.12 *Let us fix a prime p. Suppose that $\lambda \in \mathbb{Q}^{8n}$ is a reduced element in the maximal lattice $(\mathbb{Z}_p^{8n}, J_{4n})$ (or $(\mathbb{Z}_p^{8n}, -S)$). For this we remark that $(\mathbb{Z}_p^{8n}, -S)$ is verified to be $GL_{8n}(\mathbb{Z}_p)$-equivalent to $(\mathbb{Z}_p^{8n}, J_{4n})$ by a standard argument using the theory of quadratic forms over p-adic fields.*

1. *As a function on G_p, $A_\lambda(g) \in \mathcal{W}_\lambda^{\mathcal{M}}$ for $g \in G_p$, where we regard g as an element in $\mathcal{G}(\mathbb{A}_f)$ in the usual manner.*

2. *For non-negative integers l, m and a Hecke operator $C \in \mathcal{H}_{4n+1}$ we have*

$$(C * A_\lambda)\left(\begin{pmatrix} p^{l+m} & & \\ & M_m & \\ & & p^{-(l+m)} \end{pmatrix}\right) = p^{-4n(l+m)}(C * A_{p^{l+m}M_m^{-1}\lambda})(1_{8n+2}).$$

Proof. We can check that A_λ satisfies the local Maass relations (4.2) directly. For the proof of the part 1 we only have to prove $A_\lambda(g) \in \mathcal{W}_\lambda^{\mathcal{F}}$ for $g \in G_p$. It suffices to prove

$$A_\lambda\left(\begin{pmatrix} 1 & & \\ & h_0 h & \\ & & 1 \end{pmatrix}\right) = A_\lambda\left(\begin{pmatrix} 1 & & \\ & h & \\ & & 1 \end{pmatrix}\right) \quad \forall (h_0, h) \in H_\lambda \times \mathcal{H}(\mathbb{Q}_p).$$

To this end write $h_0 h$ as $h_0 h = a_0 u_0^{-1}$ with $a_0 \in GL_{8n}(\mathbb{Q})$ and $u_0 \in \prod_{p<\infty} SL_{8n}(\mathbb{Z}_p) \times SL_{8n}(\mathbb{R})$. Then, for every prime p, the condition $a_0^{-1}\lambda = u_0^{-1}(h_0 h)^{-1}\lambda \in \mathbb{Z}_p^{8n}$ implies the greatest power of p dividing the entries in $a_0^{-1}\lambda$ is the same as that of $(h_0h)^{-1}\lambda = h^{-1}\lambda$ (which equals to "$a^{-1}\lambda \in \mathbb{Z}^{8n}$" in the notation of Section 3.3). We thereby see that the greatest common divisor d_λ for L_{h_0h} coincides with that for L_h, and this proves part 1 of the lemma.

In view of the right K_p-invariance of A_λ and the Iwasawa decomposition of G_p, the part 2 of the lemma is reduced to showing

$$A_\lambda\left(\begin{pmatrix} p^{l+m} & & \\ & M_m & \\ & & p^{-(l+m)}\end{pmatrix} n(x)\begin{pmatrix}1 & & \\ & h & \\ & & 1\end{pmatrix}\right) = p^{-4n(l+m)}A_{p^{l+m}M_m^{-1}\lambda}\left(n(x)\begin{pmatrix}1 & & \\ & h & \\ & & 1\end{pmatrix}\right)$$

for any $(x,h) \in \mathbb{Q}_p^{2n} \times \mathcal{H}(\mathbb{Q}_p)$. Reviewing the definition of A_λ, this is verified by a direct calculation. ■

To prove the first assertion of the theorem it suffices to prove that F_f is a Hecke eigenform with respect to the Hecke operator $C^{(1)}_{4n+1}$ by virtue of Proposition 4.5 (or Proposition 4.6) and part 1 of Lemma 4.12.

Proposition 4.13 *For $\lambda \in \mathbb{Q}^{8n}\setminus\{0\}$ we have*

$$(C^{(1)}_{4n+1} * A_\lambda)(1_{8n+2}) = p^{4n}(\lambda_p^2 + p^{4n-1} + \cdots + p + p^{-1} + \cdots + p^{-(4n-1)})A_\lambda(1_{8n+2}).$$

Proof. We fix an arbitrary prime p and may assume that $\lambda \in \mathbb{Z}_p^{8n}$. For the proof of the proposition, the following lemma is crucial.

Lemma 4.14 *(1) The Fourier coefficients $c(n)$ of a Hecke-eigen cusp form f satisfy the following relations:*

$$pc(p^2n) = (\lambda_p^2-1)c(n) - \begin{cases} p^{-\frac{1}{2}}\lambda_p c(n/p) & (p|n) \\ 0 & (p\nmid n)\end{cases},$$
$$pc(p^2n) = (\lambda_p^2-2)c(n) - p^{-1}c(n/p^2),$$

where we assume $p^2|n$ for the second formula.

(2) For a reduced λ we have $(C^{(1)}_{4n+1} * A_\lambda)(\begin{pmatrix} p^{l+m} & & \\ & M_m & \\ & & p^{-(l+m)}\end{pmatrix})$

$$\begin{aligned}
=&p^{8n}\cdot p^{-4n(l+m+1)}A_{p^{l+1}(p^mM_m^{-1}\lambda)}(1_{8n+2}) + p^2 f_{4n-1,1}\cdot p^{-4n(l+m)}A_{p^l(p^mM_m^{-1}\lambda)}(1_{8n+2})\\
&+ p\cdot p^{-4n(l+m)}A_{p^{l+1}(p^{m-1}M_{m-1}^{-1}\lambda)}(1_{8n+2}) + p^{8n-1}\cdot p^{-4n(l+m)}A_{p^{l-1}(p^{m+1}M_{m+1}^{-1}\lambda)}(1_{8n+2})\\
&+ p^{-4n(l+m-1)}A_{p^{l-1}(p^mM_m^{-1}\lambda)}(1_{8n+2})\\
&+ \delta(m=0)\{p^{4n-1}\beta_\lambda p\cdot(p^{-4nl}A_{p^l\lambda}(1_{8n+2}) - p^{-4nl}A_{p^{l-1}(pM_1^{-1}\lambda)}(1_{8n+2}))\\
&+ p\cdot p^{-4nl}A_{p^l\lambda}(1_{8n+2})\}.
\end{aligned}$$

The first assertion is a consequence of the well-known Hecke theory for modular forms of one variable (cf. [15, Section 8.5]). Taking part 2 of Lemma 4.12 into account, we see that the second assertion is nothing but an application of Proposition 4.9 to the case of $r = 1$.

In view of Lemma 4.3 and part 2 of Lemma 4.12 we know that Lemma 4.14 (2) describes the action of $C^{(1)}_{4n+1}$ on A_λ for any $\lambda \in \mathbb{Q}^{8n} \setminus \{0\}$ (or any $\lambda \in \mathbb{Z}_p^{8n} \setminus \{0\}$). As a result we verify the proposition by using Lemma 4.14 (1) and the explicit expression of A_λ in terms of the Fourier coefficients $c(n)$s. ■

To complete the proof of the theorem, we are left with proving the formula for the other Hecke eigenvalues μ_i $(i \geq 2)$. This is an immediate consequence from Proposition 4.6 and part 1 of Lemma 4.12. As a result we are done. ■

5 Cuspidal representations generated by the lifts

This section starts with discussing some general result on automorphic representations of adelized orthogonal groups, which follows from Sugano's local theory. After this we study the cuspidal representation generated by our lift F_f in detail.

5.1 Some global consequence from Sugano's local theory

Unramified principal series representations at non-archimedean places

We now resume the setting of Section 4.1 and can relax the assumption that the characteristic F of the non-archimedean local field is not two. Recall the notations of the groups G_m, K_m and the Hecke algebra $\mathcal{H}_m$ etc. For unramified characters χ_i of $F^\times, 1 \leq i \leq m$, denote an unramified character of the standard split torus of G_m ($\simeq (F^\times)^m$) by $\chi = \mathrm{diag}(\chi_1, \chi_2, \cdots, \chi_m, \chi_m^{-1}, \cdots, \chi_2^{-1}, \chi_1^{-1})$. Let $I(\chi)$ be the normalized parabolic induction of G_m induced from the character of the minimal parabolic subgroup defined by χ. The representation of G_m given by $I(\chi)$ is called an unramified principal series representation. For us it is important to review the fundamental properties of unramified principal series representations of G_m. In many references p-adic reductive groups are often assumed to be connected for the convenience of the argument to study such representations. Though G_m is not connected, we can say that such fundamental properties are still valid for G_m. We need the following lemma:

Lemma 5.1 *1. For any unramified character χ, the unramified principal series representation $I(\chi)$ has a unique irreducible subquotient with a K_m-invariant vector (called a spherical vector). Conversely, any irreducible admissible representation of G_m with a spherical vector (called an irreducible unramified representation) is given by the irreducible subquotient of an unramified principal series representation.*

2. Two irreducible unramified representations are isomorphic to each other if and only if the Hecke eigenvalues of the spherical vectors of the two representations are the same.

Proof. A point of the proof for this lemma is the fact that, as an admissible G_m-module, every irreducible unramified representation is isomorphic to the G_m-module generated by a zonal spherical function and that, up to scalars, every zonal spherical function is uniquely parametrized by an unramified character of the split torus of G_m modulo the conjugation by the Weyl group (cf. [5, Theorem 4.3], [36, Theorem 2]). This implies the second assertion of part 1 of the lemma since the G_m-module generated by a zonal spherical function can be embedded into an unramified principal series representation with the unramified character parametrizing the zonal spherical function. Part 2 of the lemma follows from the bijective correspondence between unitary algebra homomorphisms of the Hecke algebra $\mathcal{H}_m$ to $\mathbb{C}$ and the equivalence classes of unramified characters of the split torus by the Weyl group conjugation (cf. [5, Corollary 4.2]), the latter of which parametrize the equivalence classes of the irreducible unramified representations. We should note that the disconnected-ness of G_m has no influence for these consequences (see [36, Theorem 5, Remark 1 after Theorem 3] and [5, Section 4,4]). In fact, Satake's theory on the Hecke algebras and the zonal spherical functions hold also for G_m, which has the commutative Hecke algebra $\mathcal{H}_m$ and admits the Iwasawa and Cartan decompositions.

We are now left with the first assertion of part 1 of the lemma. The Frobenius reciprocity of induced representations implies that the unramified principal series representation restricted to K_m has the trivial representation of K_m with multiplicity one. Thus there is a contradiction to this multiplicity one condition unless the uniqueness of the irreducible subquotient with a spherical vector holds. Now note that every irreducible unramified representation has a unique spherical vector, up to constant multiples, since it admits a unique zonal spherical function. This leads to the first assertion of part 1 of the lemma, ∎

We call the unique irreducible subquotient of an unramified principal series representation the *spherical constituent*.

Some consequence for a global theory of a general orthogonal group

Let E be a number field and $O(Q_T)$ denote an orthogonal group over E defined by a symmetric matrix $Q_T := \begin{pmatrix} & & 1 \\ & T & \\ 1 & & \end{pmatrix}$, where T denotes a non-degenerate symmetric matrix of degree $l \geq 2$ with entries in E. As the definition indicates the real rank of $O(Q_T)$ is greater than or equal to one, and the group $O(Q_T)$ covers a general class of orthogonal groups including those of real rank one or two, which are in our concern.

For a non-archimedean place v let $\mathcal{O}_v$ denote the completion of the integer ring $\mathcal{O}$ of E at v. For almost all non-archimedean places v's the group $O(Q_T)(E_v)$ of E_v-rational points of $O(Q_T)$ is isomorphic to G_m with a suitable choice of S_0 and has K_m as a maximal compact subgroup with $F = E_v$, $\mathfrak{o} = \mathcal{O}_v$ and $2m + n_0 = l + 2$ (for the notations F, $\mathfrak{o}$, m, n_0 and S_0, see Section 4.1), for which note that we have assumed that $(\mathfrak{o}^{2m+n_0}, Q_m)$ is a maximal lattice at Section 4.1. In fact, it is well-known that, for almost all non-archemedean places v's, the v-adic completion of an arbitrary lattice with the quadratic form defined by Q_T in E^{l+2} is isomorphic to ${\mathcal{O}_v}^{l+2}$ with the quadratic form defined by Q_m.

The group $O(Q_T)$ has a maximal E-parabolic subgroup with the unipotent radical $\mathcal{N}$ defined by the group of E-rational points

$$\mathcal{N}(E) := \left\{ n_T(x) := \begin{pmatrix} 1 & -{}^t xT & -\frac{1}{2}{}^t xTx \\ 0_{l,1} & 1_l & x \\ 0 & 0_{1,l} & 1 \end{pmatrix} \;\middle|\; x \in E^l \right\}.$$

As usual $O(Q_T)(\mathbb{A}_E)$ and $\mathcal{N}(\mathbb{A}_E)$ stand for the adele groups of $O(Q_T)$ and $\mathcal{N}$ respectively. Let Λ_E be a fixed non-trivial additive character of $\mathbb{A}_E/E$. An automorphic form Φ on $O(Q_T)(\mathbb{A}_E)$

admits a Fourier expansion

$$\Phi(g)=\sum_{\lambda\in E^l}\Phi_\lambda(g),\quad \Phi_\lambda(g):=\int_{\mathcal{N}(E)\backslash\mathcal{N}(\mathbb{A}_E)}\Phi(n_T(x)g)\Lambda_E({}^t\lambda Tx)dx.$$

In terms of the Ramanujan conjecture it is important to know how to characterize cuspidal automorphic representations with non-tempered local components. Based on Sugano's local theory, we can provide a following general class of automorphic forms or automorphic representations with a non-tempered local component, which include our non-holomorphic lifts and the Oda-Rallis-Schiffmann lifts (in the appendix).

Theorem 5.2 *Let Φ be an automorphic form on $O(Q_T)(\mathbb{A}_E)$ and π be the automorphic representation generated by Φ. Assume the following:*

1. *π is irreducible and thus decomposes into the restricted tensor product $\pi=\otimes'_{v\le\infty}\pi_v$ of irreducible admissible representations of π_vs at places $v\le\infty$.*
2. *At a non-archimedean place v, the group $O(Q_T)(E_v)$ of the E_v-rational points is isomorphic to G_m with $F=E_v$ and $m\ge 2$, and has K_m as a maximal open compact subgroup.*
3. *Regard Φ as a function in $O(Q_T)(E_v)\simeq G_m$. Suppose that Φ is left K_m-invariant and there exists $\lambda\in E^l\setminus\{0\}$ reduced as an element in E_v^l such that Φ_λ belong to $\mathcal{W}_\lambda^{\mathcal{M}}$.*

The π is non-tempered at the non-archimedean place v.

Proof. The three assumptions imply that the local component π_v is a spherical constituent of an unramified principal series representation. It is well known that the K_m-invariant vector of π_v is a Hecke eigenvector with respect to the Hecke algebra $\mathcal{H}_m$. Thus Φ is a Hecke eigenform with respect to $\mathcal{H}_m$ at the non-archimedean place v. Let $\mu_i(\Phi)$ be the Hecke eigenvalue for the Hecke operator $C_m^{(i)}$ with $1\le i\le m$. From the third assumption and Proposition 4.6 we deduce

$$\mu_i(\Phi)=|R_{m-1}^{(i-1)}|(\mu_1(\Phi)-\frac{q^{i-1}-1}{q^i-1}f_{m,1}) \tag{5.1}$$

for $i\ge 2$, where q denotes the cardinality of the residue field at v. This is also a formula for the Hecke eigenvalue of π_v.

Let $\chi=\mathrm{diag}(\chi_1,\chi_2,\cdots,\chi_m,\chi_m^{-1},\cdots,\chi_2^{-1},\chi_1^{-1})$ be the unramified character such that π_v is the spherical constituent of the unramified principal series representation $I(\chi)$. Let ϖ_v denote a uniformizer of E_v. We are going to show that

$$\chi_i(\varpi_v)=q^{m-i+\frac{n_0}{2}} \tag{5.2}$$

holds for $i\ge 2$. This is nothing but a formula for the Satake parameter $\mathrm{diag}(\chi_1(\varpi_v),\chi_2(\varpi_v),\cdots,\chi_m(\varpi_v),\chi_m(\varpi_v)^{-1},\cdots,\chi_2(\varpi_v)^{-1},\chi_1(\varpi_v)^{-1})$ of π_v. Assuming this we see that $\pi_v|_{G_{m-1}}$ is the trivial representation of G_{m-1}, where we regard G_{m-1} as a subgroup of G_m by embedding G_{m-1} into the middle $2(m-1)+s_0$ block of G_m (for this and the notation s_0 see Section 4.1). In fact, $(\chi_2(\varpi_v),\cdots,\chi_m(\varpi_v),\chi_m(\varpi_v)^{-1},\cdots,\chi_2(\varpi_v)^{-1})$ is the Satake parameter for the trivial representation of G_{m-1}. Then there are a couple of ways to verify that π_v is non-tempered. For example we can show that the matrix coefficient of the spherical vector of π_v is not $(2+\epsilon)$-integrable for any $\epsilon > 0$. This integrability condition is nothing but a definition of the temperedness.

What remains for the proof of the theorem is to verify the formula for χ_is $(2 \le i \le m)$ as above. For that purpose we need the following lemma:

Lemma 5.3 *For $1 \le r \le m$ and $1 \le i \le r$ let $\phi_r(C_r^{(i)})$ denote the $C_r^{(i)}$-action on the spherical vector of π_v by $\pi_v|_{G_r}$, where we regard G_r as a subgroup of G_m by embedding G_r into the middle $2r + s_0$ block of G_m (as we have remarked for G_{m-1} as above). For $i \ge 2$ we have*

$$\phi_m(C_m^{(i)}) = q^{m-1+\frac{n_0}{2}}(\chi_1(\varpi_v) + \chi_1(\varpi_v)^{-1})\phi_{m-1}(C_{m-1}^{(i-1)}) + (q^{i-1}-1)f_{m-1,i-1}\phi_{m-1}(C_{m-1}^{(i-2)}) + q^{i-1}(q^{\partial}-1)\phi_{m-1}(C_{m-1}^{(i-1)}) + q^i\phi_{m-1}(C_{m-1}^{(i)}).$$

Proof. Noting the parabolic induction $I(\chi)$ which contains π_v as the spherical constituent, we can deduce this from the coset decomposition of $K_m c_m^{(i)} K_m$ in [39, Lemma 7.1]. ∎

To complete the proof of the theorem we now assume that the Satake parameter of π_v satisfies the condition as in (5.2). If we can deduce from this assumption that the Hecke eigenvalues of π_v coincide with the formula (5.1), the proof is completed in view of Lemma 5.1, part 2. For the proof we first note that

$$\phi_m(C_m^{(1)}) = q^{m-1+\frac{n_0}{2}}(\chi_1(\varpi_v) + \chi_1(\varpi_v)^{-1}) + (q^{\partial}-1) + qf_{m-1,1}$$

follows from Lemma 5.3. Reviewing the normalization of the invariant measure as in the beginning of Section 4, we next see

$$\phi_{m-1}(C_{m-1}^{(i)}) = |R_{m-1}^{(i)}| \ (1 \le i \le m-1)$$

since the action $\pi_v|_{G_r}$ on the spherical vector is trivial for $r \le m-1$ as is explained above. In addition, we note

$$\frac{|R_{m-1}^{(i)}|}{|R_{m-1}^{(i-1)}|} = f_{m-1,i} \ (1 \le i \le m-1).$$

Hence, from this lemma we get

$$\mu_i(\Phi) = |R_{m-1}^{(i-1)}|(\mu_1(\Phi) + q^{\partial}(q^{i-1}-1) + q^i f_{m-1,i} - qf_{m-1,1}) \quad (i \ge 2).$$

Without difficulty we then verify by a direct calculation that the value of $\mu_i(\Phi)$ with $i \ge 2$ coincides with the formula for $\mu_i(\Phi)$ as in (5.1). Consequently we have proved the theorem. ∎

5.2 The archimedean component of a cuspidal representation generated by a Maass cusp form on $\mathcal{G}(\mathbb{A})$

Let us work over $\mathbb{Q}$ again. We let Γ be an arithmetic subgroup of $O(Q)$ and suppose that $F \in M(\Gamma, r)$ can be adelized to be a cusp form of the adele group $O(Q)(\mathbb{A})$ (e.g. when the strong approximation theorem holds for $O(Q)$), for which we do not have to assume for a moment that $\Gamma = \Gamma_S$ with an even unimodular $(\mathbb{Z}^N, S)$.

By π_F we denote the cuspidal representation generated by F. We are interested in determination of the archimedean representation of π_F. To this end, let $\delta_s : A_\infty \to \mathbb{C}^\times$ be the quasi-character parametrized by $s \in \mathbb{C}$ given by the formula $\delta_s(y) = y^s$. We trivially extend δ_s to a quasi character of the standard proper parabolic subgroup P_∞, which admits a Langlands decomposition $N_\infty A_\infty M_\infty$ with $M_\infty := \left\{ \begin{pmatrix} 1 & {}^t0_N & 0 \\ 0_N & m & 0_N \\ 0 & {}^t0_N & 1 \end{pmatrix} \;\middle|\; m \in U_\infty(= O(S)(\mathbb{R})) \right\}$. By $I_{P_\infty}^{G_\infty}(\delta_s)$ we denote the normalized parabolic induction defined by δ_s. We remark that every spherical principal series representation of G_∞ is of this form. The representation $I_{P_\infty}^{G_\infty}(\delta_s)$ is not always irreducible. In fact, according to [12, Proposition 2.4], it can be reducible when $s + \frac{N}{2} \in \mathbb{Z}$ while otherwise it is always irreducible. A direct calculation of the action of the Casimir operator on the spherical vector, i.e. the K_∞-invariant vector, yields the eigenvalue $\frac{1}{2N}(s^2 - \frac{N^2}{4})$ of $I_{P_\infty}^{G_\infty}(\delta_s)$.

Lemma 5.4 *(1) The spherical principal series representation $I_{P_\infty}^{G_\infty}(\delta_s)$ has a unique irreducible subquotient with a K_∞-invariant vector (i.e. spherical constituent). Every irreducible spherical representation of G_∞, namely an irreducible admissible representation with a K_∞-invariant vector, is an irreducible subquotient of $I_{P_\infty}^{G_\infty}(\delta_s)$ with some s.*
(2) Let π be an irreducible spherical representation of G_∞ with a fixed eigenvalue with respect to the Casimir operator. Assume that π admits a generalized Whittaker model (or Bessel model) with respect to a non-trivial character η of N_∞, i.e. $\mathrm{Hom}_{(\mathfrak{g},K_\infty)}(\pi, \mathrm{Ind}_{N_\infty}^{G_\infty}\eta) \neq \{0\}$ (recall that $\mathfrak{g}$ denotes the Lie algebra of G_∞). Such π is unique, up to isomorphisms.

Proof. (1) This assertion can be said to be a special case of the Langlands classification [21]. In fact, in view of the Langlands classification, an irreducible admissible representation with the trivial K_∞-representation can be embedded into a spherical principal series representation, which is verified to has the trivial K_∞-representation with multiplicity one by means of the Frobenius reciprocity theorem of compact groups. We thus see the uniqueness of the spherical constituent of the spherical principal series and know that every irreducible spherical representation is given as such a constituent.
(2) From (1) we see that π can be embedded into some spherical principal series $I_{P_\infty}^{G_\infty}(\delta_s)$. As we have remarked before the lemma, the eigenvalue of π with respect to the Casimir operator is written as $\frac{1}{2N}(s^2 - \frac{N^2}{4})$ with some s. For $\Phi \in \mathrm{Hom}_{(\mathfrak{g},K_\infty)}(\pi, \mathrm{Ind}_{N_\infty}^{G_\infty}\eta)$, the function $\Phi(\pi(g)v)$ with a spherical vector v of π is a right K_∞-invariant C^∞-function on G_∞ satisfying the left η-equivariance and the same eigenvalue condition with respect to the Casimir operator. Then $\Phi(\pi(g)v)$ is unique up to constant multiples and given explicitly in terms of the K-Bessel function $K_{\sqrt{-1}s}$ as is essentially proved in Proposition 2.3. The assumption $\mathrm{Hom}_{(\mathfrak{g},K_\infty)}(\pi, \mathrm{Ind}_{N_\infty}^{G_\infty}\eta) \neq \{0\}$ (indeed, one dimensional) implies that π is isomorphic to the $(\mathfrak{g}, K_\infty)$-module generated by the right translations of $\Phi(\pi(g)v)$ by G_∞. Now note the basic property $K_r = K_{-r}$ of the K-Bessel function with parameter $r \in \mathbb{C}$. The eigenvalue condition of π implies that there is another possibility that π is embedded into $I_{P_\infty}^{G_\infty}(\delta_{-s})$. Even if it is, the $(\mathfrak{g}, K_\infty)$-module generated by $\Phi(\pi(g)v)$ remains the same by the aforementioned property of the K-Bessel function. As a result we are done. ∎

Proposition 5.5 *(1) If F is a Hecke eigenvector at every finite place (for the definition see [26, Theorem 3.1]), π_F is irreducible.*
(2) Suppose that $F \in M(\Gamma, \sqrt{-1}r)$ with $r \in \mathbb{R}$. The archimedean component of π_F is isomorphic to $I_{P_\infty}^{G_\infty}(\delta_{\sqrt{-1}r})$ as admissible G_∞-modules, and is irreducible. When N is even, it is tempered.

Proof. (1) We use [26, Theorem 3.1], which reduces the problem to the irreducibility of the archimedean local representation of π_F. We first note that, as is well-known, each irreducible cuspidal representation occurs with finite multiplicity in the cuspidal spectrum, which implies that π_F is a finite sum of irreducible cuspidal representations. We therefore see that its archimedean representation is also a finite sum of irreducible admissible representations of G_∞ $(= O(Q)(\mathbb{R}))$. Let us now note that F is right K_∞-invariant, which means that F generates the trivial representation as a K_∞-module. We furthermore note the eigenvalue condition of F with respect to the Casimir operator. A non-zero cusp form F generating an irreducible spherical representation π at the archimedean place provides a non-zero element of $\mathrm{Hom}_{(\mathfrak{g},K_\infty)}(\pi, \mathrm{Ind}_{N_\infty}^{G_\infty}\eta)$ by the Fourier transform of F with respect to η for η such that the Fourier coefficient of F for η is non-zero. We then know from Lemma 5.4 that the archimedean component of π_F is at most a finite copy of one irreducible spherical representation. In fact, F should be inside only one irreducible spherical representation of G_∞ since the infinitesimal action $\mathfrak{g}$-action on F remains the same even if it is a finite sum of the spherical vectors with a fixed eigenvalue of the Casimir operator. As a result π_F should be irreducible.
(2) In view of the first assertion we are left with the irreducibility of $I_{P_\infty}^{G_\infty}(\delta_{\sqrt{-1}r})$. For $r \in \mathbb{R}\setminus\{0\}$ this has been remarked just before Lemma 5.4. For the case of $r = 0$ we remark that it is outside the points of the reducibility for the spherical principal series (cf. [12, p19]). This irreducibility was also proved by Harish Chandra [11, Section 41, Theorem 1]. For the temperedness property see [6, Remark (2.1.13)]. For this we remark that, for an even N, our spherical principal series are the fundamental series representations in the sense of Harish Chandra [11]. ∎

5.3 Cuspidal representation generated by F_f

We resume the setting that $(\mathbb{Z}^N, S)$ with $N = 8n$ is even unimodular. We are now able to show the result on the explicit determination of the cuspidal representation generated by F_f as follows:

Theorem 5.6 *Let f be a Hecke eigenform and let π_{F_f} be the cuspidal representation generated by F_f.*

1. *The representation π_{F_f} is irreducible and thus has the decomposition into the restricted tensor product $\otimes'_{v\le\infty}\pi_v$ of irreducible admissible representations π_v.*

2. *For $v = p < \infty$, π_p is the spherical constituent of the unramified principal series representation of G_p with the Satake parameter*

$$\mathrm{diag}\left(\left(\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}\right)^2, p^{4n-1}, \cdots, p, 1, 1, p^{-1}, \cdots, p^{-(4n-1)}, \left(\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}\right)^{-2}\right).$$

3. For every finite prime $p < \infty$, π_p is non-tempered while π_∞ is tempered.

Proof. This first assertion is a consequence of Theorem 4.11, Propositions 5.5 (1). Let us prove the other two assertions. Since F_f is right K_p-invariant for each finite prime p, π_p has to be the spherical constituent of an unramified principal series representation (cf. Lemma 5.1 part 1). The Hecke eigenvalue μ_1 of F_f enables us to suppose that

$$\mathrm{diag}(\chi_1(p), \chi_2(p), \cdots, \chi_{4n+1}(p), \chi_{4n+1}(p)^{-1}, \cdots, \chi_2(p)^{-1}, \chi_1(p)^{-1}) =$$
$$\mathrm{diag}\left(\left(\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}\right)^2, p^{4n-1}, \cdots, p, 1, 1, p^{-1}, \cdots, p^{-(4n-1)}, \left(\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}\right)^{-2}\right)$$

as the second assertion indicates. Since irreducible unramified representations are determined by Hecke eigenvalues of the spherical vectors up to equivalence (cf. Lemma 5.1 part 2) we need to show the Hecke eigenvalue of a spherical vector of π_p for $C_{4n+1}^{(i)}$ coincides with μ_i for each $i \geq 1$. This is verified by following the proof of Theorem 5.2, which yields the non-temperedness of π_p for all $p < \infty$. The temperedness of π_∞ is a consequence of Proposition 5.5 (2). ∎

As a result of this theorem (or as a result of Theorem 4.11 and [39, Corollary 7.9]) we can write down the standard L-function of π_{F_f} (or F_f).

Corollary 5.7 *For any prime p the local p-factor $L_p(\pi_{F_f}, \mathrm{St}, s)$ of the standard L-function for π_{F_f} (or F_f) is written as*

$$\begin{aligned} L_p(\pi_{F_f}, \mathrm{St}, s) &= \zeta_p(s)(1 - (\lambda_p^2 - 2)p^{-s} + p^{-2s})^{-1} \prod_{j=0}^{8n-2} \zeta_p(s + j - (4n-1)) \\ &= L_p(\mathrm{sym}^2(f), s) \prod_{j=0}^{8n-2} \zeta_p(s + j - (4n-1)), \end{aligned}$$

where ζ_p denotes the p-factor of the Riemann zeta function and $L_p(\mathrm{sym}^2(f), s)$ is the p-factor of the symmetric square L-function for f.

6 Appendix: Cuspidal representations generated by Oda-Rallis-Schiffmann lifts

We have used Sugano's non-archimedean local theory in [39, Section 7] to study the Hecke theory of the cusp forms given by our lifting and the cuspidal representations generated by them. His local theory is originally motivated by studying non-archimedean local aspect of the lifting theory of Oda [28] and Rallis-Schiffmann [35]. We can therefore expect that the results in Section 4 and 5 naturally hold also for the lifting by Oda and Rallis-Schiffmann. In this appendix, still restricting ourselves to "the case of even unimodular lattices", we carry out the argument similar to Sections 4 and 5 to deduce similar results for the case of Oda-Rallis-Schiffmann lifting.

6.1 Basic notation

Let $(\mathbb{Z}^{8n}, S)$ be an even unimodular lattice with a positive definite symmetric matrix S and put $Q_1 := \begin{pmatrix} & & 1 \\ & -S & \\ 1 & & \end{pmatrix}$. We then let $Q_2 := \begin{pmatrix} & & 1 \\ & Q_1 & \\ 1 & & \end{pmatrix}$ and let $\mathcal{G} = O(Q_2)$ (respectively $\mathcal{H} = O(Q_1)$) be the orthogonal group over $\mathbb{Q}$ defined by Q_2 (respectively Q_1). We introduce several algebraic subgroups of $\mathcal{G}$. We first introduce the maximal parabolic subgroup $\mathcal{P}$ with a Levi decomposition $\mathcal{P} = \mathcal{N}_1 \rtimes \mathcal{L}_1$, where $\mathcal{N}_1$ and $\mathcal{L}_1$ are defined by the groups of $\mathbb{Q}$-rational points as follows:

$$\mathcal{N}_1(\mathbb{Q}) = \left\{ n_{Q_1}(x) = \begin{pmatrix} 1 & -{}^t x Q_1 & -\frac{1}{2}{}^t x Q_1 x \\ & 1_{8n+2} & x \\ & & 1 \end{pmatrix} \;\middle|\; x \in \mathbb{Q}^{8n+2} \right\},$$

$$\mathcal{L}_1(\mathbb{Q}) = \left\{ \begin{pmatrix} a & & \\ & h & \\ & & a^{-1} \end{pmatrix} \;\middle|\; a \in \mathbb{Q}^\times,\ h \in O(Q_1)(\mathbb{Q}) \right\}.$$

For $w \in \mathbb{Q}^{8n}$ let $n_0(w) := \begin{pmatrix} 1 & {}^t w S & \frac{1}{2}{}^t w S w \\ & 1_{8n} & w \\ & & 1 \end{pmatrix}$ and $n_1(w) := \begin{pmatrix} 1 & & \\ & n_0(w) & \\ & & 1 \end{pmatrix}$. We then introduce the maximal unipotent subgroup $\mathcal{N}$ of $\mathcal{G}$ defined by its group of $\mathbb{Q}$-rational points

$$\mathcal{N}(\mathbb{Q}) := \{ n(x, w) \mid x \in \mathbb{Q}^{8n+2},\ w \in \mathbb{Q}^{8n} \},$$

where $n(x, w) := n_{Q_1}(x) n_1(w)$.

Let G_∞ be the real Lie group $\mathcal{G}(\mathbb{R})$. To describe an Iwasawa decomposition of G_∞ we introduce

$$A_\infty := \left\{ \begin{pmatrix} a_1 & & & & \\ & a_2 & & & \\ & & 1_{8n} & & \\ & & & a_2^{-1} & \\ & & & & a_1^{-1} \end{pmatrix} \;\middle|\; a_1,\ a_2 \in \mathbb{R}_+^\times \right\}$$

and a maximal compact subgroup

$$K_\infty := \{ k \in G_\infty \mid {}^t k R k = R \}$$

of G_∞, where $R = \begin{pmatrix} 1_2 & & \\ & S & \\ & & 1_2 \end{pmatrix}$ is the majorant of Q_2. We then have an Iwasawa decomposition of G_∞ as follows:

$$G_\infty = \mathcal{N}(\mathbb{R}) A_\infty K_\infty.$$

We next introduce the symmetric domain of type IV, which is identified with the quotient G_∞ / K_∞. We follow [25, Section 1.4] to describe it. Let B_{Q_1} be the bilinear form on $V \times V$ defined by Q_1 with $V = \mathbb{R}^{8n+2}$ and let (V, τ) be the Euclidean Jordan algebra equipped with the trace form

$$\tau : V \times V \ni (x, y) \mapsto \tau(x, y) = 2B_{Q_1}(x \circ y, e),$$

where

$$x \circ y := B_{Q_1}(x,e)y + B_{Q_1}(y,e)x - B_{Q_1}(x,y)e, \quad (x, y \in V)$$

with ${}^t e = (\frac{1}{\sqrt{2}}, 0, \cdots, 0, \frac{1}{\sqrt{2}})$. This Euclidean Jordan algebra has the determinant Δ given by

$$\Delta(v) := \frac{1}{2} B_{Q_1}(v,v) \quad (v \in V).$$

Let us introduce the symmetric cone $\Omega := \{v \in V \mid B_{Q_1}(v,v) > 0,\ B_{Q_1}(v,e) > 0\}$ of V. Then the symmetric domain of type IV corresponding to G_∞ is realized as $\mathcal{D} := V + \sqrt{-1}\Omega$. The Lie group G_∞ acts on $\mathcal{D}$ by the linear fractional transformation, for which we use the notation $g \cdot z$ for $(g, z) \in G_\infty \times \mathcal{D}$. Let $J(g, z) \in \mathbb{C}$ be the automorphy factor for $(g, z) \in G_\infty \times \mathcal{D}$. For the definition of $g \cdot z$ and $J(g, z)$ see [10, Section 1]. We can identify G_∞/K_∞ with $\mathcal{D}$ by $G_\infty \ni g \mapsto g \cdot (\sqrt{-1}e) \in \mathcal{D}$.

6.2 Review on Oda-Rallis-Schiffmann lifting

By $S_\kappa(SL_2(\mathbb{Z}))$ we denote the space of holomorphic cusp forms on the complex upper half plane $\mathfrak{h}$ of weight κ with respect to $SL_2(\mathbb{Z})$. To review the Oda-Rallis-Schiffmann lift from these holomorphic cusp forms we introduce the archimedean Whittaker function $W_{\lambda,\kappa}$ on G_∞ with $\lambda \in \Omega$ and a positive integer κ by

$$\begin{aligned} &W_{\lambda,\kappa}(n(x,w)ak) \\ &\quad := J(k_\infty, \sqrt{-1}e)^{-\kappa} \Delta(\mathrm{Im}(n_1(w)a \cdot \sqrt{-1}e))^{\frac{\kappa}{2}} \exp(2\pi\sqrt{-1}\tau(\lambda, x + \sqrt{-1}\mathrm{Im}(n_1(w)a \cdot \sqrt{-1}e)) \end{aligned}$$

for $(x, w, a, k) \in \mathbb{R}^{8n+2} \times \mathbb{R}^{8n} \times A_\infty \times K_\infty$, where $\mathrm{Im}(z)$ denotes the imaginary part of $z \in \mathcal{D}$.

Let $f \in S_{\kappa-4n+2}(SL_2(\mathbb{Z}))$ be given by the q-expansion $f(\tau) = \sum_{m \ge 1} c(m)q^m$ (thus κ has to be even and $\kappa - 4n + 2 \ge 12$). We put $|\lambda|_{Q_1} := \sqrt{\frac{1}{2}{}^t\lambda Q_1 \lambda} = \sqrt{\Delta(\lambda)}$ for $\lambda \in V$. We introduce a smooth function F_f on G_∞ by

$$F_f(g_\infty) = \sum_{\lambda \in \mathbb{Z}^{8n+2} \cap \Omega} C_\lambda W_{\lambda,\kappa}(g_\infty),$$

where

$$C_\lambda = \sum_{d \mid d_\lambda} d^{\kappa-1} c(\frac{|\lambda|^2_{Q_1}}{d^2})$$

with the greatest common divisor d_λ of the entries of λ. For the maximal lattice $\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}^{8n} \oplus \mathbb{Z}^{\oplus 2}$ with respect to Q_2 we introduce an arithmetic subgroup

$$\Gamma_S := \{\gamma \in \mathcal{G}(\mathbb{Q}) \mid \gamma(\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}^{8n} \oplus \mathbb{Z}^{\oplus 2}) = \mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}^{8n} \oplus \mathbb{Z}^{\oplus 2}\}.$$

We now state the following theorem by Oda [28, Corollary to Theorem 5] and Rallis-Schiffmann [35, Theorem 5.1].

Theorem 6.1 (Oda, Rallis-Schiffmann) *For $\kappa > 8n + 4$ the smooth function F_f is a holomorphic cusp form of weight κ with respect to Γ_S, lifted from the domain $\mathcal{D}$ to the group G_∞.*

6.3 Cuspidal representations generated by Oda-Rallis-Schiffmann lifts.

(1) Adelization of F_f.

To consider the cuspidal representation generated by F_f we adelize F_f. We carry out it following the argument similar to Section 3.3.

Let $K_f := \prod_{p<\infty} K_p$ with $K_p := \{g \in \mathcal{G}(\mathbb{Q}_p) \mid g\mathbb{Z}_p^{8n+4} = \mathbb{Z}_p^{8n+4}\}$. We remark that the strong approximation theorem of $\mathcal{G}(\mathbb{A})$ with respect to the maximal compact subgroup K_f holds, from which we deduce that the set of Γ_S-cusps is in bijection with $\mathcal{H}(\mathbb{Q})\backslash\mathcal{H}(\mathbb{A})/\mathcal{H}(\mathbb{R})U_f$ with $U_f := \prod_{p<\infty} U_p$ ($U_p := \{h \in \mathcal{H}(\mathbb{Q}_p) \mid h\mathbb{Z}_p^{8n+2} = \mathbb{Z}_p^{8n+2}\}$). This is nothing but Lemma 2.1 for the case of $\mathcal{G} = O(Q_2)$.

For $h = (h_p)_{p\le\infty} \in \mathcal{H}(\mathbb{A}_f)$ we put $L_h := (\prod_{p<\infty} h_p\mathbb{Z}_p^{8n+2} \times \mathbb{R}^{8n+2}) \cap \mathbb{Q}^{8n+2}$ and write $h = au^{-1}$ with $(a,u) \in GL_{8n+2}(\mathbb{Q}) \times (\prod_{p<\infty} SL_{8n+2}(\mathbb{Z}_p) \times SL_{8n+2}(\mathbb{R}))$. For $\lambda \in L_h \setminus \{0\}$ we denote by d_λ the greatest common divisor of the entries of $a^{-1}\lambda \in \mathbb{Z}^{8n+2}$, which is checked to be well-defined by the same argument as the proof of Lemma 3.2.

We introduce a function A_λ indexed by $\lambda \in \mathbb{Q}^{8n+2} \setminus \{0\}$ as follows:

$$A_\lambda\left(\begin{pmatrix}1 & & \\ & h & \\ & & 1\end{pmatrix}\right) := \begin{cases}\sum_{d|d_\lambda} d^{\kappa-1}c(\frac{|\lambda|^2_{Q_1}}{d^2}) & (\lambda \in L_h)\\ 0 & (\lambda \in \mathbb{Q}^{8n+2}\setminus L_h)\end{cases},$$

$$A_\lambda\left(\begin{pmatrix}\beta & & \\ & h & \\ & & \beta^{-1}\end{pmatrix}\right) := ||\beta||_{\mathbb{A}}^{\kappa} A_{||\beta||_{\mathbb{A}}^{-1}\lambda}\left(\begin{pmatrix}1 & & \\ & h & \\ & & 1\end{pmatrix}\right) \quad \forall(\beta,h) \in \mathbb{A}_f^\times \times \mathcal{H}(\mathbb{A}_f),$$

$$A_\lambda(n_2(x)gk) := \Lambda({}^t\lambda Q_1 x)A_\lambda(g) \quad \forall(x,g,k) \in \mathbb{A}_f^{8n+2} \times \mathcal{G}(\mathbb{A}_f) \times K_f,$$

where Λ denotes the standard additive character of $\mathbb{A}/\mathbb{Q}$. This A_λ is verified to be well-defined function on $\mathcal{G}(\mathbb{A}_f)$ similarly as in the proof of Lemma 3.2. With this A_λ we adelize F_f by

$$F_f(g) = \sum_{\lambda \in \mathbb{Q}^{8n+2}\setminus\{0\}} A_\lambda(g_f)W_{\lambda,\kappa}(g_\infty)$$

for $g = g_f g_\infty \in \mathcal{G}(\mathbb{A})$ with $(g_f, g_\infty) \in \mathcal{G}(A_f) \times G_\infty$. By the definition of the adelized F_f, F_f is right K_f-invariant. By the standard argument in terms of the strong approximation theorem the left Γ_S-invariance of the non-adelic F_f then implies the left $\mathcal{G}(\mathbb{Q})$-invariance of the adelic F_f. The adelized F_f is a cusp form on $\mathcal{G}(\mathbb{A})$.

(2) Cuspidal representation generated by F_f

To determine explicitly the cuspidal representation of $\mathcal{G}(\mathbb{A})$ generated by F_f we first provide an explicit formula for Hecke eigenvalues of the adelized F_f. We can apply the non-archimedean local theory in Section 4.1 to our situation that $m = 4n+2$, $q = p$, $F = \mathbb{Q}_p$, and $\partial = n_0 = 0$. The p-adic group $\mathcal{G}(\mathbb{Q}_p)$ is viewed as G_{4n+2} in the notation of Section 4.1. We need the lemma as follows:

Lemma 6.2 *As a function on $\mathcal{G}(\mathbb{Q}_p)(\simeq G_{4n+2})$, $A_\lambda(g) \in \mathcal{W}_\lambda^{\mathcal{M}}$, where we regard $g \in \mathcal{G}(\mathbb{Q}_p)$ as an element in $\mathcal{G}(\mathbb{A})$ in the usual manner.*

We then state the theorem on the Hecke eigenvalues of F_f.

Theorem 6.3 *Suppose that f is a Hecke eigenform and let λ_p be the Hecke eigenvalue of f at $p < \infty$.*
(1) F_f is a Hecke eigenform.
(2) Let μ_i be the Hecke eigenvalue of the Hecke operator for $C^{(i)}_{4n+2}$ with $1 \le i \le 4n+2$. We have

$$\mu_i = \begin{cases} p^{4n+1}(p^{-(\kappa-4n-1)}\lambda_p^2 + p^{4n} + \cdots + p + p^{-1} + \cdots + p^{-4n}) & (i=1) \\ |R^{(i-1)}_{4n+1}|(\mu_1 - \dfrac{p^{i-1}-1}{p^i-1} f_{4n+2,1}) & (2 \le i \le 4n+2) \end{cases}.$$

Proof. We give just an overview of the proof since it is quite similar to the case of the lifting from the Maass cusp forms. The only difference is the recurrence relation for the Fourier coefficients of the holomorphic cusp form f as follows:

Lemma 6.4 *Let $f(\tau) = \sum_{n\ge 1} c(m)q^m \in S_{\kappa-(4n-2)}(SL_2(\mathbb{Z}))$ be a Hecke eigenform. We have*

$$c(p^2m) = (\lambda_p^2 - p^{\kappa-(4n-1)})c(m) - \begin{cases} p^{\kappa-(4n-1)}\lambda_p c(m/p) & (p|m) \\ 0 & (p \nmid m) \end{cases},$$
$$c(p^2m) = (\lambda_p^2 - 2p^{\kappa-(4n-1)})c(m) - p^{2(\kappa-(4n-1))}c(m/p^2),$$

where we assume $p^2|m$ for the second formula.

This follows from the well known recurrence relation of the Fourier coefficients (cf. [37, Chapitre VII, Section 5.3, Corollaire 2]).

With this lemma and [39, Theorem 7.4] for $\mathcal{W}^{\mathcal{F}}_\lambda$ on G_{4n+2} (similar to Proposition 4.9) , we get the explicit formula for μ_1 by the proof similar to that of Proposition 4.13. The formula for μ_i with $i \ge 2$ is then an immediate consequence from Proposition 4.6. ∎

Cuspidal representation generated by F_f

We now state the theorem quite similar to Theorem 5.6.

Theorem 6.5 *Let π_{F_f} be the cuspidal representation generated by F_f and suppose that f is a Hecke eigenform.*
(1) The representation π_{F_f} is irreducible and thus has the decomposition into the restricted tensor product $\otimes'_{v\le\infty}\pi_v$ of irreducible admissible representations π_v.
(2) For $v = p < \infty$, π_p is the spherical constituent of the unramified principal series representation of G_p with the Satake parameter

$$\mathrm{diag}\left(\left(\frac{\lambda'_p + \sqrt{\lambda'^2_p - 4}}{2}\right)^2, p^{4n}, \cdots, p, 1, 1, p^{-1}, \cdots, p^{-4n}, \left(\frac{\lambda'_p + \sqrt{\lambda'^2_p - 4}}{2}\right)^{-2}\right),$$

where $\lambda'_p := p^{-\frac{\kappa-(4n-1)}{2}}\lambda_p$.
(3) For every finite prime $p < \infty$, π_p is non-tempered while π_∞ is tempered.

Proof. Also for this theorem we sketch the proof since it is similar to that of Theorem 5.6. Let $\mathfrak{g}_\infty$ be the Lie algebra of G_∞. The right translations of F_f by G_∞ generate the anti-holomorphic discrete series representation π_κ with minimal K_∞-type given by

$$K_\infty \ni k \mapsto J(k, \sqrt{-1}e)^{-\kappa}$$

as a $(\mathfrak{g}_\infty, K_\infty)$-module, and it is irreducible. This is known as a classical fact due to Rallis-Schiffmann (for instance see [34, Section 6]). Since F_f is a Hecke eigenform under the assumption we see the irreducibility of π_F by [26, Theorem 3.1]. This is nothing but the first assertion. Due to Theorem 6.3 the rest of the assertions are settled by the proof similar to parts 2 and 3 of Theorem 5.6. For the third assertion we remark that the discrete series representations of semi-simple real Lie groups are a well-known class of tempered representations. ∎

As we deduce Corollary 5.7 from Theorem 5.6 we have the following as an immediate consequence from Theorem 6.5.

Corollary 6.6 *For any prime p the local p-factor $L_p(\pi_{F_f}, \mathrm{St}, s)$ of the standard L-function for π_{F_f} (or F_f) is written as*

$$L_p(\pi_{F_f}, \mathrm{St}, s) = L_p(\mathrm{sym}^2(f), s) \prod_{j=0}^{8n} \zeta_p(s + j - 4n),$$

where $L_p(\mathrm{sym}^2(f), s)$ is the p-factor of the symmetric square L-function for f.

Remark 6.7 *This result is essentially obtained in [39, Theorem 8.1], which expresses the standard L-functions of the Oda-Rallis-Schiffmann lifts in the Jacobi form formulation in terms of L-functions of Jacobi forms. Sugano has remarked that $L_p(\mathrm{sym}^2(f), s)$ is a local factor of the L-function of some Jacobi form.*

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